Constraint Satisfaction on Infinite Domains

2nd session

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Technische Universität Wien / Université Diderot - Paris 7

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Algebraic and Model Theoretical Methods in Constraint Satisfaction
Banff International Research Station

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Outline reminder

Part I: CSPs / dividing the world /
pp definitions, polymorphism clones, ω-categoricity

Part II: pp interpretations / topological clones

Part III: Canonical functions, Ramsey structures / Graph-SAT

Part IV: Model-complete cores / The infinite tractability conjecture

Reminder from 1st session

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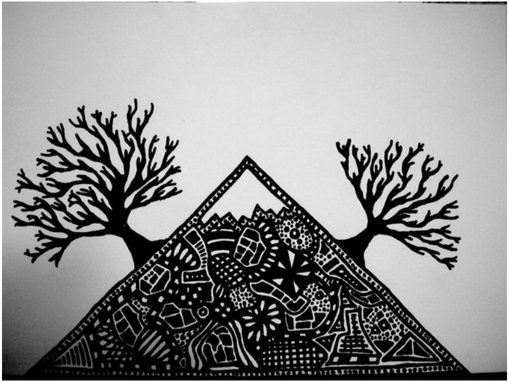
Polymorphisms preserve:

- arbitrary intersections
- directed unions



Part II:

pp interpretations / topological clones

Algebraic constructions

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CSP and HSP$^{\text{fin}}$

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For a function clone **C**:

CSP and $\mathrm{HSP}^{\mathrm{fin}}$

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- $\mathrm{H}(\mathbf{C})$... function clones obtained by factoring by congruence.
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Proof sketch

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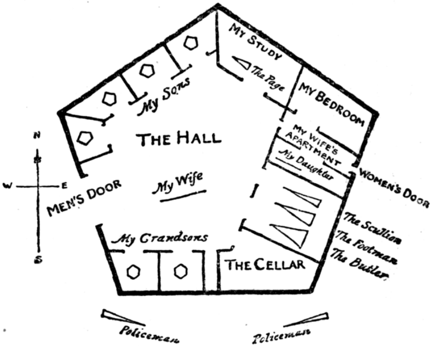
- Δ can be simulated ("pp interpreted") on pp-definable factor of pp-definable subset of finite power of Γ.



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Set of all finitary functions $\bigcup_n D^{D^n}\ldots$ sum space.

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For finite function clones: topology discrete.

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Theorem ("Topological Birkhoff" Bodirsky + MP '12)

Let Δ, Γ be ω-categorical or finite. TFAE:

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Let Δ, Γ be ω-categorical or finite and such that $\mathrm{Pol}(\Delta) \cong \mathrm{Pol}(\Gamma)$. Then $\mathrm{CSP}(\Delta)$ and $\mathrm{CSP}(\Gamma)$ are polynomial-time equivalent.

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Let Δ, Γ be ω-categorical or finite. TFAE:

- Δ has a pp interpretation in Γ;
- there exists a continuous homomorphism $\xi \colon \mathrm{Pol}(\Gamma) \to \mathrm{Pol}(\Delta)$ whose image is dense in an oligomorphic function clone.

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- there exists a continuous clone homomorphism $\xi\colon$ Pol(Γ) $\to$ Pol(Γ') for all structures Γ';
- all finite Γ' have a pp interpretation in Γ.

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Set $\xi(f)$ to be the i-th k-ary projection in **1**.

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Straightforward: $\xi : \mathrm{Pol}(\Gamma) \to \mathbf{1}$ is continuous homomorphism.

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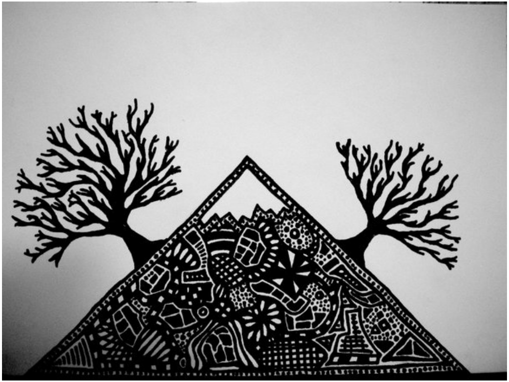
So the Betweenness problem is NP-hard.



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Complexity of CSP(Γ) only depends on Pol(Γ).

Clones of reducts

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Let Δ be ω-categorical, and let Γ be a structure on the same domain. TFAE:

- Γ is a reduct of Δ;
- $\mathrm{Aut}(\Gamma) \supseteq \mathrm{Aut}(\Delta)$;
- $\mathrm{Pol}(\Gamma) \supseteq \mathrm{Aut}(\Delta)$.

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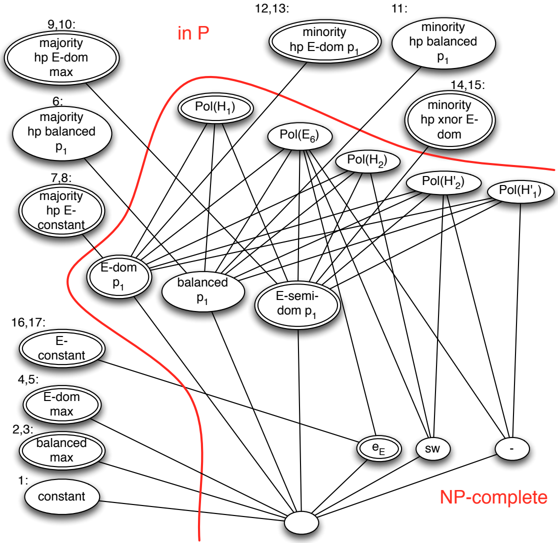
Closed function clones on fixed domain form complete lattice:

- Intersection of function clones is function clone
- Intersection of closed sets is closed.

Graph-SAT classification



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- an injective function whose image is an independent set.

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Let Δ be a structure.

$f \colon \Delta^n \to \Delta$ is canonical iff
for all tuples $t_1, \ldots, t_n$ of the same length
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Flipping edges and non-edges around a vertex $c \in G$
not canonical on G, but canonical on (G, c).

Ramsey structures

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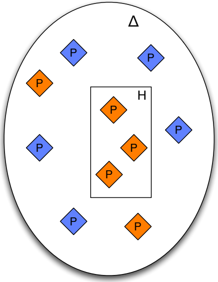

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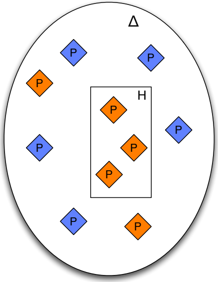

Theorem (Nešetřil + Rödl)

The random ordered graph is Ramsey.

Canonizing functions on Ramsey structures

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Proposition (Bodirsky + MP + Tsankov '11)

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Proof: Via topological dynamics (Kechris + Pestov + Todorcevic '05).

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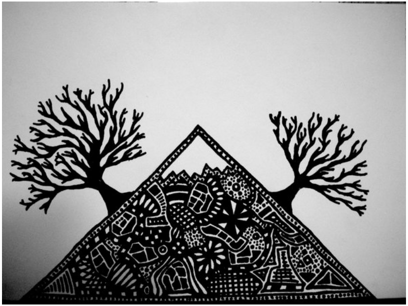
Canonical functions of same behavior belong to the same closed clones.

Conclusion: We only care about canonical functions in a function clone (in fact they are dense in the clone).

“I am indeed, in a certain sense a Circle," replied the Voice,
“and a more perfect Circle than any in Flatland;
but to speak more accurately,
I am many Circles in one."



3rd session: tomorrow 9:00