The reducts of $(\mathbb{N}, =)$ up to primitive positive interdefinability

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Outline

Reducts of relational structures

$\Gamma = (X, \mathcal{R}) \ldots$ relational structure.

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Examples

- P. J. Cameron: There are 5 reducts of $(\mathbb{Q}, <)$ up to f.o.-interdefinability.
- M. Junker and M. Ziegler: There are 116 reducts of $(\mathbb{Q}, <, a)$ up to f.o.-interdefinability.

First order definability and permutations

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Let $\mathcal{G}$ be a permutation group.
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Let Γ be ω-categorical.
Then $\mathrm{Inv}\,\mathrm{Aut}(\Gamma) = \mathit{fo}(\Gamma)$.

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$\mathcal{O} := \bigcup_{n \geq 1} \mathcal{O}^{(n)} \ldots$ finitary operations on X.

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Let $f \in \mathcal{O}^{(n)}$ and $R \subseteq X^m$.
f preserves R $\leftrightarrow f(r_1, \ldots, r_n) \in R$ for all $r_1, \ldots, r_n \in R$.

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$\mathrm{Pol}(\Gamma) := \{f \in \mathcal{O} : f$ preserves all relations of $\Gamma\}$.
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Observation

Let Γ be ω-categorical. Then $\mathrm{Inv}\,\mathrm{Pol}(\Gamma) = pp(\Gamma)$.

pp-definability and local clones

Definition

A set $\mathcal{C} \subseteq \mathcal{O}$ is a *clone* $\leftrightarrow$

- $\mathcal{C}$ is closed under composition, i.e. $f(g_1, \ldots, g_n) \in \mathcal{C}$ for all $f, g_1, \ldots, g_n \in \mathcal{C}$, and
- $\mathcal{C}$ contains the projections, i.e. for all $1 \leq k \leq n$ the operation $\pi_k^n(x_1, \ldots, x_n) = x_k$.

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The local clones are exactly the Inv Pol-closed subsets of $\mathcal{O}$.

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Conclusion

Inv (or Pol) is an antiisomorphism between the lattice of local clones above S_ω and the reducts of $(\mathbb{N}, =)$!

The local clones above S_ω

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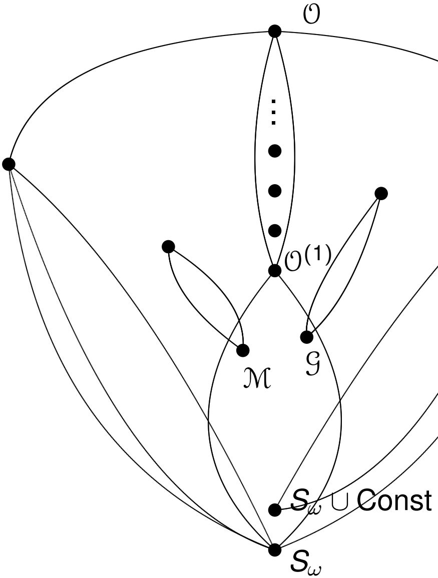

Connection to CSP and outlook

Constraint Satisfaction Problem

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Future work

Determine (up to pp interdefinability) the reducts of other ω-categorical structures.
Example: Random graph.