Reducts of the random permutation

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- a relational structure $(A; <_1, <_2)$

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Question (Cameron, 2002)

What are the closed supergroups of $\mathrm{Aut}(\Pi)$?

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- Conjecture (Simon Thomas, 1991): If Δ is a countable relational structure which is homogeneous in a finite language, then $\mathrm{Aut}(\Delta)$ has only finitely many closed supergroups.
- classifying computational complexity of CSPs involving finite permutations

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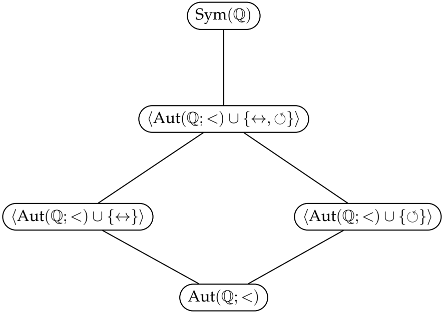
Theorem (Cameron, 1976)

The closed supergroups of $\mathrm{Aut}(\mathbb{Q};<)$ are

- $\mathrm{Aut}(\mathbb{Q};<)$
- $\langle \mathrm{Aut}(\mathbb{Q};<) \cup \{\leftrightarrow\} \rangle$
- $\langle \mathrm{Aut}(\mathbb{Q};<) \cup \{\circlearrowleft\} \rangle$
- $\langle \mathrm{Aut}(\mathbb{Q};<) \cup \{\leftrightarrow, \circlearrowleft\} \rangle$
- $\mathrm{Sym}(\mathbb{Q})$



Connection with model theory

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$$\begin{aligned} \{\text{reducts of } \Delta\}/\!\sim &\to \{\text{closed supergroups of } \operatorname{Aut}(\Delta)\} \\ \Gamma/\!\sim &\mapsto \operatorname{Aut}(\Gamma) \end{aligned}$$

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Example

$\mathrm{Btw}(x, y, z) \Leftrightarrow (x < y < z) \vee (z < y < x)$
$\operatorname{Aut}(\mathbb{Q}; \mathrm{Btw}) = \langle \operatorname{Aut}(\mathbb{Q}, <) \cup \{\leftrightarrow\} \rangle$

A model of Th(Π)

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Let $D \subseteq \mathbb{Q}^2$ be

- dense
- independent: for distinct $(x_1, x_2), (y_1, y_2) \in D$, $x_i \neq y_i$

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Then $(D; <_1, <_2) \cong \Pi$.

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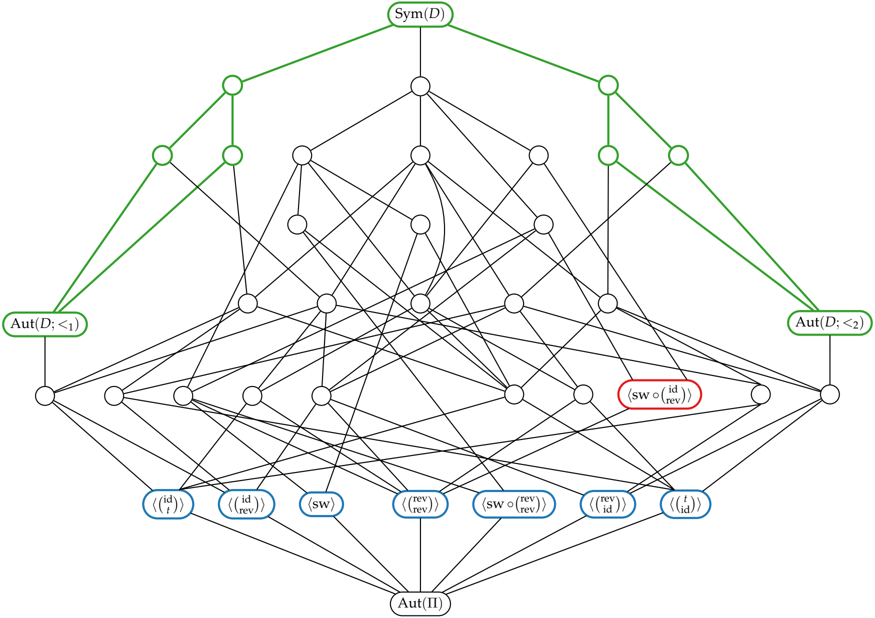
- $\binom{\mathrm{rev}}{\mathrm{id}}$
- $\binom{t}{\mathrm{id}}$



Sym(D)
Aut(D; <1)
Aut(D; <2)
⟨sw ∘ (id, rev)⟩
⟨(id, t)⟩
⟨(id, rev)⟩
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Aut(Π)

Asymmetry in the roles of $\binom{\mathrm{id}}{\mathrm{rev}}$ and $\binom{\mathrm{id}}{t}$

While $\leftrightarrow$ and $\circlearrowleft$ appear to play symmetric roles as generators of closed supergroups of $\mathrm{Aut}(\mathbb{Q};<)$, the corresponding permutations $\binom{\mathrm{id}}{\mathrm{rev}}$ and $\binom{\mathrm{id}}{t}$ of D do not.

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$$\langle \tbinom{\mathrm{rev}}{\mathrm{rev}} \rangle = \langle \tbinom{\mathrm{id}}{\mathrm{rev}} \circ \tbinom{\mathrm{rev}}{\mathrm{id}} \rangle \subsetneq \langle \tbinom{\mathrm{id}}{\mathrm{rev}}, \tbinom{\mathrm{rev}}{\mathrm{id}} \rangle$$

$$\langle \tbinom{\mathrm{id}}{t} \circ \tbinom{t}{\mathrm{id}} \rangle = \langle \tbinom{\mathrm{id}}{t}, \tbinom{t}{\mathrm{id}} \rangle$$

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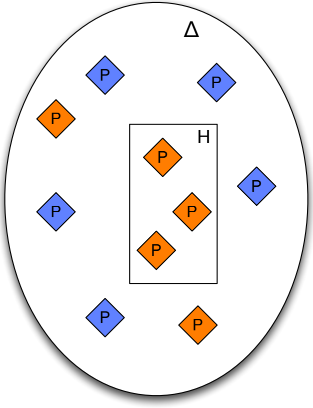

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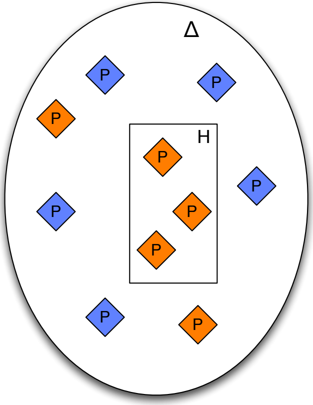

Theorem (Bodirsky, Pinsker, Tsankov, 2011)

If Δ is

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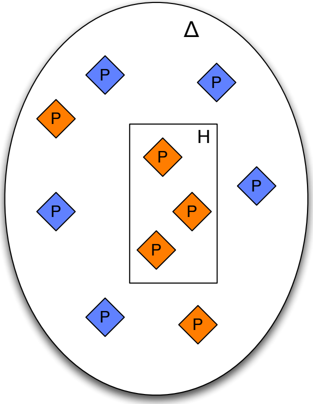

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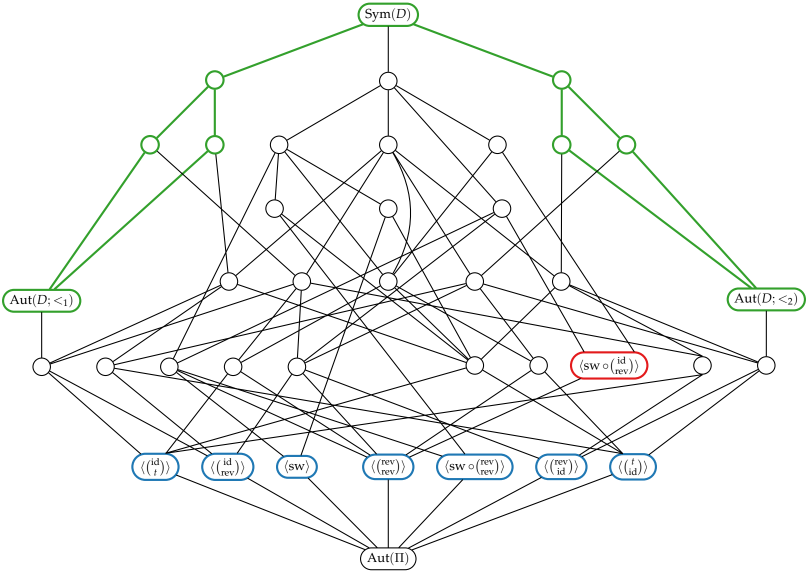
We obtain our classification by studying the behavior of such canonical functions.

Thank you!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