TBA

Michael Pinsker (Paris 7)

joint work with Manuel Bodirsky (LIX Palaiseau)

Dagstuhl 2012

Topological Birkhoff & Applications

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Outline

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- Generalization of Birkhoff's HSP$^{\text{fin}}$ theorem from finite to certain infinite algebras

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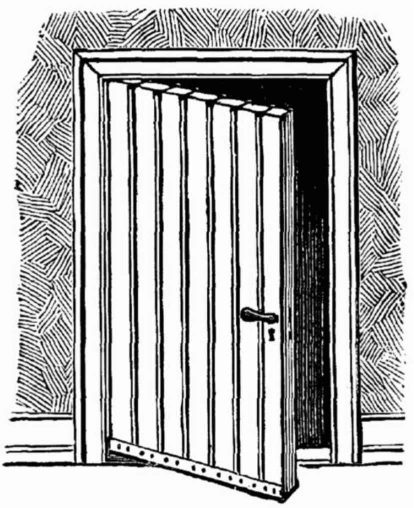
- Corollary in the purely model theoretic language: Primitive positive interpretations
- Applications to CSPs with infinite templates

Implication chain: $\downarrow$ (TBA)

Motivation chain: $\uparrow$ (ATB)



Part I: Simple cloning

CSPs

Let Γ be a relational structure with finite language τ.

CSP(Γ)

INPUT: A finite set of variables and τ-constraints on these variables.
QUESTION: Does there exists a satisfying assignment of values in Γ?

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Γ can be infinite!

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Finite simple cloning

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Let Γ, Δ be finite relational structures on the same domain. TFAE:

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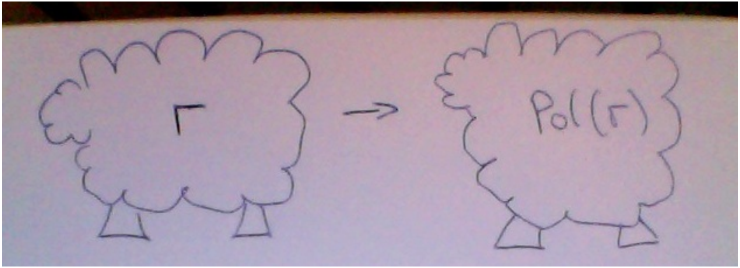

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CSP: essentially finitely many choices for n variables!

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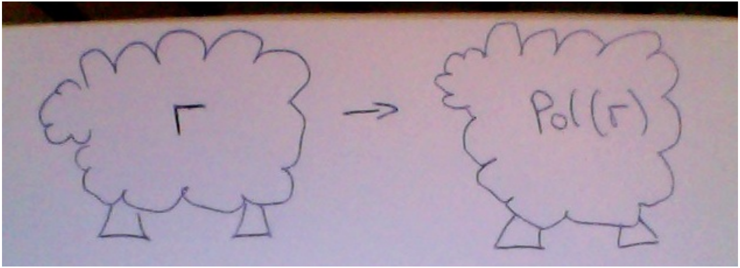

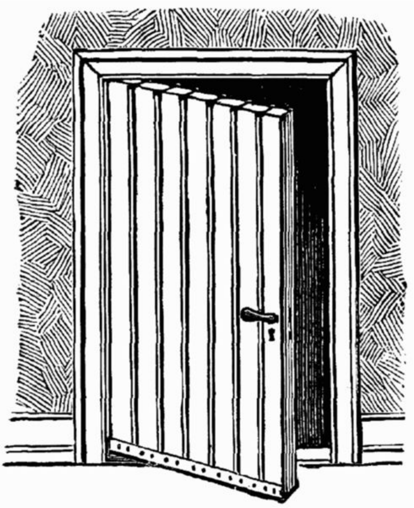

Part II: Double cloning

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Example: $(\mathbb{Q}; +, \cdot)$ has a pp-interpretation in $(\mathbb{Z}; +, \cdot)$.

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Theorem

Let Γ, Δ *be finite. TFAE:*

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- *there exists* $\mathfrak{B} \in \mathrm{HSP}^{\mathrm{fin}}(\mathrm{Pol}(\Gamma))$ *whose functions are elements of* $\mathrm{Pol}(\Delta)$.

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Theorem (Birkhoff)

Let $\mathfrak{A}$, $\mathfrak{B}$ be finite τ-algebras. TFAE:

- $\mathfrak{B} \in \mathsf{HSP}^{\mathsf{fin}}(\mathfrak{A})$.
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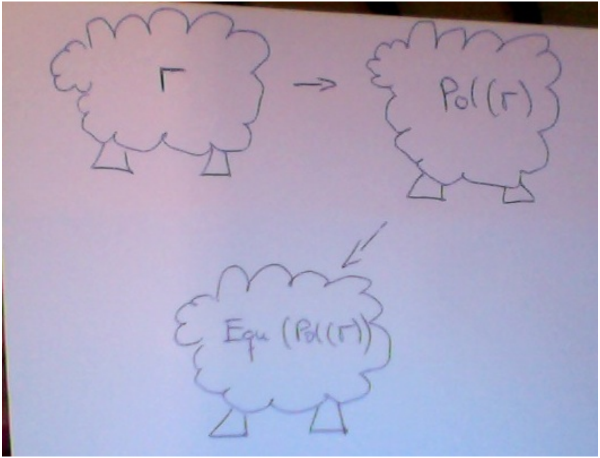

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Conjecture (Bulatov+Jeavons+Krokhin; Feder+Vardi)

For finite idempotent cores Γ this is the unique reason for NP-hardness.

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Let Γ *be* ω*-categorical, and* Δ *be arbitrary. TFAE:*

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Thm. A relational structure Γ is ω-categorical iff $\mathrm{Pol}(\Gamma)$ is oligomorphic.

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- $\mathfrak{B} \in \mathrm{HSP}^{\mathrm{fin}}(\mathfrak{A})$.
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Let Γ, Δ *be* ω*-categorical or finite relational structures. TFAE:*

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Let Γ, Δ *be* ω*-categorical or finite relational structures. TFAE:*

- Δ *has a pp-interpretation in* Γ*;*
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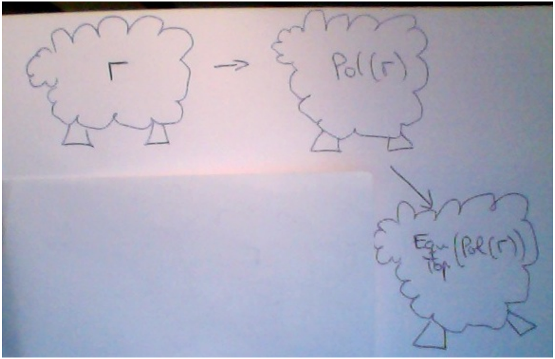

Cool hardness proofs

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Corollary

Let Γ be ω-categorical. TFAE:

- positive 1-in-3-SAT has a pp-interpretation in Γ;
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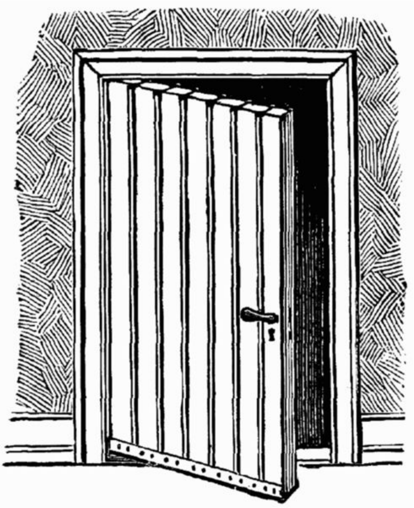
Straightforward: $\xi : \text{Pol}(\Gamma) \to \mathbf{1}$ is continuous homomorphism.



Part III: Infinite triple cloning

Automatic continuity

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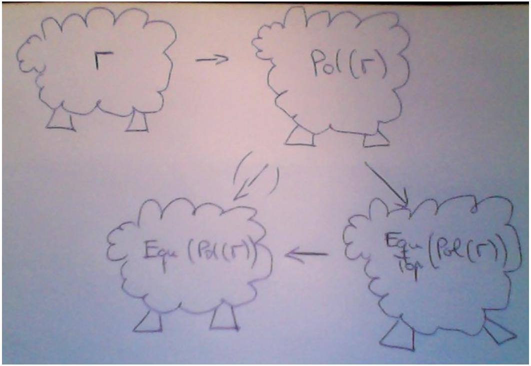

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- $(\mathbb{N}; =)$ (Dixon+Neumann+Thomas'86)
- $(\mathbb{Q}; <)$ (Truss'89)
- the random graph (Hodges+Hodkinson+Lascar+Shelah'93)

Reference

Topological Birkhoff

Manuel Bodirsky and Michael Pinsker

Transactions of the AMS / arXiv.

Thank you!