Clones from ideals (Part I)

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Outline

The clone lattice

$X \ldots$ base set.

$\mathcal{O}^{(n)} = X^{X^n} = \{f : X^n \to X\} \ldots$ n-ary functions on X.

$\mathcal{O} = \bigcup_{n \geq 1} \mathcal{O}^{(n)} \ldots$ finitary operations on X.

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$Cl(X) = \{\mathcal{C} \subseteq \mathcal{O} : \mathcal{C} \text{ clone}\} \ldots$ lattice of clones (with inclusion).

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Fact

$\mathcal{C}_I$ clone for all ideals I.

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Corollary

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Remark

Support of I: $\text{supp}(I) := \bigcup\{A \subseteq X : A \in I\}$.
If we know the position of $\mathcal{C}_I$ in $\text{Cl}(\text{supp}(I))$, then we know its position in $\text{Cl}(X)$.
We assume: $\text{supp}(I) = X$ and I has at least one but not all infinite sets.

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I countably generated $\rightarrow \mathcal{C}_I$ maximal.

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Definition

$I^d := \{A \subseteq X : \forall B \in I\,(A \cap B \text{ is finite})\}$.
$\hat{I} := (I^d)^d$.
Alternatively: $\hat{I} := \{A \subseteq X : \forall B \subseteq A \exists C \subseteq B\,(C \in I)\}$.
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Theorem

$I \subseteq \hat{I}$.
$\mathcal{C}_I \subseteq \mathcal{C}_{\hat{I}}$.
$I = J \leftrightarrow \mathcal{C}_I = \mathcal{C}_J$.
$\mathcal{C}_I \subseteq \mathcal{C}_J \rightarrow I \subseteq J \subseteq \hat{I}$. The implication cannot be reversed.

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Assume $X \notin \hat{I}$.
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Theorem (Answer to the problem of Czédli and Heindorf)

- Every ideal clone can be extended to a maximal ideal clone.
- Every maximal clone extending an ideal clone is an ideal clone.

$2^{2^{\aleph_0}}$ maximal clones on ω without the Axiom of Choice

Let $X = 2^{<\omega}$ ... set of finite $0-1$-sequences.

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The ideals are distinct, i.e. if $A_1 \neq A_2$, then $(X_{A_1})^d \neq (X_{A_2})^d$:

For $f \in A_1 \setminus A_2$ we have $B_f \in (X_{A_2})^d \setminus (X_{A_1})^d$.

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Therefore the corresponding clones are distinct and maximal.