Algebraic and model-theoretic methods in constraint satisfaction

3rd session

Michael Pinsker

Technische Universität Wien / Université Diderot - Paris 7

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Outline reminder

Part I: CSPs / dividing the world /
pp definitions, polymorphism clones, ω-categoricity

Part II: pp interpretations / topological clones

Part III: Canonical functions, Ramsey structures / Graph-SAT

Part IV: Model-complete cores / The infinite tractability conjecture

Summary of last session

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Let Γ be a structure in a finite relational language τ.

Definition

CSP(Γ) is the decision problem:

INPUT: variables $x_1, \ldots, x_n$ and atomic τ-statements about them.

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Reason: HSP$^{\text{fin}}$ / pp interpretations.

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Corollary

Let Γ be ω-categorical. TFAE:

- $(\{0,1\};\{(1,0,0),(0,1,0),(0,0,1)\})$ has a pp-interpretation in Γ
- there exists a continuous homomorphism from $\mathrm{Pol}(\Gamma)$ onto **1**
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Straightforward: $\xi : \mathrm{Pol}(\Gamma) \to \mathbf{1}$ is continuous homomorphism.

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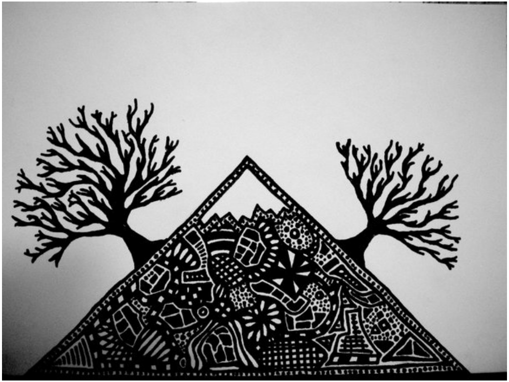
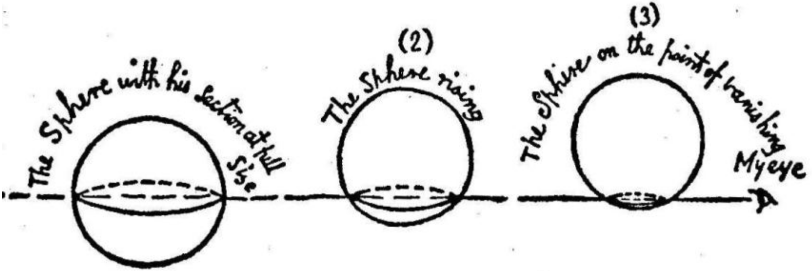
Blackboard



Part III:

Canonical functions, Ramsey structures / Graph-SAT



Canonical functions and Ramsey structures

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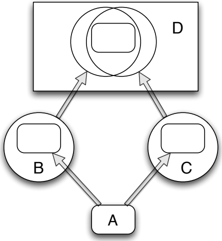

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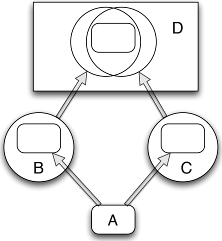

Δ homogeneous $\leftrightarrow$ for all finite $A, B \subseteq \Delta$, for all isomorphisms $i : A \to B$ there exists $\alpha \in \mathrm{Aut}(\Delta)$ extending i.

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Let $\Gamma = (D; R_1, \ldots, R_n)$ be a reduct of Δ
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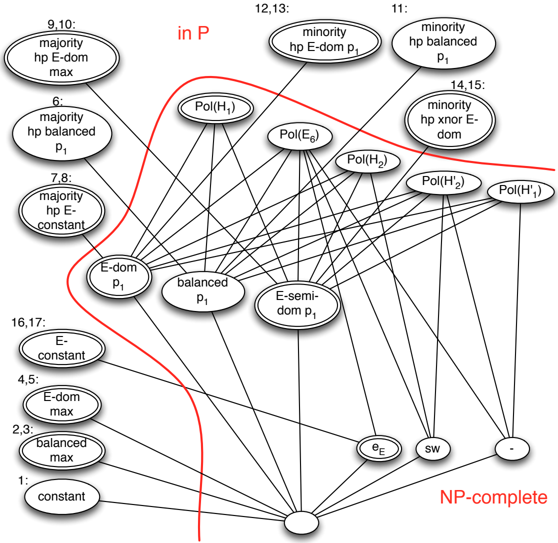
Examples: Graph-SAT and Temp-SAT always in P / NP-complete.

Graph-SAT classification



Clones of reducts

Theorem (follows from Ryll-Nardzewski, Engeler, Svenonius)

Let Δ be ω-categorical, and let Γ be a structure on the same domain. TFAE:

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Closed function clones on fixed domain form complete lattice:

- Intersection of function clones is function clone
- Intersection of closed sets is closed.

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Flipping edges and non-edges around a vertex $c \in G$
not canonical on G, but canonical on (G, c).

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For all finite substructures P, H of Δ:
Whenever we color the copies of P in Δ with 2 colors
then there is a monochromatic copy of H in Δ.

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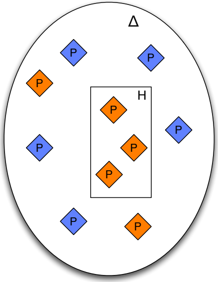
For all finite substructures P, H of Δ:
Whenever we color the copies of P in Δ with 2 colors
then there is a monochromatic copy of H in Δ.



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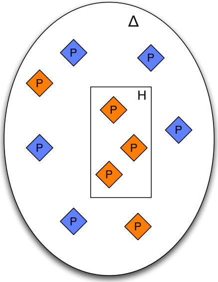

Theorem (Nešetřil + Rödl)

The random ordered graph is Ramsey.

Canonizing functions on Ramsey structures

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Proposition (Bodirsky + MP + Tsankov '11)

Let

- Δ be ordered Ramsey homogeneous finite relational language
- $f : \Delta^n \to \Delta$
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Proof: Via topological dynamics (Kechris + Pestov + Todorcevic '05).

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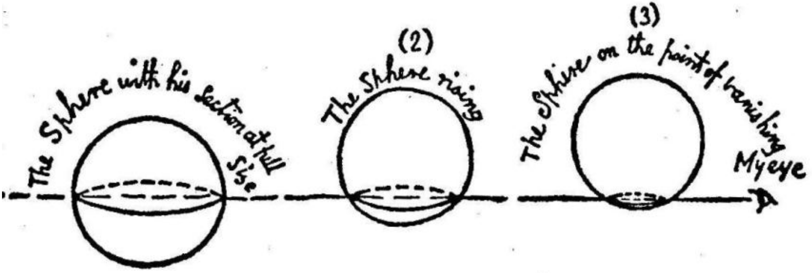
Conclusion: We only care about canonical functions in a function clone (in fact they are dense in the clone).


The Sphere with his section at full Size
(2)
The Sphere rising
(3)
The Sphere on the point of vanishing
My eye
A

Graph-SAT

Complexity of CSP for reducts of G

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Theorem (Bodirsky + MP '10)

Let Γ be a reduct of the random graph. Then:

- Either Γ has one out of 17 canonical polymorphisms, and CSP(Γ) is tractable,
- or CSP(Γ) is NP-complete.

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Theorem (Bodirsky + MP '10)

Let Γ be a reduct of the random graph. Then:

- Either Γ pp-defines one out of 5 hard relations, and CSP(Γ) is NP-complete,
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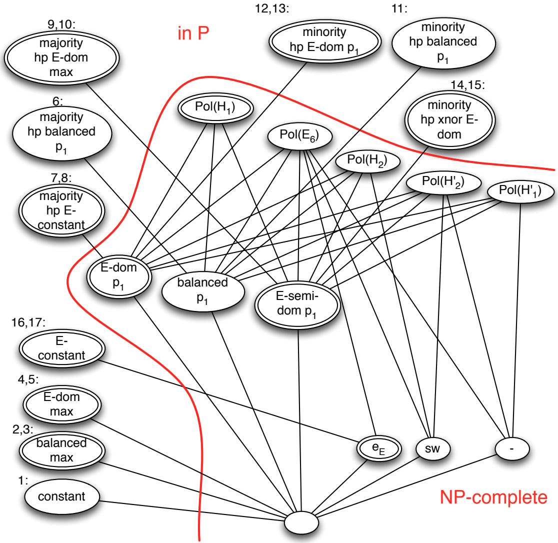
Graph-SAT classification



Theorem

The following 17 distinct clones are precisely the minimal tractable closed function clones containing Aut(G):

1. The clone generated by a constant operation.
2. The clone generated by a balanced binary injection of type max.
3. The clone generated by a balanced binary injection of type min.
4. The clone generated by an E-dominated binary injection of type max.
5. The clone generated by an N-dominated binary injection of type min.
6. The clone generated by a function of type majority which is hyperplanely balanced and of type projection.
7. The clone generated by a function of type majority which is hyperplanely E-constant.
8. The clone generated by a function of type majority which is hyperplanely N-constant.
9. The clone generated by a function of type majority which is hyperplanely of type max and E-dominated.
10. The clone generated by a function of type majority which is hyperplanely of type min and N-dominated.

The Meta Problem

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Meta-Problem of Graph-SAT(Ψ)

INPUT: A finite set Ψ of graph formulas.

QUESTION: Is Graph-SAT(Ψ) in P?

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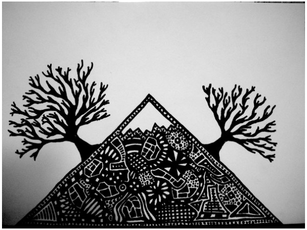
Theorem (Bodirsky + MP '10)

The Meta-Problem of Graph-SAT(Ψ) is decidable.

From dreams I proceed to facts.



Part IV: In 10 minutes