Reducts of Homogeneous Structures I: The Ramsey Property

Michael Pinsker

Université Denis Diderot - Paris 7 (60%)
Technische Universität Wien (30%)
Hebrew University of Jerusalem (10%)

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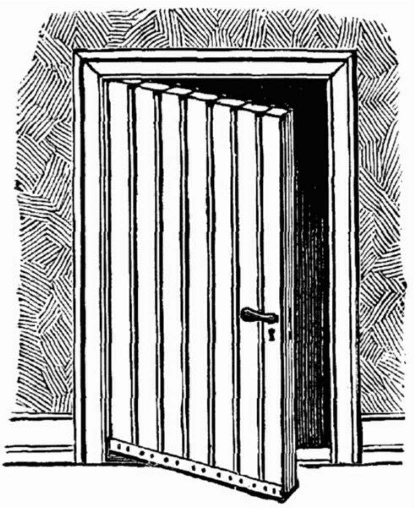
Outline



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Problem

Classify the reducts of Δ.

We call Δ the *base structure*.

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We factor this quasiorder
by the equivalence relation of fo-interdefinability,
and obtain a complete lattice.

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Observe:

Primitive positive (pp) interdefinability is finer than
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existential (ex) interdefinability is finer than
first order (fo) interdefinability.

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In fact:

The lattice corresponding to fo-definability is a factor of
the lattice corresponding to ex-definability is a factor of
the lattice corresponding to ep-definability is a factor of
the lattice corresponding to pp-definability.

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STOP!

In practice helps also for fo.

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For our method, we will need even “more” than homogeneity in a finite language:

The Ramsey property

Example: The dense linear order

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Denote by $(\mathbb{Q};<)$ be the order of the rationals, and set

$$\begin{aligned}
\text{betw}(x,y,z) &:= \{(x,y,z) \in \mathbb{Q}^3 : x < y < z \text{ or } z < y < x\} \\
\text{cycl}(x,y,z) &:= \{(x,y,z) \in \mathbb{Q}^3 : x < y < z \text{ or } z < x < y \\
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5. Γ is first-order interdefinable with $(\mathbb{Q}; =)$.

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Let $G = (V; E)$ be the random graph, and set for all $k \geq 2$

$$R^{(k)} := \{(x_1, \ldots, x_k) \subseteq V^k : x_i \text{ distinct, number of edges odd}\}.$$

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5. Γ is first-order interdefinable with $(V; =)$.

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Theorem (Junker, Ziegler '08)

$(\mathbb{Q}; <, 0)$ has 116 reducts up to fo-interdefinability.

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Depressing fact (Horváth, Pongrácz, P. '11)

The random graph with a constant has too many reducts up to fo-interdefinability.

Thomas' conjecture

Conjecture (Thomas '91)

Let Δ be homogeneous in a finite language.

Then Δ has finitely many reducts up to fo-interdefinability.

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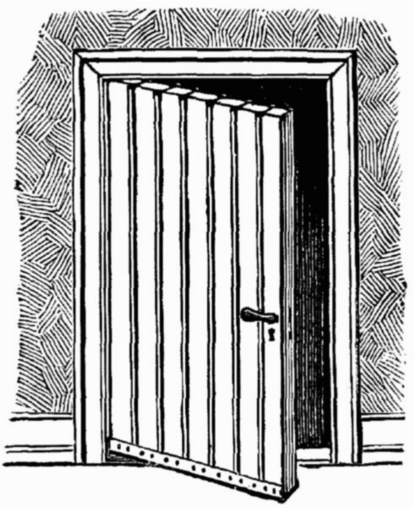

Functions on homogeneous structures

Permutation groups

Permutation groups

Theorem (Ryll-Nardzewski)

Let Δ be ω-categorical.

The mapping

$$\Gamma \mapsto \mathrm{Aut}(\Gamma)$$

is a one-to-one correspondence between

the *first-order closed* reducts of Δ and

the *closed permutation groups* containing $\mathrm{Aut}(\Delta)$.

first order closed = contains all fo-definable relations

Monoids

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Theorem (follows from the Homomorphism preservation thm)

Let Δ be ω-categorical.

The mapping

$$\Gamma \mapsto \text{End}(\Gamma)$$

is a one-to-one correspondence between
the *existential positive closed* reducts of Δ and
the *closed transformation monoids* containing $\text{Aut}(\Delta)$.

A monoid of functions from Δ to Δ is *closed* iff
it is closed in the Baire space Δ^Δ.

Clones

Clones

Theorem (Bodirsky, Nešetřil '03)

Let Δ be ω-categorical. Then

$$\Gamma \mapsto \text{Pol}(\Gamma)$$

is a one-to-one correspondence between
the *primitive positive closed* reducts of Δ and
the *closed clones* containing $\text{Aut}(\Delta)$.

A clone is a set of finitary operations on Δ which

- contains all projections $\pi_i^n(x_1, \ldots, x_n) = x_i$, and
- is closed under composition.

$\text{Pol}(\Gamma)$ is the clone of all homomorphisms from finite powers of Γ to Γ.

A clone C is closed if for each $n \geq 1$, the set of n-ary operations in C is a closed subset of the Baire space Δ^{Δ^n}.

For ω-categorical Δ:

Reducts up to **fo-interdefinability** $\leftrightarrow$
closed **permutation groups** $\supseteq \mathrm{Aut}(\Delta)$;

Reducts up to **ep-interdefinability** $\leftrightarrow$
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Reducts up to **pp-interdefinability** $\leftrightarrow$
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Let $\bar{G}$ be the graph that arises by switching edges and non-edges.

Let $- : V \to V$ be an isomorphism between G and $\bar{G}$.

For $c \in V$, let G_c be the graph that arises by switching all edges and non-edges containing c.

Let $\mathrm{sw}_c : V \to V$ be an isomorphism between G and G_c.

Theorem (Thomas '91)

The closed groups containing $\mathrm{Aut}(G)$ are the following:

1. $\mathrm{Aut}(G)$
2. $\langle \{-\} \cup \mathrm{Aut}(G) \rangle$
3. $\langle \{\mathrm{sw}_c\} \cup \mathrm{Aut}(G) \rangle$
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5. The full symmetric group S_V.

How to classify all reducts up to . . .-interdefinability?

Climb up the lattice!

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Conclusion: Every finite graph has a copy in G on which f is canonical.

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Problem: Identity.

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By topological closure, f generates a function which:

- behaves like f on $\{c, d\}$, and
- is canonical as a function from $(V; E, c, d)$ to $(V; E)$.

The minimal monoids on the random graph

Theorem (Thomas '96)

Let $f : G \to G$ a function
which does not locally look like an automorphism.

(that is, it violates at least one edge or a non-edge.)

Then f generates one of the following:

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We thus know the *minimal closed monoids* containing $\text{Aut}(G)$.

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Theorem (Bodirsky, P. ’10)

Let f be a finitary operation on G
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Then f generates one of the following:

- One of the five minimal unary functions of the previous theorem;
- One of 9 canonical binary injections.

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More involved argument: Extend G by a random dense linear order.

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Definition

A class $\mathcal{C}$ of τ-structures is called a *Ramsey class* iff
for all $H, P \in \mathcal{C}$ there exists S in $\mathcal{C}$ such that $S \to (H)^P$.

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Let Δ now be an arbitrary structure.

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for all tuples $(x_1, \ldots, x_n), (y_1, \ldots, y_n)$ of the same type
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Why don't you just do it?

Adding constants to Ramsey classes

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If Γ is Ramsey, is $(\Gamma, c_1, \ldots, c_n)$ still Ramsey?

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An ordered homogeneous structure Δ is Ramsey iff
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Corollary

If Γ is ordered, homogeneous, and Ramsey, then so is $(\Gamma, c_1, \ldots, c_n)$.

Minimal monoids above Ramsey structures

Thus: If Γ is ordered Ramsey, $f : \Gamma \to \Gamma$, and $c_1, \ldots, c_n \in \Gamma$, then f generates a function which

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- Every closed supermonoid of End(Γ) contains a minimal closed supermonoid of End(Γ).

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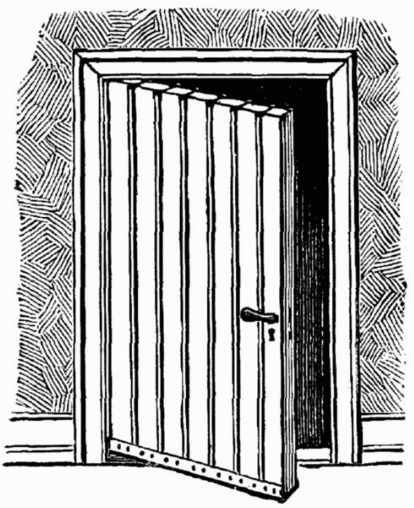

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Theorem (Bodirsky, P., Tsankov '10)

Let Δ be

- ordered
- homogeneous
- Ramsey
- with finite language
- *finitely bounded*.

Then the following problem is decidable:

INPUT: Two finite language reducts Γ, Γ' of Δ.
QUESTION: Are Γ, Γ' pp (ep-) interdefinable?

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- Is fo-interdefinability decidable?

Reference

Reducts of Ramsey structures

by Manuel Bodirsky and Michael Pinsker