THE MINIMAL CLONES ABOVE THE PERMUTATIONS

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ABSTRACT. We determine the atoms of the interval of the clone lattice consisting of those clones which contain all permutations, on an infinite base set. This is equivalent to the description of the atoms of the lattice of transformation monoids above the permutations.

1. The problem and the result

Let X be an infinite set of cardinality $\kappa = \aleph_\alpha$, let $\mathscr{O}$ be the set of all finitary operations on X, and for all natural numbers $n \geq 1$ let $\mathscr{O}^{(n)}$ be the set of n-ary operations on X. A set of operations $\mathscr{C} \subseteq \mathscr{O}$ is called a *clone* if and only if it is closed under composition of functions and contains all projections, i.e. the functions $\pi^n_k \in \mathscr{O}^{(n)}$ satisfying $\pi^n_k(x_1, \ldots, x_n) = x_k$, for all $n \geq 1$ and all $1 \leq k \leq n$. Ordering the set of all clones on X by set-theoretical inclusion, one obtains a complete algebraic lattice $\mathrm{Cl}(X)$. The cardinality of $\mathrm{Cl}(X)$ is easily seen to equal 2^{2^κ}, and the lattice seems to be too complicated to ever be fully described. In fact, it has been shown recently that $\mathrm{Cl}(X)$ contains all algebraic lattices which do not have more compact elements than it as complete sublattices [Pin]. A recent survey of clones on infinite sets is [GP].

Because of the complexity of $\mathrm{Cl}(X)$, it has been tried to investigate interesting parts of the lattice, such as the atoms, referred to as *minimal* clones, or the dual atoms, called *maximal* or *precomplete* clones. However, at least on infinite X, even describing the minimal clones or the maximal clones seems unrealistic, since despite considerable efforts the minimal clones are not even known in the much smaller clone lattice over a finite base set, and since the number of maximal clones on an infinite base set has been proven to equal 2^{2^κ} ([Ros76], see also [GS02]). Successful research has been done on intervals of the clone lattice, for example on $[\langle \mathscr{O}^{(1)} \rangle, \mathscr{O}]$ in [Gav65], [GS02], [GSb], and [Pin04], where $\langle \mathscr{O}^{(1)} \rangle$ denotes the clone generated by $\mathscr{O}^{(1)}$, and on the interval above the clone of idempotent functions in [GSa]. Several results could also be obtained on the intervals $[\langle \mathscr{S} \rangle, \mathscr{O}]$ and $[\langle \mathscr{S} \rangle, \langle \mathscr{O}^{(1)} \rangle]$,

2000 *Mathematics Subject Classification.* Primary 08A40; secondary 08A05.

Key words and phrases. clone lattice, permutation group, monoid lattice, minimal clone, atom, interval.

The second author is grateful for support through the Postdoctoral Fellowship of the Japan Society for the Promotion of Science (JSPS) and later through grant P17812 of the Austrian Science Foundation (FWF).

where $\mathscr{S}$ is the monoid of all permutations on X: In [Hei02] a complete list of the maximal clones of $[\langle\mathscr{S}\rangle, \mathscr{O}]$, and in [Gav65] one of the clones maximal in $[\langle\mathscr{S}\rangle, \langle\mathscr{O}^{(1)}\rangle]$ were provided on a countably infinite base set X (the latter being a list of monoids, since clones below $\langle\mathscr{O}^{(1)}\rangle$ consist of functions which depend on at most one variable and therefore correspond to monoids in an obvious way; we shall for this reason drop the brackets and talk about the interval $[\mathscr{S}, \mathscr{O}^{(1)}]$ of the monoid lattice). In [Pin05a], the author extended the first result to all sets of infinite regular cardinality, and the second result to all infinite sets. It turned out that there exist $\max\{|\alpha|, \aleph_0\}$ maximal clones in $[\langle\mathscr{S}\rangle, \mathscr{O}]$, and $2\cdot|\alpha| + 5$ maximal monoids in $[\mathscr{S}, \mathscr{O}^{(1)}]$. Those numbers are relatively small considering the size of the clone lattice (or the monoid lattice, which is as large as the clone lattice), but the author proved in [Pin05b] that the cardinality of $[\mathscr{S}, \mathscr{O}^{(1)}]$ is $2^{2^{\max\{|\alpha|,\aleph_0\}}}$, so rather large.

In this article, we determine all clones minimal in $[\langle\mathscr{S}\rangle, \mathscr{O}]$. It turns out quickly that all such clones are in fact monoids, that is, they only contain functions depending on at most one variable. Therefore, the problem reduces to finding the minimal monoids of $[\mathscr{S}, \mathscr{O}^{(1)}]$, which is interesting in itself. We will see that there exist $\max\{|\alpha|, \aleph_0\}$ such monoids. Surprisingly, this implies that if $|X| < \aleph_\omega$, in particular on countably infinite X, there exist only finitely many maximal but infinitely many minimal elements in $[\mathscr{S}, \mathscr{O}^{(1)}]$.

For a monoid $\mathscr{G} \subseteq \mathscr{O}^{(1)}$, define $\mathrm{Pol}(\mathscr{G})$ to consist of all $f \in \mathscr{O}$ for which $f(g_1, \ldots, g_n) \in \mathscr{G}$ whenever $g_1, \ldots, g_n \in \mathscr{G}$. Call a clone $\mathscr{C}$ *collapsing* iff it is uniquely determined by its unary part $\mathscr{C}^{(1)} = \mathscr{C} \cap \mathscr{O}^{(1)}$, that is, there exist no other clones with the same unary part. Equivalently, $\mathscr{C}$ is collapsing iff all functions in $\mathrm{Pol}(\mathscr{C}^{(1)})$ are *essentially unary*, that is, they depend on at most one variable.

Lemma 1. *$\langle\mathscr{S}\rangle$ is collapsing.*

Proof. Let $f \in \mathrm{Pol}(\mathscr{S}) \cap \mathscr{O}^{(2)}$. Then $\gamma(x) = f(x,x)$ is a permutation. Now let $a, b \in X$ be distinct. There exists $c \in X$ with $\gamma(c) = f(a,b)$. If $c \notin \{a,b\}$, then we can find $\alpha, \beta \in \mathscr{S}$ with $\alpha(a) = a$, $\alpha(b) = c$, $\beta(a) = b$, and $\beta(b) = c$. But then $f(\alpha,\beta)(a) = f(a,b) = f(c,c) = f(\alpha,\beta)(b)$, so $f(\alpha,\beta)(x)$ is not a permutation. Thus, $c \in \{a,b\}$, so we have shown that $f(x,y) \in \{f(x,x), f(y,y)\}$ for all $x, y \in X$.
Next we claim that for all $a, b \in X$, if $f(a,b) = f(a,a)$, then $f(b,a) = f(b,b)$. Indeed, consider any permutation α which has a cycle (ab). Then $f(a,\alpha(a)) = f(a,b) = f(a,a)$, so $f(b,\alpha(b)) = f(b,a)$ has to be different from $f(a,a)$, because otherwise the function $f(x,\alpha(x))$ is not injective. Therefore, $f(b,a) = f(b,b)$.
Assume without loss that $f(a,b) = f(a,a)$, for some distinct $a, b \in X$. We first claim that $f(a,c) = f(a,a)$ for all $c \in X$. For assume not; then $f(a,c) = f(c,c)$, and therefore $f(c,a) = f(a,a)$. Let $\beta \in \mathscr{S}$ map a to b and c to a. Then $f(a,\beta(a)) = f(a,b) = f(a,a)$, but also $f(c,\beta(c)) = f(c,a) = f(a,a)$, a contradiction since $f \in \mathrm{Pol}(\mathscr{S})$. Hence, $f(a,c) = f(a,a)$ for all $c \in X$.

Now if $f(\tilde{a},\tilde{b}) \neq f(\tilde{a},\tilde{a})$ for some $\tilde{a},\tilde{b} \in X$, then $\tilde{a} \neq a$ by the observation we just made, and $f(\tilde{a},\tilde{b}) = f(\tilde{b},\tilde{b})$ and so $f(\tilde{b},\tilde{a}) = f(\tilde{a},\tilde{a})$; thus, $\tilde{b} \neq a$. Therefore, the conditions $f(\tilde{a},\tilde{b}) = f(\tilde{b},\tilde{b})$ but $f(a,\tilde{b}) = f(a,a) \neq f(\tilde{b},\tilde{b})$ lead to a similar contradiction as before. Hence, $f(x,y) = f(x,x)$ for all $x,y \in X$, and we have shown that f depends on at most one variable. Since $f \in \mathrm{Pol}(\mathscr{S}) \cap \mathscr{O}^{(2)}$ was arbitrary, all binary functions of $\mathrm{Pol}(\mathscr{S})$ are essentially unary. By a result of Grabowski [Gra97], this implies that $\langle \mathscr{S} \rangle$ is collapsing. (The mentioned result was proved for finite base sets of at least three elements but the same proof works on infinite sets.) □

Lemma 2. *If $\mathscr{C}$ is a clone that is minimal in $[\langle \mathscr{S} \rangle, \mathscr{O}]$, then it contains only essentially unary functions.*

Proof. Since $\mathscr{C} \supsetneq \langle \mathscr{S} \rangle$ we have $\mathscr{C}^{(1)} \supseteq \mathscr{S}$. If $\mathscr{C}^{(1)} = \mathscr{S}$, then $\mathscr{C} = \langle \mathscr{S} \rangle$ since $\langle \mathscr{S} \rangle$ is collapsing, contradicting the assumption that $\langle \mathscr{S} \rangle$ be a proper subset of $\mathscr{C}$. Therefore, $\mathscr{C}^{(1)}$ properly contains $\mathscr{S}$, so $\mathscr{C} = \langle \mathscr{C}^{(1)} \rangle$ since $\mathscr{C}$ is minimal above $\mathscr{S}$. Hence $\mathscr{C}$ contains only essentially unary functions. □

By the preceding lemma, when looking for the minimal clones above $\langle \mathscr{S} \rangle$, it suffices to determine the minimal monoids above $\mathscr{S}$. Clearly, such monoids are generated by a single non-permutation together with $\mathscr{S}$. We call functions which generate a minimal monoid above the permutations *$\mathscr{S}$-minimal*.

Definition 3. Set $\mathscr{K} = \{1 \leq \xi \leq \kappa : \xi \text{ a cardinal}\}$; then $|\mathscr{K}| = \max\{|\alpha|, \aleph_0\}$. Define for every $f \in \mathscr{O}^{(1)}$ a function

$$s_f : \begin{array}{ccc} \mathscr{K} & \to & \mathscr{K} \cup \{0\} \\ \xi & \mapsto & |\{y \in X : |f^{-1}[y]| = \xi\}| \end{array}$$

In words, the function assigns to every $1 \leq \xi \leq \kappa$ the number of equivalence classes in the kernel of f which have cardinality ξ. We call s_f the *kernel sequence* of f. The *support* $\mathrm{supp}(s_f)$ of s_f is the set of all $\xi \leq \kappa$ for which $s_f(\xi) \neq 0$. The *strong support* $\mathrm{supp}'(s_f)$ of s_f is the set of those cardinals $\xi \leq \kappa$ for which $s_f(\xi) \cdot \xi > |X \setminus f[X]|$. The *weak support* of s_f is defined to equal $\mathrm{supp}(s_f) \setminus \mathrm{supp}'(s_f)$. The restriction of s_f to its strong support is denoted by s'_f. We write $s'_f = s'_g$ iff $\mathrm{supp}'(s_f) = \mathrm{supp}'(s_g)$ and s_f and s_g agree on $\mathrm{supp}'(s_f)$. For $\psi_1, \psi_2 \leq \kappa$ we set $s_f(> \psi_1) = \sum_{\psi_1 < \zeta \leq \kappa} s_f(\zeta)$, and $s_f(> \psi_1, < \psi_2) = \sum_{\psi_1 < \zeta < \psi_2} s_f(\zeta)$, and similarly with $\leq$ and $\geq$.

Definition 4. For $f \in \mathscr{O}^{(1)}$ we define the following cardinals:

(1) $\mu_f = \min \mathrm{supp}(s_f)$.
(2) $\sigma_f = s_f(\mu_f)$.
(3) $\rho_f = s_f(> \mu_f)$.
(4) $\nu_f = |X \setminus f[X]|$.
(5) $\varepsilon_f = \sup \mathrm{supp}(s_f)$.
(6) $\varepsilon'_f = \sup \mathrm{supp}'(s_f)$, if $\mathrm{supp}'(s_f) \neq \emptyset$, and $\varepsilon'_f = \mu_f$ otherwise.

(7) $\lambda'_f = \sup\{\xi \in \operatorname{supp}'(s_f) : \xi \leq \nu_f\}$, if that set is non-void, and $\lambda'_f = \mu_f$ otherwise.
(8) $\chi_f = \min\{1 \leq \xi \leq \kappa : \exists \zeta \in \operatorname{supp}(s_f) : s_f(\geq \xi) \leq \zeta\}$.

So $1 \leq \mu_f \leq \lambda'_f \leq \varepsilon'_f \leq \varepsilon_f \leq \kappa$, and $\operatorname{supp}(s_f)$ is a subset of the interval $[\mu_f, \varepsilon_f]$ and $\operatorname{supp}'(s_f)$ one of the interval $[\mu_f, \varepsilon'_f]$. The size of the complement ν_f is independent of the other cardinals, and it will be important in our proof whether or not $\varepsilon_f > \nu_f$, that is, whether or not there exists a kernel class larger than the complement of the range of f. If $\varepsilon_f > \nu_f$, then $\varepsilon'_f = \varepsilon_f$ so we can forget about ε'_f and have either $\mu_f \leq \lambda'_f \leq \nu_f < \varepsilon_f$ or $\nu_f < \mu_f \leq \varepsilon_f$; in the latter case we left away λ'_f as it equals μ_f. If $\varepsilon_f \leq \nu_f$, then we have $\mu_f \leq \varepsilon'_f \leq \varepsilon_f \leq \nu_f$; λ'_f is irrelevant as it equals ε'_f. In that case, χ_f will play a role and in the relevant situations (e.g. if f is $\mathscr{S}$-minimal, or if it satisfies conditions (σ) and (χ) of Theorem 5) we have $\varepsilon'_f < \chi_f \leq \varepsilon_f^+$, where ε_f^+ is the successor cardinal of ε_f.

Theorem 5. *The constant functions are $\mathscr{S}$-minimal. If $f \in \mathscr{O}^{(1)} \backslash \mathscr{S}$ is nonconstant, then f is $\mathscr{S}$-minimal if and only if all of the following hold:*

(μ) *$\mu_f = 1$ or μ_f is infinite.*
(ν) *If μ_f is finite, then ν_f is infinite or zero.*
(σ) *$\sigma_f = \kappa$.*
(ρ) *$\rho_f < \kappa$.*
(s'dec) *s'_f is strictly decreasing.*
(n) *$n \notin \operatorname{supp}'(s_f)$ for all $1 < n < \aleph_0$.*
(ε) *$\varepsilon_f = 1$ or ε_f is infinite.*
(scont) *$s_f(\geq \xi) = \min\{s_f(\geq \zeta) : \zeta < \xi\}$ for all singular $\xi \leq \chi_f$ and $s_f(\geq n) = s_f(\geq 2)$ for all finite $2 \leq n \leq \chi_f$.*
(χ) *If $\varepsilon_f \leq \nu_f$, then $s_f(\geq \chi_f)$ is finite.*
($\#\varepsilon$) *If $\varepsilon_f > \nu_f$, then $s_f(\varepsilon_f)$ is infinite.*
(λ') *If $\varepsilon_f > \nu_f$, then $s_f(\xi) = 0$ for all $\lambda'_f < \xi \leq \nu_f$.*

The following theorem describes the clones generated by $\mathscr{S}$-minimal functions. It says that the clone an $\mathscr{S}$-minimal function f generates contains those non-permutations g which satisfy the conditions of Theorem 5, have the same characteristic values as f as defined in Definition 4, agree with f on the strong support, have the same inversely-accumulated kernel sequence $s_g(\geq \xi)$ below χ_g as f, and for which ε_g is obtained as a maximum of the support of s_g iff it is a maximum of the support of s_f.

Theorem 6. *Let f, g be $\mathscr{S}$-minimal and non-constant. Then $\langle \{f\} \cup \mathscr{S} \rangle = \langle \{g\} \cup \mathscr{S} \rangle$ if and only if all of the following hold:*

(1) $\mu_g = \mu_f$
(2) $\nu_g = \nu_f$
(3) $s'_g = s'_f$
(4) $\chi_g = \chi_f$
(5) $s_g(\geq \xi) = s_f(\geq \xi)$ *for all $\xi < \chi_f$*

(6) $\varepsilon_g = \varepsilon_f$
(7) $s_g(\varepsilon_g) = 0$ *iff* $s_f(\varepsilon_f) = 0$.

Corollary 7. *The number of clones (monoids) minimal in* $[\langle \mathscr{S} \rangle, \mathscr{O}]$ *(in* $[\mathscr{S}, \mathscr{O}^{(1)}]$*) on an infinite set of cardinality* $\aleph_\alpha$ *is* $\max\{|\alpha|, \aleph_0\}$.

Let X be countably infinite. For all $\nu < \aleph_0$, define a monoid $\mathscr{I}_\nu$ to consist of $\mathscr{S}$ plus all functions $f \in \mathscr{O}^{(1)}$ with $\mu_f = \aleph_0$ and $\nu_f = \nu$. Denote by $\mathscr{H}$ the monoid containing $\mathscr{S}$ and all functions with $\varepsilon_f = 1$ and $\nu_f = \aleph_0$, and by Const the monoid of all constant operations plus the permutations.

Corollary 8. *On countably infinite* X*, the minimal monoids above* $\mathscr{S}$ *are exactly the monoids* $\mathscr{I}_\nu$ *(*$\nu < \aleph_0$*),* Const*, and* $\mathscr{H}$*.*

1.1. **Notation and notions.** The smallest clone containing a set of functions $\mathscr{F} \subseteq \mathscr{O}$ is denoted by $\langle \mathscr{F} \rangle$. For $\xi \leq \kappa$ a cardinal and $f \in \mathscr{O}^{(1)}$ we set $Y^f_\xi = \{y \in X : |f^{-1}[y]| = \xi\}$ and $Y^f_{>\xi} = \bigcup_{\zeta > \xi} Y^f_\zeta$. Similarly we use notations like $Y^f_{\geq \xi}$ and so on; a less self-explanatory one is $Y^f_{>\xi, <\zeta}$ which we define to be $Y^f_{>\xi} \cap Y^f_{<\zeta}$. We say that a set $Y \subseteq X$ is *large* iff $|Y| = \kappa$, and that it is *small* otherwise. Y is *co-large* (*co-small*) iff its complement in X is large (small). For a function $f \in \mathscr{O}^{(1)}$, we denote the image of $Y \subseteq X$ under f by $f[Y]$ and the preimage by $f^{-1}[Y]$; if $Y = \{y\}$ is a singleton, then we cut short and write $f^{-1}[y]$ rather than $f^{-1}[\{y\}]$. Since we are interested in cardinals as arguments and values of kernel sequences, a statement like "for all $\psi_1 < \xi < \psi_2$" or "for all ξ in the interval (ψ_1, ψ_2)" will usually refer to all cardinals between ψ_1 and ψ_2, not all ordinals; occasionally, however, we will enumerate a set Z of cardinality ξ by something like $Z = \{z_\zeta : \zeta < \xi\}$, in which case ζ refers to all ordinals below ξ. We shall mention explicitly whenever this is the case.

1.2. **A gimmick for the quest.**

Lemma 9. *If* $f, g \in \mathscr{O}^{(1)}$ *are unary functions satisfying* $s_f = s_g$ *and* $\nu_f = \nu_g$*, then there exist* $\beta, \gamma \in \mathscr{S}$ *such that* $f = \beta \circ g \circ \gamma$.

Proof. The assumption $s_f = s_g$ implies that there is $\gamma \in \mathscr{S}$ such that $f(x) = f(y)$ iff $g \circ \gamma(x) = g \circ \gamma(y)$, for all $x, y \in X$. Obviously, $|f[X]| = |g[X]| = |g \circ \gamma[X]|$ as $s_f = s_g$. Together with the fact that $|X \setminus f[X]| = |X \setminus g[X]|$ this implies that we can find $\beta \in \mathscr{S}$ such that $f[X] = \beta \circ g \circ \gamma[X]$, and even so that $f = \beta \circ g \circ \gamma$. ☐

2. Sufficiencies for $\mathscr{S}$-minimality

We prove that the conditions of Theorem 5 are sufficient for a function to be $\mathscr{S}$-minimal.

2.1. **Descendants.** In this section we derive properties of functions generated by operations which satisfy all or some conditions of Theorem 5.

2.1.1. *The man who wasn't there.*

Lemma 10. *Let $f \in \mathscr{O}^{(1)}$ and $g \in \langle \{f\} \cup \mathscr{S} \rangle \setminus \mathscr{S}$. If ν_f is infinite or zero, then $\nu_g = \nu_f$. If ν_f is finite, then $\nu_g \geq \nu_f$ is finite as well.*

Proof. It is enough to show that if $h \in \mathscr{O}^{(1)}$, then $\nu_f \leq \nu_{f \circ h} \leq \nu_f + \nu_h$: The assertion then follows by induction over complexity of terms. Since $f[X] \supseteq f \circ h[X]$ we have that $|X \setminus f[X]| \leq |X \setminus f \circ h[X]|$, so $\nu_f \leq \nu_{f \circ h}$. Also, $\nu_{f \circ h} = |X \setminus f \circ h[X]| \leq |(X \setminus f[X]) \cup f[X \setminus h[X]]| \leq |X \setminus f[X]| + |X \setminus h[X]| = \nu_f + \nu_h$ and we are done. □

Lemma 11. *Let $f \in \mathscr{O}^{(1)}$ satisfy (ν), and let $g \in \langle \{f\} \cup \mathscr{S} \rangle \setminus \mathscr{S}$. Then $\nu_g = \nu_f$.*

Proof. If ν_f is infinite or zero, then we can refer to Lemma 10, so assume that ν_f is finite and nonzero. Then μ_f is infinite by (ν), in particular $\mu_f > \nu_f$. Using induction over terms, it is enough to show that if $h \in \mathscr{O}^{(1)}$ satisfies $\nu_h = \nu_f$, then $\nu_{f \circ h} = \nu_f$. Indeed, since $\mu_f > |X \setminus f[X]| = |X \setminus h[X]|$, h hits every class of the kernel of f, so that $f \circ h[X] = f[X]$ and we are done. □

2.1.2. *The dwarf-box.*

Lemma 12. *Let $f, g \in \mathscr{O}^{(1)}$, where g satisfies (ν), and set $h = f \circ g$. If $1 \leq n < \aleph_0$, then $s_h(n) \leq s_f(n) + s_g(>1, \leq n) + \min(\nu_g, s_f(>n, \leq \nu_g))$.*

Proof. If μ_g is infinite, then $s_h(n) = 0$ for all $1 \leq n < \aleph_0$ so there is nothing to show. So assume that μ_g is finite; then ν_g is zero or infinite, by (ν). If $|h^{-1}[y]| = n$ for some $y \in X$, then $|f^{-1}[y]| \geq n$ or there exists $z \in f^{-1}[y]$ with $1 < |g^{-1}[z]| \leq n$. The latter case occurs at most $s_g(>1, \leq n)$ times. $|f^{-1}[y]| = n$ occurs exactly $s_f(n)$ times. If $|f^{-1}[y]| > n$ then there exists $z \in f^{-1}[y]$ not in the range of g, which happens at most ν_g times. Also, in that case we cannot have $|f^{-1}[y]| > \nu_g$, for otherwise $|h^{-1}[y]| \geq |f^{-1}[y] \cap g[X]| = |f^{-1}[y]|$, the latter equality holding as ν_g is zero or infinite. Hence, $s_h(n) \leq s_f(n) + s_g(>1, \leq n) + \min(\nu_g, s_f(>n, \leq \nu_g))$. □

Lemma 13. *Let $f \in \mathscr{O}^{(1)}$ satisfy (ν). Then we have for all $g \in \langle \{f\} \cup \mathscr{S} \rangle$: For all $1 < n < \aleph_0$, if $s_f(>1, \leq n) + \min(\nu_f, s_f(>n, \leq \nu_f))$ is zero or infinite, then $s_g(n) \leq s_f(>1, \leq n) + \min(\nu_f, s_f(>n, \leq \nu_f))$; otherwise $s_g(n)$ is finite.*

Proof. We prove this by induction over terms. We can obviously assume that μ_f is finite, hence ν_f is zero or infinite by (ν). The statement is clear if $g \in \{f\} \cup \mathscr{S}$, so assume $g = f \circ h$, with $h \in \langle \{f\} \cup \mathscr{S} \rangle$ satisfying the induction hypothesis; by Lemma 10, $\nu_h = \nu_f$ and in particular h satisfies (ν). Therefore, $s_g(n) \leq s_f(n) + s_h(>1, \leq n) + \min(\nu_f, s_f(>n, \leq \nu_f))$ by Lemma 12. Now observe that if $1 < k \leq n$, then $s_f(>1, \leq k) + \min(\nu_f, s_f(>k, \leq \nu_f)) \leq s_f(>1, \leq n) + \min(\nu_f, s_f(>n, \leq \nu_f))$. Therefore, if $s_f(>1, \leq n) + \min(\nu_f, s_f(>n, \leq \nu_f))$ is finite, then so is $s_f(>1, \leq k) + \min(\nu_f, s_f(>k, \leq \nu_f))$, and thus $s_h(k)$ is finite by induction hypothesis, for all $1 < k \leq n$.

Hence, $s_g(n)$ is finite. If on the other hand $s_f(>1,\leq n)+\min(\nu_f, s_f(>n,\leq \nu_f))$ is infinite or zero, then we have $s_h(k) \leq s_f(>1,\leq n)+\min(\nu_f, s_f(>n,\leq \nu_f))$ for all $1<k\leq n$ by induction hypothesis, finishing the proof. □

Lemma 14. *Let $f \in \mathscr{O}^{(1)}$ satisfy (n) and (ν), and let $g \in \langle\{f\}\cup\mathscr{S}\rangle\backslash\mathscr{S}$. Then g satisfies (n) as well.*

Proof. By (ν) and Lemma 11 we have that $\nu_g = \nu_f$. If $1 < \nu_f < \aleph_0$, then μ_f is infinite and so is μ_g, so there is nothing to show. If ν_f is zero or infinite, then since $s_f(n) \leq \nu_f$ for all $1<n<\aleph_0$ by (n), we have that the same holds for s_g by Lemma 13. Hence, $n \notin \mathrm{supp}'(s_g)$ for all $1<n<\aleph_0$. □

2.1.3. *Upper bounds.*

Lemma 15. *Let $f \in \mathscr{O}^{(1)}$, and let $\xi \leq \kappa$ be infinite. Then we have for all $g \in \langle\{f\}\cup\mathscr{S}\rangle$: If $s_f(>\xi)$ is infinite or zero, then $s_g(>\xi) \leq s_f(>\xi)$; if $s_f(>\xi)$ is finite, then $s_g(>\xi)$ is finite as well.*

Proof. Using induction over terms, it is sufficient to show that if $h \in \mathscr{O}^{(1)}$, then $s_{f\circ h}(>\xi) \leq s_f(>\xi)+s_h(>\xi)$. Indeed, let $y \in X$ with $|(f\circ h)^{-1}[y]| > \xi$. Then $|f^{-1}[y]| > \xi$ or there exists $z \in f^{-1}[y]$ with $|h^{-1}[z]| > \xi$. The first possibility occurs for $s_f(>\xi)$ elements $y \in X$, and the second one for $s_h(>\xi)$ elements $y \in X$ and we are done. □

Lemma 16. *Let $f, g \in \mathscr{O}^{(1)}$, and set $h = f \circ g$. Let $\xi \leq \kappa$ be infinite and regular. Then $s_h(\xi) \leq s_f(\xi)+s_g(\xi)+\min(\nu_g, s_f(>\xi,\leq \nu_g))$.*

Proof. If $|h^{-1}[y]| = \xi$, then either there exists $z \in f^{-1}[y]$ with $|g^{-1}[z]| = \xi$, or $|f^{-1}[y]| \geq \xi$, because $\xi = \sum_{z\in f^{-1}[y]} |g^{-1}[z]|$ is infinite and regular. The first case can occur at most $s_g(\xi)$ times. That $|f^{-1}[y]| = \xi$ occurs $s_f(\xi)$ times, so let us consider the last possibility, $|f^{-1}[y]| > \xi$. If $|f^{-1}[y]| > \nu_g$, then $|h^{-1}[y]| \geq |f^{-1}[y] \cap g[X]| = |f^{-1}[y]| > \xi$, so we must have $\xi < |f^{-1}[y]| \leq \nu_g$. Only $s_f(>\xi,\leq \nu_g)$ elements $y \in X$ have this property. Moreover, if $|f^{-1}[y]| > \xi$ but $|h^{-1}[y]| = \xi$, then there exists $z \in f^{-1}[y]\backslash g[X]$, which happens at most ν_g times. □

Lemma 17. *Let $f \in \mathscr{O}^{(1)}$, and let $\xi \leq \kappa$ be infinite and regular. Then for all $g \in \langle\{f\}\cup\mathscr{S}\rangle$ we have: If $s_f(\xi)+\min(\nu_f, s_f(>\xi,\leq \nu_f))$ is infinite or zero, then $s_g(\xi) \leq s_f(\xi)+\min(\nu_f, s_f(>\xi,\leq \nu_f))$. If $s_f(\xi)+\min(\nu_f, s_f(>\xi,\leq \nu_f))$ is finite, then $s_g(\xi)$ is finite as well.*

Proof. We use induction over terms. The lemma is clear if $g = f$, so say $g = t \circ q$, with $q \in \langle\{f\}\cup\mathscr{S}\rangle$ satisfying the induction hypothesis, and $t \in \{f\}\cup\mathscr{S}$. There is nothing to show if $t \in \mathscr{S}$ so say $t = f$. By Lemma 16 we have $s_g(\xi) \leq s_f(\xi)+s_q(\xi)+\min(\nu_q, s_f(>\xi,\leq \nu_q))$. We distinguish two cases:

Case 1. If ν_f is infinite, then $\nu_q = \nu_f$ by Lemma 10, and thus $s_g(\xi) \leq s_f(\xi)+s_q(\xi)+\min(\nu_f, s_f(>\xi,\leq \nu_f))$. Now if $s_f(\xi)+\min(\nu_f, s_f(>\xi,\leq \nu_f))$ is infinite or zero, then using the induction hypothesis for q we get

$s_g(\xi) \leq 2 \cdot (s_f(\xi) + \min(\nu_f, s_f(> \xi, \leq \nu_f)))$; the assertion clearly follows. If $s_f(\xi) + \min(\nu_f, s_f(> \xi, \leq \nu_f))$ is finite, then $s_q(\xi)$ is finite too and so is $s_g(\xi)$.

Case 2. If ν_f is finite, then so is ν_q by Lemma 10, so $s_g(\xi) \leq s_f(\xi) + s_q(\xi) + \min(\nu_q, s_f(> \xi, \leq \nu_q)) = s_f(\xi) + s_q(\xi)$ as $\xi > \nu_q$. Now if $s_f(\xi)$ is infinite, then $s_q(\xi) \leq s_f(\xi) + \min(\nu_f, s_f(> \xi, \leq \nu_f)) = s_f(\xi)$ by induction hypothesis, so $s_g(\xi) \leq s_f(\xi)$. If $s_f(\xi)$ is finite, then so is $s_q(\xi)$ by induction hypothesis; hence, $s_g(\xi) \leq s_f(\xi) + s_q(\xi)$ is finite. □

Lemma 18. *Let $f, g \in \mathscr{O}^{(1)}$, and set $h = f \circ g$. Let $\xi \leq \kappa$ be infinite, and let $\lambda < \xi$. Then $s_h(\xi) \leq s_f(\xi) + s_g(> \lambda, \leq \xi) + \min(\nu_g, s_f(> \xi, \leq \nu_g))$.*

Proof. If $|h^{-1}[y]| = \sum_{z \in f^{-1}[y]} |g^{-1}[z]| = \xi$, then either there exists $z \in f^{-1}[y]$ with $\lambda < |g^{-1}[z]| \leq \xi$, or $|f^{-1}[y]| \geq \xi$. The first case can occur at most $s_g(> \lambda, \leq \xi)$ times. In the second case, observe that if $|f^{-1}[y]| > \xi$, then there must exist $z \in f^{-1}[y]$ which is not in the range of g; this can happen at most ν_g times. Also, in that case we must have $|f^{-1}[y]| \leq \nu_g$, for otherwise $|h^{-1}[y]| \geq |f^{-1}[y] \cap g[X]| = |f^{-1}[y]| > \xi$; the two conditions for f are satisfied by $\min(\nu_g, s_f(> \xi, \leq \nu_g))$ elements $y \in X$. The last possibility is that $|f^{-1}[y]| = \xi$, which happens at most $s_f(\xi)$ times. □

Lemma 19. *Let $f \in \mathscr{O}^{(1)}$, and let $\xi \leq \kappa$ be infinite. Then we have for all $g \in \langle \{f\} \cup \mathscr{S} \rangle$: If $s_f(> \xi, \leq \nu_f)$ is infinite or zero, then $s_g(> \xi, \leq \nu_f) \leq s_f(> \xi, \leq \nu_f)$; if $s_f(> \xi, \leq \nu_f)$ is finite, then $s_g(> \xi, \leq \nu_f)$ is finite as well.*

Proof. Using induction over terms, it is sufficient to show that if $h \in \mathscr{O}^{(1)}$, then $s_{h \circ f}(> \xi, \leq \nu_f) \leq s_h(> \xi, \leq \nu_f) + s_f(> \xi, \leq \nu_f)$. Indeed, let $y \in X$ with $\xi < |(h \circ f)^{-1}[y]| \leq \nu_f$. Then $|h^{-1}[y]| > \xi$ or there exists $z \in h^{-1}[y]$ with $\xi < |f^{-1}[z]| \leq \nu_f$. The latter possibility occurs for at most $s_f(> \xi, \leq \nu_f)$ elements $y \in X$. If $|h^{-1}[y]| > \xi$ and $|h^{-1}[y]| > \nu_f$, then $|(h \circ f)^{-1}[y]| \geq |h^{-1}[y] \cap f[X]| = |h^{-1}[y]| > \nu_f$, so this is impossible and we have to have $|h^{-1}[y]| \leq \nu_f$. Therefore, the first case happens at most $s_h(> \xi, \leq \nu_f)$ times. □

Lemma 20. *Let $f \in \mathscr{O}^{(1)}$, and let $\xi \leq \kappa$ be infinite. Let moreover $\lambda < \xi$. Then for all $g \in \langle \{f\} \cup \mathscr{S} \rangle$ we have: If $s_f(> \lambda, \leq \xi) + \min(\nu_f, s_f(> \xi, \leq \nu_f))$ is infinite or zero, then $s_g(\xi) \leq s_f(> \lambda, \leq \xi) + \min(\nu_f, s_f(> \xi, \leq \nu_f))$. If $s_f(> \lambda, \leq \xi) + \min(\nu_f, s_f(> \xi, \leq \nu_f))$ is finite, then $s_g(\xi)$ is finite as well.*

Proof. We use induction over terms. The lemma is clear if $g = f$, so say $g = q \circ t$, with $q \in \langle \{f\} \cup \mathscr{S} \rangle$ satisfying the induction hypothesis, and $t \in \{f\} \cup \mathscr{S}$. There is nothing to show if $t \in \mathscr{S}$ so say $t = f$. By Lemma 18, we have $s_g(\xi) \leq s_q(\xi) + s_f(> \lambda, \leq \xi) + \min(\nu_f, s_q(> \xi, \leq \nu_f))$. We distinguish three cases:

Case 1. Assume first $s_f(> \lambda, \leq \xi) + \min(\nu_f, s_f(> \xi, \leq \nu_f)) = 0$; we have to show $s_g(\xi) = 0$. We have $s_q(\xi) = 0$ by induction hypothesis, so $s_g(\xi) \leq s_f(> \lambda, \leq \xi) + \min(\nu_f, s_q(> \xi, \leq \nu_f)) = \min(\nu_f, s_q(> \xi, \leq \nu_f))$. Now if $\nu_f = 0$, then we have $s_g(\xi) = 0$ and we are done. If $\nu_f > 0$, then $s_f(> \xi, \leq \nu_f) = 0$ since $\min(\nu_f, s_f(> \xi, \leq \nu_f)) = 0$, and Lemma 19 implies $s_q(> \xi, \leq \nu_f) = 0$, so $s_g(\xi) = 0$.

Case 2. Now assume that $s_f(> \lambda, \leq \xi) + \min(\nu_f, s_f(> \xi, \leq \nu_f))$ is infinite. Then $s_q(\xi) \leq s_f(> \lambda, \leq \xi) + \min(\nu_f, s_f(> \xi, \leq \nu_f))$ by induction hypothesis, so $s_g(\xi) \leq 2 \cdot s_f(> \lambda, \leq \xi) + \min(\nu_f, s_f(> \xi, \leq \nu_f)) + \min(\nu_f, s_q(> \xi, \leq \nu_f))$. If $s_f(> \xi, \leq \nu_f)$ is infinite, then by Lemma 19 we have $s_q(> \xi, \leq \nu_f) \leq s_f(> \xi, \leq \nu_f)$, so $s_g(\xi) \leq 2 \cdot (s_f(> \lambda, \leq \xi) + \min(\nu_f, s_f(> \xi, \leq \nu_f)))$ and we are done. If $s_f(> \xi, \leq \nu_f)$ is finite, then $s_q(> \xi, \leq \nu_f)$ is finite as well by Lemma 19. Also in that case, $s_f(> \lambda, \leq \xi)$ must be infinite, so $s_g(\xi) \leq 2 \cdot s_f(> \lambda, \leq \xi) + \min(\nu_f, s_f(> \xi, \leq \nu_f)) + \min(\nu_f, s_q(> \xi, \leq \nu_f)) = 2 \cdot s_f(> \lambda, \leq \xi) = s_f(> \lambda, \leq \xi)$.

Case 3. We consider the case where $s_f(> \lambda, \leq \xi) + \min(\nu_f, s_f(> \xi, \leq \nu_f))$ is finite. By induction hypothesis, $s_q(\xi)$ is finite. Now if ν_f is finite, then $s_g(\xi) \leq s_q(\xi) + s_f(> \lambda, \leq \xi) + \min(\nu_f, s_q(> \xi, \leq \nu_f))$ is finite, too. If ν_f is infinite, then $s_f(> \xi, \leq \nu_f)$ must be finite, and thus $s_q(> \xi, \leq \nu_f)$ is finite by Lemma 19; again, we have that $s_g(\xi) \leq s_q(\xi) + s_f(> \lambda, \leq \xi) + \min(\nu_f, s_q(> \xi, \leq \nu_f))$ is finite. □

2.1.4. *Lower bounds.*

Lemma 21. *Let $f, g \in \mathscr{O}^{(1)}$, and set $h = f \circ g$. Let $\xi \in (\nu_g, \kappa]$ be infinite, and assume that either $s_g(> \xi) = 0$ or $s_g(> \xi) < s_f(\xi)$ and $s_f(\xi)$ is infinite. Then $s_h(\xi) \geq s_f(\xi)$.*

Proof. Fix some $y \in Y^f_\xi$. Then $|h^{-1}[y]| \geq |f^{-1}[y] \cap g[X]|$. Since $\nu_g < \xi$ we have that $|f^{-1}[y] \cap g[X]| = |f^{-1}[y]| = \xi$. Now $|h^{-1}[y]| > \xi$ if and only if there exists $z \in f^{-1}[y]$ with $|g^{-1}[z]| > \xi$. This happens only for $s_g(> \xi)$ elements $y \in Y^f_\xi$, so it does not happen for $s_f(\xi)$ elements $y \in Y^f_\xi$, since either $s_g(> \xi) = 0$ or $s_f(\xi)$ is infinite and $s_g(> \xi) < s_f(\xi)$. Hence, $|h^{-1}[y]| = \xi$ for at least $s_f(\xi)$ elements $y \in Y^f_\xi$. □

Lemma 22. *Let $f \in \mathscr{O}^{(1)}$, let $\xi \in (\nu_f, \kappa]$ be infinite, and assume that either $s_f(> \xi) = 0$ or $s_f(> \xi) < s_f(\xi)$ and $s_f(\xi)$ is infinite. Then for all $g \in \langle \{f\} \cup \mathscr{S} \rangle \backslash \mathscr{S}$ we have $s_g(\xi) \geq s_f(\xi)$.*

Proof. We use induction over terms: Let $q \in \langle \{f\} \cup \mathscr{S} \rangle \backslash \mathscr{S}$ satisfy the induction hypothesis, and let $g = f \circ q$. Because $\xi > \nu_f$ is infinite, we have $\xi > \nu_q$ by Lemma 10. If $s_f(> \xi) = 0$, then $s_q(> \xi) = 0$ by Lemma 15. Also, by the same lemma, we have that if $s_f(\xi)$ is infinite, then $s_f(> \xi) < s_f(\xi)$ implies $s_q(> \xi) < s_f(\xi)$. Now $s_g(\xi) \geq s_f(\xi)$ by Lemma 21. □

Lemma 23. *Let $f, g \in \mathscr{O}^{(1)}$, and set $h = f \circ g$. Let $\xi \leq \nu_g$ be infinite, and assume $s_g(> \xi) + \nu_g < s_f(\xi)$ and that $s_f(\xi)$ is infinite. Then $s_h(\xi) \geq s_f(\xi)$.*

Proof. Fix some $y \in Y^f_\xi$. Then $|h^{-1}[y]| \geq |f^{-1}[y] \cap g[X]|$. We have that $|f^{-1}[y] \cap g[X]| \geq \xi$ for at least $s_f(\xi)$ elements $y \in Y^f_\xi$, since $|f^{-1}[y] \cap g[X]| < \xi$

implies that $f^{-1}[y]\backslash g[X]$ is non-empty, which happens at most $\nu_g < s_f(\xi)$ times. Now $|h^{-1}[y]| > \xi$ if and only if there exists $z \in f^{-1}[y]$ with $|g^{-1}[z]| > \xi$. By assumption $s_g(> \xi) < s_f(\xi)$ this happens for fewer than $s_f(\xi)$ elements $y \in Y^f_\xi$, so that $|h^{-1}[y]| = \xi$ for $s_f(\xi)$ elements $y \in Y^f_\xi$. □

Lemma 24. *Let $f \in \mathscr{O}^{(1)}$, let $\xi \leq \nu_f$ be infinite, and assume $s_f(> \xi) + \nu_f < s_f(\xi)$ and that $s_f(\xi)$ is infinite. Then for all $g \in \langle \{f\} \cup \mathscr{S} \rangle \backslash \mathscr{S}$ we have $s_g(\xi) \geq s_f(\xi)$.*

Proof. Using induction over terms, we assume $g = f \circ q$, with $q \in \langle \{f\} \cup \mathscr{S} \rangle \backslash \mathscr{S}$ satisfying the induction hypothesis. Because $s_f(\xi) > \nu_f$ is infinite, we have $s_f(\xi) > \nu_q$ by Lemma 10. By Lemma 15, since $s_f(> \xi) < s_f(\xi)$, and since $s_f(\xi)$ is infinite, we have $s_q(> \xi) < s_f(\xi)$. Thus, reference to Lemma 23 completes the proof. □

2.1.5. *The king.*

Lemma 25. *Let $f \in \mathscr{O}^{(1)}$ satisfy (ε). Then $\varepsilon_g = \varepsilon_f$ for all $g \in \langle \{f\} \cup \mathscr{S} \rangle \backslash \mathscr{S}$.*

Proof. If $\varepsilon_f = 1$, then f is injective and so is g, hence $\varepsilon_g = 1$. Otherwise ε_f is infinite. Fix $\xi \leq \kappa$; if $s_f(\geq \xi) > 0$, then clearly also $s_g(\geq \xi) > 0$ since kernel classes cannot become smaller, so $\varepsilon_g \geq \varepsilon_f$. On the other hand, $s_g(> \varepsilon_f) \leq s_f(> \varepsilon_f) = 0$ by Lemma 15, so $\varepsilon_g \leq \varepsilon_f$. □

Definition 26. We say that $f \in \mathscr{O}^{(1)}$ satisfies (εreg) iff $s_f(\varepsilon_f) > 0$ or ε_f is regular.

Lemma 27. *If $f \in \mathscr{O}^{(1)}$ satisfies (s'dec), (scont), and (χ), then it satisfies (εreg).*

Proof. If $\varepsilon_f > \nu_f$, then the support of s_f above ν_f is finite by (s'dec), so $s_f(\varepsilon_f) > 0$. If $\varepsilon_f \leq \nu_f$ and $\varepsilon_f > \chi_f$, then $0 < s_f(\geq \chi_f) < \aleph_0$ by (χ), so again $s_f(\varepsilon_f) > 0$. If $\varepsilon_f \leq \nu_f$ and $\varepsilon_f \leq \chi_f$, then we have that if ε_f is singular, then $s_f(\varepsilon_f) = s_f(\geq \varepsilon_f) = \min\{s_f(\geq \zeta) : \zeta < \varepsilon_f\} > 0$, by (scont). □

2.1.6. *Farmers.*

Definition 28. We say that $f \in \mathscr{O}^{(1)}$ satisfies (κ) iff $\nu_f = \kappa$ implies $s_f(\kappa) = 0$.

Lemma 29. *If $f \in \mathscr{O}^{(1)}$ satisfies (σ) and (χ), then it satisfies (κ).*

Proof. Observe that $\nu_f = \kappa$ implies $\varepsilon_f \leq \nu_f$. If $s_f(\kappa) > 0$, then $\chi_f = 1$. But then $s_f(\geq \chi_f) = s_f(\geq \mu_f) \geq \sigma_f = \kappa$ by (σ), contradicting (χ). □

Lemma 30. *Let $f \in \mathscr{O}^{(1)}$ satisfy (μ), (ν), (σ), (ρ), (ε), (εreg), and (κ). Then g satisfies $\mu_g = \mu_f$, $\nu_g = \nu_f$, $\varepsilon_g = \varepsilon_f$, and ($\mu$), ($\nu$), ($\sigma$), ($\rho$), ($\varepsilon$), ($\varepsilon$reg), and ($\kappa$), for all $g \in \langle \{f\} \cup \mathscr{S} \rangle \backslash \mathscr{S}$.*

Proof. Using induction over terms, we assume $g = f \circ h$, with $h \in \langle \{f\} \cup \mathscr{S} \rangle$ having all asserted properties. We are going to prove $\mu_g = \mu_f$, $\nu_g = \nu_f$, $\varepsilon_g = \varepsilon_f$, ($\sigma$), ($\rho$), ($\varepsilon$reg), and ($\kappa$); conditions ($\mu$), ($\nu$) and ($\varepsilon$) will follow automatically from $\mu_g = \mu_f$, $\nu_g = \nu_f$ and $\varepsilon_g = \varepsilon_f$. By Lemmas 11 and 25 and conditions (ν) and (ε) we have $\nu_g = \nu_f$ and $\varepsilon_g = \varepsilon_f$. We prove ($\varepsilon$reg). If $\varepsilon_g = \varepsilon_h$ is singular, then there exists $y \in Y^h_{\varepsilon_h}$, by ($\varepsilon$reg); but then $f(y) \in Y^g_{\varepsilon_g}$, and hence g satisfies (εreg). We show (κ). If $\nu_g = \nu_f = \kappa$, then $s_f(\kappa) = 0$ by (κ). Now if $\varepsilon_f < \kappa$, then $\varepsilon_g = \varepsilon_f < \kappa$, so $s_g(\kappa) = 0$. If $\varepsilon_f = \kappa$, then κ is regular by (εreg). But then $s_g(\kappa) \leq s_f(\kappa) = 0$, by Lemma 17. Hence, g satisfies (κ).
We now claim that $g[X]$ is large. Indeed, this is trivial if $\nu_g < \kappa$, so assume $\nu_g = \nu_f = \kappa$. Then by (κ) we have $s_g(\kappa) = s_f(\kappa) = 0$. Since $|\bigcup_{y \in g[X]} g^{-1}[y]| = \kappa$, $\varepsilon_g < \kappa$ immediately implies $|g[X]| = \kappa$, so consider the case $\varepsilon_g = \varepsilon_f = \kappa$. Because $s_f(\kappa) = 0$ and by (εreg) we have that κ is regular. Hence, $|\bigcup_{y \in g[X]} g^{-1}[y]| = \kappa$ again implies $|g[X]| = \kappa$.

Now if $g^{-1}[y] > \mu_f$, then by (μ) we have that $y \in Y^f_{>\mu_f}$ or there exists $z \in f^{-1}[y] \cap Y^h_{>\mu_f}$, which happens for at most $\rho_f + \rho_h < \kappa$ times. Therefore $Y^g_{>\mu_f}$ is small, and hence $Y^g_{\mu_f}$ must be large since $g[X] = Y^g_{\mu_f} \cup Y^g_{>\mu_f}$ is large. Whence, $\mu_g = \mu_f$ and g satisfies (σ) and (ρ). □

2.1.7. *The valley of giants.*

Definition 31. We say that $f \in \mathscr{O}^{(1)}$ satisfies (s'inf) iff $s'_f(\xi)$ is infinite for all $\xi \in \mathrm{supp}'(s_f)$.

Lemma 32. *If $f \in \mathscr{O}^{(1)}$ satisfies (ν), (s'dec), and ($\#\varepsilon$), then it satisfies (s'inf).*

Proof. Let $\xi \in \mathrm{supp}'(s_f)$. If $\xi > \nu_f$, then $s_f(\xi) \geq s_f(\varepsilon_f) \geq \aleph_0$, by (s'dec) and ($\#\varepsilon$). If $\xi \leq \nu_f$, then $s_f(\xi) > \nu_f$. If ν_f was finite, then we would have $1 \leq \xi \leq \nu_f < \aleph_0$, so in particular μ_f would be finite and $0 < \nu_f < \aleph_0$, contradicting (ν). Hence ν_f and thus also $s_f(\xi)$ are infinite. □

Lemma 33. *Let $f \in \mathscr{O}^{(1)}$ satisfy (μ), (ν), (σ), (ρ), (s'dec), (n), (ε), (s'inf), (εreg), and (κ). Then $s'_g = s'_f$ for all $g \in \langle \{f\} \cup \mathscr{S} \rangle \setminus \mathscr{S}$.*

Proof. By Lemma 30, we have that $\mu_g = \mu_f$, $\sigma_g = \sigma_f = \kappa$, $\varepsilon_g = \varepsilon_f$, and $\nu_g = \nu_f$. Let $\xi \in \mathrm{supp}'(s_f)$ so that $\xi > 1$; then ξ is infinite by (n). Choose $\lambda < \xi$ such that $\mathrm{supp}'(s_f) \cap (\lambda, \xi)$ is empty; this is possible since the strong support is finite by condition (s'dec). Also, if $\xi > \nu_f$, then we can choose $\lambda \geq \nu_f$. By Lemma 20 and since $s_f(\xi)$ is infinite by (s'inf) we have $s_g(\xi) \leq s_f(>\lambda, \leq \xi) + \min(\nu_f, s_f(>\xi, \leq \nu_f))$. But the latter expression equals $s_f(\xi)$, since if $\xi > \nu_f$, then $s_f(>\lambda, <\xi) = 0$ by the choice of λ and also $s_f(>\xi, \leq \nu_f) = 0$, and if $\xi \leq \nu_f$, then $s_f(>\lambda, <\xi) \leq \nu_f$ so that the equality follows from the fact that $s_f(\xi) > \nu_f$. Hence, $s_g(\xi) \leq s_f(\xi)$. By (s'dec), (s'inf) and Lemmas 24 and 22 we have $s_g(\xi) \geq s_f(\xi)$, so $s_g(\xi) = s_f(\xi)$. If $1 \in \mathrm{supp}'(s_f)$, then $\mu_f = 1$ and so $s_g(1) = s_f(1) = \kappa$, since $\mu_g = \mu_f = 1$ and

$\sigma_g = \sigma_f = \kappa$.
Now let $\xi \notin \operatorname{supp}'(s_f)$; then $\xi \leq \nu_f = \nu_g$. Consider first the case where ξ is infinite, and choose λ as before. If $s_f(>\lambda, \leq \xi) + \min(\nu_f, s_f(>\xi, \leq \nu_f))$ is infinite, then Lemma 20 implies $s_g(\xi) \leq s_f(>\lambda, \leq \xi) + \min(\nu_f, s_f(>\xi, \leq \nu_f)) \leq \nu_f$, and thus $\xi \notin \operatorname{supp}'(s_g)$. If on the other hand $s_f(>\lambda, \leq \xi) + \min(\nu_f, s_f(>\xi, \leq \nu_f))$ is finite, then the same holds for $s_g(\xi)$ and hence $s_g(\xi) \leq \nu_f$ as $\nu_f \geq \xi$ is infinite; again, $\xi \notin \operatorname{supp}'(s_g)$. Now consider the case where $\xi > 1$ is finite. Then conditions (n) and (ν) together with Lemma 14 guarantee that $\xi \notin \operatorname{supp}'(s_g)$. Finally, assume $\xi = 1$. If ξ is in the weak support of s_f, then $\mu_f = \xi = 1$ and we have $s_g(\xi) = s_f(\xi) = \kappa$. Because $1 \notin \operatorname{supp}'(s_f)$ we must have $\nu_f = \nu_g = \kappa$, so $1 \notin \operatorname{supp}'(s_g)$. If $\xi = 1 \notin \operatorname{supp}(s_f)$, then $1 \notin \operatorname{supp}(s_g)$, so in particular $1 \notin \operatorname{supp}'(s_g)$.
Therefore, we have shown that $\operatorname{supp}'(s_g) = \operatorname{supp}'(s_f)$, and that $s_g(\xi) = s_f(\xi)$ for all $\xi \in \operatorname{supp}'(s_f)$. □

2.2. **Gambling back the loss.** We investigate which functions are generated by operations satisfying (some of) the conditions of Theorem 5, the ultimate goal being to show that if $g \in \langle \{f\} \cup \mathscr{S} \rangle \setminus \mathscr{S}$, where $f \in \mathscr{O}^{(1)}$ satisfies the conditions of Theorem 5, then $f \in \langle \{g\} \cup \mathscr{S} \rangle$, proving $\mathscr{S}$-minimality.

2.2.1. *When the king is larger than the man who wasn't there.* We modify functions $f \in \mathscr{O}^{(1)}$ below λ'_f. This finishes the case $\varepsilon_f > \nu_f$.

Lemma 34. *Let $f \in \mathscr{O}^{(1)}$ satisfy (μ), (ν), (σ), (s'dec), and (n). Then there exists $g \in \langle \{f\} \cup \mathscr{S} \rangle$ such that $s_g(\xi) = 0$ for all $\xi < \lambda'_f$ with $\xi \notin \operatorname{supp}'(s_f)$, and $s_g(\xi) = s_f(\xi)$ for all other $\xi \leq \kappa$. In particular, there are no elements below λ'_f in the weak support of s_g.*

Proof. We may assume that $\lambda'_f > \mu_f$, for the lemma is trivial otherwise; thus, $\lambda'_f > 1$ and so λ'_f is infinite by condition (n). Also, from $\lambda'_f > \mu_f$ it follows that $\mu_f \leq \nu_f$, which together with (ν) implies that ν_f is infinite. Set $Y = \bigcup \{Y^f_\zeta : \zeta < \lambda'_f \wedge \zeta \notin \operatorname{supp}'(s_f)\}$; then $|Y| \leq \nu_f$. Observe that Y does not contain $Y^f_{\mu_f}$, as $\lambda'_f > \mu_f$ implies that the strong support of s_f is non-empty and thus $\nu_f < \kappa$, so $\mu_f \in \operatorname{supp}'(s_f)$ by (σ). Let α map all $y \in Y^f_{\lambda'_f}$ into $f^{-1}[y]$. Because λ'_f is infinite we have $|f^{-1}[Y^f_{\lambda'_f}] \setminus \alpha[Y^f_{\lambda'_f}]| = |f^{-1}[Y^f_{\lambda'_f}]| > \nu_f$, so we can extend α by mapping Y into $f^{-1}[Y^f_{\lambda'_f}]$ in such a way that α stays injective. Now extend α again, mapping y into $f^{-1}[y]$ for all $y \in Y^f_{>\mu_f}$ for which α has not yet been defined, and let α map a suitable part of $X \setminus f[X]$ bijectively onto $f^{-1}[Y]$. By (σ) we can choose $S \subseteq Y^f_{\mu_f}$ large such that $Y^f_{\mu_f} \setminus S$ is still large, and let α map S bijectively onto $f^{-1}[S]$. Extend α to a bijection; this is possible as the domain of α is disjoint from $Y^f_{\mu_f} \setminus S$ and its range is disjoint from $f^{-1}[Y^f_{\mu_f} \setminus S]$.
We calculate $|g^{-1}[y]|$ for all $y \in X$. If $y \in Y^f_{\mu_f}$, then $|(\alpha \circ f)^{-1}[z]| \in \{0, \mu_f\}$ for

all $z \in f^{-1}[y]$, so $|g^{-1}[y]| \in \{0, \mu_f\}$ by (μ); if $y \in S \subseteq Y^f_{\mu_f}$, then $|g^{-1}[y]| = \mu_f$ as $\alpha(y) \in f^{-1}[y]$. If $y \in Y$, then $g^{-1}[y]$ is empty as $f^{-1}[y] \subseteq \alpha[X \backslash f[X]]$. If $y \in Y^f_{>\mu_f} \backslash Y$, then $g^{-1}[y] \supseteq f^{-1}[y]$; also, $|(\alpha \circ f)^{-1}[z]| \leq |f^{-1}[y]|$ for all $z \in f^{-1}[y]$, so $|g^{-1}[y]| = |f^{-1}[y]|$ since $y \notin Y$ implies that $|f^{-1}[y]|$ is infinite. Therefore we have $s_g(\xi) = 0$ for all $\xi < \lambda'_f$ outside the strong support of s_f, and $s_g(\xi) = s_f(\xi)$ for all other $\xi \leq \kappa$. In particular, since $\nu_g = \nu_f$ by (ν) and Lemma 11, there are no elements below λ'_f in the weak support of s_g. □

Lemma 35. *Let $f \in \mathscr{O}^{(1)}$ satisfy (μ), (ν), (σ), (s'dec), (n), and (s'inf). Let $p \in \mathscr{O}^{(1)}$ be so that $s'_p = s'_f$, $\mu_p = \mu_f$ and $\nu_p = \nu_f$. Then there exists $g \in \langle \{f\} \cup \mathscr{S} \rangle$ such that $s_g \restriction_{[1,\lambda'_f)} = s_p \restriction_{[1,\lambda'_f)}$, and $s_g \restriction_{[\lambda'_f,\kappa]} = s_f \restriction_{[\lambda'_f,\kappa]}$.*

Proof. We assume that the strong support of s_f below ν_f is non-void, for otherwise $\lambda'_f = \mu'_f$ by definition and the lemma is trivial. For the same reason, we may assume that $\mu_f < \lambda'_f$; then λ'_f and hence also ν_f are infinite. By Lemma 34 there exists $h \in \langle \{f\} \cup \mathscr{S} \rangle$ with the property that $s_h(\xi) = 0$ for all $\xi < \lambda'_f$ which are not in the strong support of s_f and such that $s_h(\xi) = s_f(\xi)$ for all other $\xi \leq \kappa$; since f satisfies (ν), Lemma 11 implies that $\nu_h = \nu_f$. Clearly, $s'_h = s'_f$ and $\varepsilon_h = \varepsilon_f$. Now fix for every $\xi < \lambda'_f = \lambda'_h$ outside the strong support of s_p a set $Z_\xi \subseteq Y^h_{\lambda'_h}$ such that $|Z_\xi| = s_p(\xi)$ and such that $Z_{\xi_1} \cap Z_{\xi_2} = \emptyset$ whenever $\xi_1 \neq \xi_2$. This is possible since the sum over all $s_p(\xi)$, where $\xi < \lambda'_h$ is not an element of the strong support of s_p, is at most $\nu_h < s_h(\lambda'_h)$. Fix for every $y \in Z_\xi$ a set $B_y \subseteq h^{-1}[y]$ with $|B_y| = \xi$, and set $C_y = h^{-1}[y] \backslash B_y$. Set $Z = \bigcup\{Z_\xi : \xi \notin \mathrm{supp}'(s_h) \wedge \xi < \lambda'_h\}$, $B = \bigcup\{B_y : y \in Z\}$, and $C = \bigcup\{C_y : y \in Z\}$. Choose $S \subseteq Y^h_{\mu_h}$ large such that $Y^h_{\mu_h} \backslash S$ is still large. Now let α map S bijectively onto $f^{-1}[S] \cup B$, a suitable part of $X \backslash h[X]$ bijectively onto C, and $Y^h_{\lambda'_h}$ onto $h^{-1}[Y^h_{\lambda'_h}] \backslash (B \cup C)$; the latter can be done as $|h^{-1}[Y^h_{\lambda'_h}] \backslash (B \cup C)| = |h^{-1}[Y^h_{\lambda'_h} \backslash Z]| = |h^{-1}[Y^h_{\lambda'_h}]| = \lambda'_h \cdot |Y^h_{\lambda'_h}| = |Y^h_{\lambda'_h}|$. For all $y \in Y^h_{>\mu_h} \backslash Y^h_{\lambda'_h}$ we let $\alpha(y) \in f^{-1}[y]$. We extend α to a bijection and set $g = h \circ \alpha \circ h$.

Now if $y \in Y^h_{\mu_h}$, then $|g^{-1}[y]| \in \{0, \mu_h\}$, and if $y \in S$, then $|g^{-1}[y]| = \mu_h$. If $y \in Y^h_{>\mu_h} \backslash Z$, then $|g^{-1}[y]| = |f^{-1}[y]|$. Indeed, (n) and the fact that s_h vanishes outside its strong support below λ'_h imply that $|f^{-1}[y]|$ is infinite; the equation then follows since $\alpha(y) \in f^{-1}[y]$ and $|(\alpha \circ f)^{-1}[z]| \leq |f^{-1}[y]|$ for all $z \in f^{-1}[y]$ by construction of α. If $y \in Z_\xi$ for some $\mu_h < \xi < \lambda'_h$, then $|g^{-1}[y]| = |(\alpha \circ h)^{-1}[B_y \cup C_y]| = |(\alpha \circ h)^{-1}[B_y]| = |B_y| \cdot \mu_h = |B_y| = \xi$. Therefore, $s_g(\mu_f) = |S| = \kappa = s_p(\mu_f)$, $s_g(\xi) = |Z_\xi| = s_p(\xi)$ for all $\mu_f < \xi < \lambda'_f$ outside the strong support of s_f, and $s_g(\xi) = s_h(\xi) = s_f(\xi)$ for all $\xi \geq \lambda'_f$ and all $\xi \in \mathrm{supp}'(s_f) = \mathrm{supp}'(s_p)$. □

Proposition 36. *Let $f \in \mathscr{O}^{(1)}$ be so that $\nu_f < \varepsilon_f$. If f moreover satisfies (μ), (ν), (σ), (ρ), (s'dec), (n), (ε), (#ε), and (λ'), then it is $\mathscr{S}$-minimal.*

Remark 37. Under those conditions, f automatically satisfies (χ) and (scont): Condition (χ) is trivial as $\varepsilon_f \nleq \nu_f$. For (scont), observe that $s_f(\geq \xi) = s_f(\geq \psi)$, where $\psi = \min\{\zeta \in \mathrm{supp}'(s_f) : \zeta \geq \xi\}$, if the latter set is not empty, which is the case for all $\xi \leq \varepsilon_f$ as $\nu_f < \varepsilon_f$. Indeed, if ψ exists, then $s_f(\geq \xi) = s_f(\geq \psi) + s_f(\geq \xi, < \psi)$. Now if $\xi > \lambda'_f$, then $s_f(\geq \xi, < \psi) = 0$ by (λ') and the definition of the strong support, and we have $s_f(\geq \xi) = s_f(\geq \psi)$; if $\xi \leq \lambda'_f$, then $\psi \leq \lambda'_f$ and the equation follows from $s_f(\geq \psi) > \nu_f$ and $s_f(\geq \xi, < \psi) \leq \nu_f$ and the fact that $s_f(\geq \xi) \geq s_f(\psi)$ is infinite since f satisfies (s'inf) by Lemma 32. Therefore, the function $s_f(\geq \xi)$ drops only at successor cardinals of elements of the strong support, and (scont) follows from the fact that the strong support is finite by (s'dec) and does not contain any finite cardinals greater than 1 by (n).

Proof. Let f satisfy all the conditions, and let $g \in \langle\{f\} \cup \mathscr{S}\rangle \setminus \mathscr{S}$. By Lemmas 27, 29 and 32, f satisfies (εreg), (κ), and (s'inf). Therefore we have $\mu_g = \mu_f$, $\nu_g = \nu_f$ and $\varepsilon_g = \varepsilon_f$ by Lemma 30. Moreover by the same lemma, g satisfies (μ), (ν) and (σ). By Lemma 33, $s'_g = s'_f$, in particular g satisfies (s'dec), (s'inf), and $\lambda'_g = \lambda'_f$. From Lemma 14 and the fact that f satisfies (n) and (ν) we infer that (n) holds for g as well. Therefore by Lemma 35, there exists $h \in \langle\{g\} \cup \mathscr{S}\rangle$ such that $s_h \restriction_{[1,\lambda'_f)} = s_f \restriction_{[1,\lambda'_f)}$ and $s_h \restriction_{[\lambda'_f,\kappa]} = s_g \restriction_{[\lambda'_f,\kappa]}$; thus, $s_h \restriction_{(\nu_f,\kappa]} = s_f \restriction_{(\nu_f,\kappa]}$ as $s'_g = s'_f$. Now $\mathrm{supp}(s_h) \cap (\lambda'_f, \nu_f]$ is empty. Indeed, $s_f(\xi) = 0$ for all $\xi \in (\lambda'_f, \nu_f]$ by (λ'); therefore, $s_h(\xi) = 0$ for all infinite $\xi \in (\lambda'_f, \nu_f]$ by Lemma 20. If $\xi \in (\lambda'_f, \nu_f]$ is finite, then we must have $\lambda'_f = \mu_f = 1$, by (n) and (μ). Thus, s_f yields constantly zero on $(1, \nu_f]$, and so $s_h(\xi) = 0$ by Lemma 13. Hence, both s_h and s_f vanish on $(\lambda'_f, \nu_f]$. Therefore, $s_h = s_f$ so that since also $\nu_h = \nu_g = \nu_f$ by Lemma 11, we conclude $f \in \langle\{g\} \cup \mathscr{S}\rangle$. ☐

2.2.2. *Beyond the giants.* First we show that if $f \in \mathscr{O}^{(1)}$ satisfies some of the conditions of Theorem 5, then $\chi_g = \chi_f$ and $s_g(\geq \xi) = s_f(\geq \xi)$ for all $\xi < \chi_f$ and all $g \in \langle\{f\} \cup \mathscr{S}\rangle$. Assuming $\varepsilon_f \leq \nu_f$, we then modify functions $f \in \mathscr{O}^{(1)}$ above ε'_f and below χ_f.

Lemma 38. *Let $f \in \mathscr{O}^{(1)}$ and $\xi \leq \kappa$ be infinite and regular or $\xi \leq 2$, and let $g \in \langle\{f\} \cup \mathscr{S}\rangle$. If $s_f(\geq \xi)$ is infinite or zero, then $s_g(\geq \xi) \leq s_f(\geq \xi)$. If $s_f(\geq \xi)$ is finite, then $s_g(\geq \xi)$ is finite as well.*

Proof. It is enough to show that if $h \in \mathscr{O}^{(1)}$, then $s_{f \circ h}(\geq \xi) \leq s_f(\geq \xi) + s_h(\geq \xi)$; the lemma then clearly follows by induction over terms. Indeed, if $|(f \circ h)^{-1}[y]| \geq \xi$, then $|f^{-1}[y]| \geq \xi$ or there exists $z \in f^{-1}[y]$ such that $|h^{-1}[z]| \geq \xi$, since ξ is infinite and regular or $\xi \leq 2$. The first possibility happens $s_f(\geq \xi)$ and the second possibility $s_h(\geq \xi)$ times. ☐

Lemma 39. *Let $f \in \mathscr{O}^{(1)}$ satisfy (ε) and (εreg), and let $1 \leq \xi < \chi_f$. Then $s_g(\geq \xi) \geq s_f(\geq \xi)$ for all $g \in \langle\{f\} \cup \mathscr{S}\rangle \setminus \mathscr{S}$.*

Proof. We can assume $\varepsilon_f > 1$, so ε_f is infinite. Using induction over terms, it is enough to show that if $h \in \langle\{f\} \cup \mathscr{S}\rangle \backslash \mathscr{S}$ satisfies $s_h(\geq \xi) \geq s_f(\geq \xi)$, then also $s_{h \circ f}(\geq \xi) \geq s_f(\geq \xi)$. By Lemma 25, we have $\varepsilon_h = \varepsilon_f$. Consider $Y^f_{\geq\xi}$; we claim that $|h[Y^f_{\geq\xi}]| = |Y^f_{\geq\xi}| = s_f(\geq \xi)$. To see this, observe first that $Y^f_{\geq\xi} = \bigcup_{y \in h[Y^f_{\geq\xi}]}(h^{-1}[y] \cap Y^f_{\geq\xi})$. Now $|h^{-1}[y]| < s_f(\geq \xi)$ for all $y \in X$. Indeed, otherwise we would have $\varepsilon_f = \varepsilon_h \geq |h^{-1}[y]| \geq s_f(\geq \xi) \geq \varepsilon_f$, the last inequality holding since $\xi < \chi_f$; thus, $s_h(\varepsilon_f) > 0$. But then $s_f(\varepsilon_f) > 0$ by (εreg) and Lemma 17, so $\varepsilon_f \in \mathrm{supp}(s_f)$ and $\varepsilon_f = s_f(\geq \xi)$, contradicting $\xi < \chi_f$. Therefore, $|h^{-1}[y] \cap Y^f_{\geq\xi}| \leq |h^{-1}[y]| < s_f(\geq \xi)$ for all $y \in h[Y^f_{\geq\xi}]$. Thus if we had $|h[Y^f_{\geq\xi}]| < s_f(\geq \xi)$, then we could conclude that $s_f(\geq \xi)$ is singular and the supremum of a set of cardinals of kernel classes of h, the latter fact implying $s_f(\geq \xi) \leq \varepsilon_h = \varepsilon_f$. But since $\xi < \chi_f$ we would have $s_f(\geq \xi) \geq \varepsilon_f$ and hence $s_f(\geq \xi) = \varepsilon_f$, and therefore ε_f would be singular. Also, we would have $\zeta < s_f(\geq \xi) = \varepsilon_f$ for all ζ in the support of s_f, so $s_f(\varepsilon_f) = 0$, in contradiction with (εreg). So we must have $|h[Y^f_{\geq\xi}]| = s_f(\geq \xi)$, and since $|(h \circ f)^{-1}[y]| \geq \xi$ for all $y \in h[Y^f_{\geq\xi}]$ we are done. $\square$

Lemma 40. *Let $f \in \mathscr{O}^{(1)}$ satisfy (ε), (scont), and (εreg), and let $1 \leq \xi < \chi_f$. Then $s_g(\geq \xi) = s_f(\geq \xi)$ for all $g \in \langle\{f\} \cup \mathscr{S}\rangle \backslash \mathscr{S}$.*

Proof. If $\varepsilon_f = 1$ then there is nothing to show, so we may assume that ε_f is infinite, by (ε). Then $s_f(\geq \xi)$ is infinite for all $\xi < \chi_f$. Now if ξ is infinite and regular, or if $\xi \leq 2$, then the assertion is a direct consequence of Lemmas 38 and 39. If ξ is singular or finite and greater than 2, then there exists $\zeta < \xi$ infinite and regular or equal to 2 such that $s_f(\geq \zeta) = s_f(\geq \xi)$, by (scont). We have $s_g(\geq \xi) \leq s_g(\geq \zeta) = s_f(\geq \zeta) = s_f(\geq \xi)$, and $s_g(\geq \xi) \geq s_f(\geq \xi)$ by Lemma 39. $\square$

Lemma 41. *If $f \in \mathscr{O}^{(1)}$ satisfies (scont), then χ_f is infinite and regular, or $\chi_f \leq 2$.*

Proof. If χ_f was singular or finite and greater than two, then (scont) would imply that there exists $\zeta < \chi_f$ such that $s_f(\geq \zeta) = s_f(\geq \chi_f)$, contradicting that χ_f is the minimal $1 \leq \xi \leq \kappa$ such that $s_f(\geq \xi) \leq \lambda$ for some $\lambda \in \mathrm{supp}(s_f)$. $\square$

Lemma 42. *Let $f \in \mathscr{O}^{(1)}$ satisfy (ε), (scont), and (εreg). Then $\chi_g = \chi_f$ for all $g \in \langle\{f\} \cup \mathscr{S}\rangle \backslash \mathscr{S}$.*

Proof. Using (ε), we assume that ε_f is infinite. By (scont) and Lemma 41, $\chi_f \leq 2$ or χ_f is infinite and regular. Assume $\chi_g < \chi_f$. By Lemma 39, $s_g(\geq \chi_g) \geq s_f(\geq \chi_g)$. But $s_f(\geq \chi_g) > \lambda$ for all $\lambda \in \mathrm{supp}(s_f)$, and therefore also for all $\lambda \in \mathrm{supp}(s_g)$, since $\varepsilon_g = \varepsilon_f$ by (ε) and Lemma 25, and since (εreg) and Lemma 17 in addition imply that $s_g(\varepsilon_f) > 0$ only if $s_f(\varepsilon_f) > 0$. Thus, $s_g(\geq \chi_g) > \lambda$ for all $\lambda \in \mathrm{supp}(s_g)$, contradicting the definition of χ_g. Assume now that $\chi_g > \chi_f$. Then $s_g(\geq \chi_f) > \lambda$ for all $\lambda \in \mathrm{supp}(s_g)$, and hence also for all $\lambda \in \operatorname{supp}(s_f)$. In particular, $s_g(\geq \chi_f) \geq \varepsilon_f$ is infinite; thus by Lemma 38 we have that $s_f(\geq \chi_f)$ is infinite as well and $s_g(\geq \chi_f) \leq s_f(\geq \chi_f)$, so $s_f(\geq \chi_f) > \lambda$ for all $\lambda \in \operatorname{supp}(s_f)$, in contradiction with the definition of χ_f. □

Lemma 43. *Let $f \in \mathscr{O}^{(1)}$ satisfy (μ), (σ), (n), (ε), (εreg), and assume $\varepsilon_f \leq \nu_f$. There exists $g \in \langle \{f\} \cup \mathscr{S} \rangle$ such that $s_g(\xi) = s_g(\geq \xi) = s_f(\geq \xi)$ for all $\varepsilon'_f < \xi < \chi_f$ and such that $s_g(\xi) = s_f(\xi)$ for all $\xi \leq \varepsilon'_f$ and all $\xi \geq \chi_f$.*

Proof. We can assume that ε_f is infinite; for otherwise, $\operatorname{supp}(s_f) = \{1\}$ by condition (ε), and the lemma would be trivial. Also, we assume $\varepsilon'_f < \chi_f$, so in particular $\chi_f > \mu_f$. Define $\delta' \leq \kappa$ to be minimal with the property that $s_f(\zeta) < \varepsilon_f$ for all $\zeta \geq \delta'$, if such a cardinal exists, and to equal χ_f otherwise. Set $\delta = \min\{\delta', \chi_f\}$. Define $H_\xi = \{\xi \leq \zeta < \delta : s_f(\zeta) \geq \varepsilon_f\}$, for all $\xi \in (\varepsilon'_f, \delta)$. We claim that $\sum_{\zeta \in H_\xi} s_f(\zeta) = s_f(\geq \xi)$. Indeed, if $s_f(\geq \xi) = \varepsilon_f$, then there exists $\xi \leq \zeta < \delta$ with $s_f(\zeta) = \varepsilon_f$, since $\xi < \delta$, so our claim is true. If $s_f(\geq \xi) > \varepsilon_f$ and our claim did not hold, then we would have $\sum_{\zeta \in [\xi, \varepsilon_f] \setminus H_\xi} s_f(\zeta) = s_f(\geq \xi)$, since $s_f(\geq \xi) \geq \varepsilon_f$ is infinite. But then

$$s_f(\geq \xi) = \sum_{\zeta \in [\xi, \varepsilon_f] \setminus H_\xi} s_f(\zeta) \leq \sum_{\zeta \in [\xi, \varepsilon_f] \setminus H_\xi} \varepsilon_f \leq \varepsilon_f,$$

contradicting $s_f(\geq \xi) > \varepsilon_f$. For every $\varepsilon'_f < \xi < \delta$ with $s_f(\xi) \geq \varepsilon_f$, write Y^f_ξ as a disjoint union $\bigcup_{\zeta \leq \xi} Y_{\xi,\zeta}$ in such a way that $|Y_{\xi,\zeta}| = |Y^f_\xi| = s_f(\xi)$. This is possible as $s_f(\xi) \geq \varepsilon_f$ is infinite. Set $Y'_\zeta = \bigcup_{\xi \in H_\zeta} Y_{\xi,\zeta}$, for all $\zeta \in (\varepsilon'_f, \delta)$. Then $Y'_\zeta \subseteq Y^f_{\geq \zeta}$ and $|Y'_\zeta| = \sum_{\xi \in H_\zeta} |Y_{\xi,\zeta}| = \sum_{\xi \in H_\zeta} s_f(\xi) = s_f(\geq \zeta)$.
If $\delta < \chi_f$, then $s_f(\geq \delta) \geq \varepsilon_f$ by definition of χ_f, and $s_f(\zeta) < \varepsilon_f$ for all $\zeta \geq \delta$ implies $s_f(\geq \delta) \leq \varepsilon_f$, so $s_f(\geq \delta) = \varepsilon_f$; in fact, the same argument shows $s_f(\geq \xi) = \varepsilon_f$ for all $\delta \leq \xi < \chi_f$. Also, $\delta < \chi_f$ implies $s_f(\varepsilon_f) = 0$, so ε_f must be regular by (εreg). We write $Y^f_{\geq \delta}$ as a disjoint union $\bigcup_{\delta \leq \xi < \chi_f} Z_\xi$, where all Z_ξ are of cardinality ε_f, and set $Y'_\xi = Z_\xi \cap Y^f_{\geq \xi}$; then $|Y'_\xi| = \varepsilon_f = s_f(\geq \xi)$ since $|Z_\xi \setminus Y^f_{\geq \xi}| \leq Y^f_{\geq \delta, < \xi} < \varepsilon_f$ by the regularity of ε_f.
Write Y for the union over all Y'_ξ with $\varepsilon'_f < \xi < \chi_f$. Now fix for all $\varepsilon'_f < \xi < \chi_f$ and all $y \in Y'_\xi$ a set $B_y \subseteq f^{-1}[y]$ with $|B_y| = \xi$, and set $C_y = f^{-1}[y] \setminus B_y$. Write $B = \bigcup_{y \in Y} B_y$ and $C = \bigcup_{y \in Y} C_y$. Fix $S \subseteq Y^f_{\mu_f}$ large and such that $Y^f_{\mu_f} \setminus S$ is still large, and let α map S bijectively onto $B \cup f^{-1}[S]$. Let α map all $y \in Y^f_\xi$, where $\mu_f < \xi \leq \varepsilon'_f$ or $\xi \geq \chi_f$, into $f^{-1}[y]$. Write $F = Y^f_{>1, <\aleph_0} \cap (Y^f_{\leq \varepsilon'_f} \cup Y^f_{\geq \chi_f})$, and let F^* consist of those elements of $f^{-1}[F]$ which are not yet in the range of α. Set $D = Y^f_{> \varepsilon'_f, < \chi_f} \setminus Y$, and $D^* = f^{-1}[D]$. Let α map a suitable part of $X \setminus f[X]$ bijectively onto $C \cup D^* \cup F^*$. We can do that since $|C \cup D^* \cup F^*| \leq |f^{-1}[Y^f_{> \varepsilon'_f}] \cup f^{-1}[Y^f_{>1, <\aleph_0}]| \leq \nu_f + \nu_f = \nu_f$,

by the definition of ε'_f and by (n). Choose moreover $T \subseteq Y^f_{\mu_f} \backslash S$ with $|T| = |Y^f_{>\varepsilon'_f, <\chi_f}|$ and so that $Y^f_{\mu_f} \backslash (S \cup T)$ is still large, and let α map $Y^f_{>\varepsilon'_f, <\chi_f}$ into $f^{-1}[T]$ in such a way that every kernel class of $f^{-1}[T]$ is hit exactly once. Extend α to a bijection, and set $g = f \circ \alpha \circ f$; we can do that since α is not defined on $Y^f_{\mu_f} \backslash (S \cup T)$ and its range is disjoint from $f^{-1}[Y^f_{\mu_f} \backslash (S \cup T)]$. We calculate $|g^{-1}[y]|$ for all $y \in f[X]$. Assume first that $y \in Y$, and say that $y \in Y'_\xi$, where $\varepsilon'_f < \xi < \chi_f$. Then $|g^{-1}[y]| = |(\alpha \circ f)^{-1}[B_y]| = \mu_f \cdot |B_y| = |B_y| = \xi$, since μ_f is one or infinite, and $\xi \geq \mu_f$. Assume now that $y \notin Y$. If $y \in Y^f_\xi$ for some infinite $\mu_f < \xi \leq \varepsilon'_f$ or $\xi \geq \chi_f$, then $|g^{-1}[y]| = |f^{-1}[y]| = \xi$, since $\alpha(y) \in f^{-1}[y]$ and since $|(\alpha \circ f)^{-1}[z]| \in \{0, \mu_f, \xi\}$ for all $z \in f^{-1}[y]$. If $y \in Y^f_\xi$ for some finite $\mu_f < \xi \leq \varepsilon'_f$ or $\xi \geq \chi_f$, then $|g^{-1}[y]| = |f^{-1}[y]| = \xi$, since $\alpha(y) \in f^{-1}[y]$ and since $|(\alpha \circ f)^{-1}[z]| = 0$ for all $z \in f^{-1}[y]$ except $\alpha(y)$. If $y \in Y^f_\xi$ for some $\varepsilon'_f < \xi < \chi_f$ but $y \notin Y$, then $y \in D$ and therefore $|g^{-1}[y]| = 0$. If $y \in S$, then $|g^{-1}[y]| = \mu_f \cdot \mu_f = \mu_f$. If $y \in T$, then there exists exactly one $z \in f^{-1}[y] \cap \alpha[Y]$, and $|(\alpha \circ f)^{-1}[w]| \in \{0, \mu_f\}$ for all $w \in f^{-1}[y]$ except that z, so we have $\varepsilon_f < |g^{-1}[y]| < \chi_f$. If $y \in Y^f_{\mu_f} \backslash (S \cup T)$ then $|g^{-1}[y]| \in \{0, \mu_f\}$. Therefore we have that for all $\mu_f < \xi \leq \varepsilon'_f$ and all $\xi \geq \chi_f$, $Y^g_\xi = Y^f_\xi$ and thus $s_g(\xi) = s_f(\xi)$; $s_g(\mu_f) = |S| = \kappa = s_f(\mu_f)$; and finally, for all $\varepsilon'_f < \xi < \chi_f$ we have $|g^{-1}[y]| = \xi$ iff $y \in Y'_\xi$ or $y \in f \circ \alpha[Y^f_\xi] \subseteq T$, so $s_g(\xi) = |Y'_\xi| + |Y^f_\xi| = s_f(\geq \xi)$; this also implies $s_f(\geq \xi) = s_g(\xi) \leq s_g(\geq \xi) = \sum_{\xi \leq \zeta \leq \varepsilon_f} s_g(\zeta) \leq \sum_{\xi \leq \zeta \leq \varepsilon_f} s_f(\geq \zeta) \leq \varepsilon_f \cdot s_f(\geq \xi) = s_f(\geq \xi)$, so $s_g(\geq \xi) = s_g(\xi) = s_f(\geq \xi)$. □

Lemma 44. *Let $f \in \mathscr{O}^{(1)}$ satisfy (μ), (σ), (n), (ε), (scont), (εreg), and $\varepsilon_f \leq \nu_f$. Let $p \in \mathscr{O}^{(1)}$ be so that $\chi_p = \chi_f$ and $s_p(\geq \xi) = s_f(\geq \xi)$ for all $\varepsilon'_f < \xi < \chi_f$. Then there exists $g \in \langle \{f\} \cup \mathscr{S} \rangle$ such that $s_g(n) = s_p(n)$ for all finite $\varepsilon'_f < n < \chi_f$, such that $s_g(\xi) = s_g(\geq \xi) = s_f(\geq \xi)$ for all infinite $\varepsilon'_f < \xi < \chi_f$, and such that $s_g(\xi) = s_f(\xi)$ for all other $\xi \leq \kappa$.*

Proof. We assume that ε_f is infinite, using (ε); hence, $\nu_f \geq \varepsilon_f$ is infinite, too. By Lemma 43, we may assume that $s_f(\xi) = s_f(\geq \xi)$ for all $\varepsilon'_f < \xi < \chi_f$, since this modification obviously does not change the conditions f satisfies, nor the values of ε_f, ε'_f, χ_f, and ν_f, the latter one staying unchanged by Lemma 10. Then there is nothing left to show if ε'_f is infinite, so we assume it is finite and therefore $\varepsilon'_f = \mu_f = 1$ by (n) and (μ). Also, we can assume $\chi_f > 2$, so χ_f is infinite by (scont) and Lemma 41.
Because $\chi_f > 2$, we have that $s_f(\geq 2) \geq \varepsilon_f$ is infinite. Fix for every $1 < n < \aleph_0$ a set $Z_n \subseteq Y^f_n$ with $|Z_n| = s_p(n)$, and set $W_n = Y^f_n \backslash Z_n$. Set $Z = \bigcup_{1<n<\aleph_0} Z_n$ and $W = \bigcup_{1<n<\aleph_0} W_n$.
Assume first that $s_f(\geq 2) = s_f(\geq \aleph_0)$; then $\aleph_0 < \chi_f$ and hence $s_f(\geq 2) = s_f(\aleph_0)$. Let α map y into $f^{-1}[y]$, for all $y \in Z$ and all $y \in Y^f_{\geq \aleph_0}$.

Now let α map W into $f^{-1}[Y^f_{\aleph_0}]$ in such a way that it stays injective; since $|W| \leq s_p(\geq 2) = s_f(\aleph_0)$, there is enough room to do so. Let α map a suitable part of $X\backslash f[X]$ bijectively onto the set of those elements of $f^{-1}[Z \cup W]$ which are not yet in the image of α. The set $X\backslash f[X]$ is large enough as $|f^{-1}[Z\cup W]| \leq |f^{-1}[Y^f_{>\varepsilon'_f}]| \leq \nu_f$. Choose $S \subseteq Y^f_1$ large and such that $Y^f_1\backslash S$ is large, and map S bijectively onto $f^{-1}[S]$. Extend the partial injection α to a bijection and set $g = f \circ \alpha \circ f$. If $y \in Y^f_1$, then $|g^{-1}[y]| \leq 1$; if $y \in S$, then $|g^{-1}[y]| = 1$. If $y \in Z_n$, then $|g^{-1}[y]| = |f^{-1}[y]| = n$. If $y \in W$, then $f^{-1}[y] \subseteq \alpha[X\backslash f[X]]$, so $|g^{-1}[y]| = 0$. If $y \in Y^f_\xi$ for an infinite ξ, then $|g^{-1}[y]| = \xi$. Therefore, $s_g(\xi) = s_f(\xi)$ for all infinite $\xi \leq \kappa$, $s_g(n) = |Z_n| = s_p(n)$ for all $1 < n < \aleph_0$, and $s_g(1) = s_f(1) = \kappa$ and we are done.

So assume now that $s_f(\geq 2) > s_f(\geq \aleph_0)$. Then also $s_p(\geq 2) > s_p(\geq \aleph_0)$, by the assumptions $s_p(\geq \xi) = s_f(\geq \xi)$ for all $1 < \xi < \chi_f$ and $\chi_p = \chi_f$. Therefore, $|Z| = s_p(\geq 2, < \aleph_0) = s_p(\geq 2) = s_f(\geq 2) = s_f(\geq 2, < \aleph_0)$, and we can find a bijection γ from $Y^f_{>1,<\aleph_0}$ onto Z such that whenever $z \in Y^f_n$, then $\gamma(z) \in Z_j$ for some $j \geq n$. For every such z, we fix a set $B_z \subseteq f^{-1}[\gamma(z)]$ such that $|B_z| = j - n$, and an element $b_z \in f^{-1}[\gamma(z)]\backslash B_z$. Set $B = \bigcup\{B_z : z \in Y^f_{>1,<\aleph_0}\}$. Let α map every $z \in Y^f_{>1,<\aleph_0}$ to b_z. Fix a large set $S \subseteq Y^f_1$ such that $Y^f_1\backslash S$ is large, and let α map S bijectively onto $f^{-1}[S] \cup B$. Now let α map a suitable part of $X\backslash f[X]$ onto the set of those elements of $f^{-1}[Y^f_{>1,<\aleph_0}]$ which are not in the range of α; this is possible as $f^{-1}[Y^f_{>1,<\aleph_0}] \subseteq f^{-1}[Y^f_{>\varepsilon'_f}]$ is not larger than $X\backslash f[X]$. Map all $y \in Y^f_{\geq\aleph_0}$ into $f^{-1}[y]$. Extend α to a bijection and set $g = f \circ \alpha \circ f$. Now if $y \in Y^f_1$, then $|g^{-1}[y]| \leq 1$, and $|g^{-1}[y]| = 1$ for all $y \in S$. If $y \in Z_j$ for some $1 < j < \aleph_0$, then there exist $1 < n \leq j$ and $z \in Y^f_n$ with $\gamma(z) = y$, and we have $|g^{-1}[y]| = |(\alpha\circ f)^{-1}[b_z]|+|(\alpha\circ f)^{-1}[B_z]| = |f^{-1}[z]|+1\cdot|B_z| = n+(j-n) = j$. If $y \in Y^f_{>1,<\aleph_0}\backslash Z$, then $|g^{-1}[y]| = 0$. If $y \in Y^f_\xi$ for some infinite $\xi \leq \kappa$, then $|g^{-1}[y]| = |f^{-1}[y]| = \xi$. Therefore, $s_g(j) = |Z_j| = s_p(j)$ for all $1 < j < \aleph_0$, $s_g(\xi) = s_f(\xi)$ for all infinite ξ, and $s_g(1) = s_f(1) = \kappa$. □

Lemma 45. *Let $f \in \mathscr{O}^{(1)}$ satisfy (μ), (σ), (n), (ε), (scont), (εreg), and $\varepsilon_f \leq \nu_f$. Let $p \in \mathscr{O}^{(1)}$ be so that $\chi_p = \chi_f$ and $s_p(\geq \xi) = s_f(\geq \xi)$ for all $\varepsilon'_f < \xi < \chi_f$. Then there exists $g \in \langle\{f\} \cup \mathscr{S}\rangle$ such that $s_g(\xi) = s_p(\xi)$ for all $\varepsilon'_f < \xi < \chi_f$, and $s_g(\xi) = s_f(\xi)$ for all $\xi \leq \varepsilon'_f$ and all $\xi \geq \chi_f$.*

Proof. We can assume that ε_f is infinite, for otherwise there is nothing to show; hence, $\nu_f \geq \varepsilon_f$ is infinite, too. We also assume that $\varepsilon'_f < \chi_f$, so in particular $\chi_f > \mu_f$. By Lemma 44, we may assume that $s_f(n) = s_p(n)$ for all finite $\varepsilon'_f < n < \chi_f$, and that $s_f(\xi) = s_f(\geq \xi)$ for all infinite $\varepsilon'_f < \xi < \chi_f$. Fix for every infinite $\varepsilon'_f < \xi < \chi_f$ a set $Z_\xi \subseteq Y^f_\xi$ with $|Z_\xi| = s_p(\xi)$. If

$s_p(\xi) = s_f(\xi) = |Y^f_\xi|$, then we shall have $Z_\xi = Y^f_\xi$. For every $z \in Z_\xi$, write $f^{-1}[z] = \{z_\zeta : \zeta < \xi\}$ (observe that here, the $\zeta < \xi$ are of course all ordinals below ξ and not only cardinals). Write $A^\zeta_\xi = \{z_\zeta : z \in Z_\xi\} \subseteq f^{-1}[Z_\xi]$, for all $\zeta < \xi$ (so the index ξ is a cardinal, and the index ζ an ordinal below ξ). Then $|A^\zeta_\xi| = |Z_\xi|$. We define a partial injection α in the following way: Map each Z_ξ bijectively onto A^0_ξ. Next we define α on $Y^f_\xi \backslash Z_\xi$; this is only necessary if $Y^f_\xi \backslash Z_\xi \neq \emptyset$, which happens if and only if $s_p(\xi) < s_f(\xi)$. In that case, set $Z_{>\xi} = \bigcup_{\zeta>\xi} Z_\zeta$. Because $s_p(\xi) < s_f(\xi)$ but $s_p(\geq \xi) = s_f(\geq \xi) = s_f(\xi)$, we have $s_p(> \xi) = s_f(\xi)$; in particular, this implies $s_p(> \xi) = s_p(\geq \xi) > s_p(\geq \chi_p)$, and together with $\chi_p = \chi_f$ we infer $s_p(> \xi, < \chi_f) = s_p(> \xi)$. Hence, $|Z_{>\xi}| = s_p(> \xi, < \chi_f) = s_p(> \xi) = s_f(\xi)$. Therefore, $|\bigcup_{\zeta>\xi} A^\xi_\zeta| = |Z_{>\xi}| = s_f(\xi) = |Y^f_\xi \backslash Z_\xi|$. Let α map $Y^f_\xi \backslash Z_\xi$ bijectively onto $\bigcup_{\zeta>\xi} A^\xi_\zeta$. The function α is injective. Indeed, it is injective on each Z_ξ and $Y^f_\xi \backslash Z_\xi$ by definition, and $f[Z_\xi] = A^0_\xi$, and $f[Y^f_\xi \backslash Z_\xi] = \bigcup_{\zeta>\xi} A^\xi_\zeta$, so the ranges of those injective parts are disjoint. Extend α by mapping y into $f^{-1}[y]$ for all $y \in Y^f_\xi$ for which $\mu_f < \xi \leq \varepsilon'_f$ or $\xi \geq \chi_f$, or for which $\varepsilon'_f < \xi < \chi_f$ is finite. Extend α by mapping a suitable part of $X \backslash f[X]$ bijectively onto those elements of $f^{-1}[Y^f_{>\varepsilon'_f} \cup Y^f_{>1,<\aleph_0}]$ which are not yet in the range of α; this can be done as $f^{-1}[Y^f_{>\varepsilon'_f}]$ and $f^{-1}[Y^f_{>1,<\aleph_0}]$ are not larger than ν_f, the first one by definition of ε'_f and the second one by condition (n). Choose $S \subseteq Y^f_{\mu_f}$ large and such that $Y^f_{\mu_f} \backslash S$ is large, and let α map S bijectively onto $f^{-1}[S]$. Extend α to a bijection and set $g = f \circ \alpha \circ f$.

We calculate $|g^{-1}[z]|$ for all $z \in X$. If $z \in Z_\xi$ for some infinite $\varepsilon'_f < \xi < \chi_f$, then $\alpha(z) \in f^{-1}[z]$, so $g^{-1}[z] \supseteq f^{-1}[z]$ and hence $|g^{-1}[z]| \geq \xi$. On the other hand, if $\alpha(w) = z_\zeta \in f^{-1}[Z_\xi]$, then $\alpha(w) \in A^\zeta_\xi$ by definition of that set. If $\zeta = 0$, then $w \in Z_\xi$, so $|f^{-1}[w]| = \xi$, and if $\xi > \zeta > 0$, then $w \in Y^f_\zeta \backslash Z_\zeta$, so $|f^{-1}[w]| < \xi$. Thus $|g^{-1}[z]| \leq \xi$ and whence, $|g^{-1}[z]| = \xi$.

If $z \in Y^f_\xi \backslash Z_\xi$ for some infinite $\varepsilon'_f < \xi < \chi_f$, then $f^{-1}[z] \cap (\alpha \circ f[X]) = \emptyset$, and so $g^{-1}[z] = \emptyset$. If $z \in Y^f_\xi$ for some $\xi \geq \chi_f$, then $(f \circ \alpha)^{-1}[z] \cap f[X] = \{z\}$, by definition of α, and so $|g^{-1}[z]| = |f^{-1}[z]| = \xi$. If $z \in Y^f_\xi$ for some infinite $\mu_f < \xi \leq \varepsilon'_f$, then $z \in (f \circ \alpha)^{-1}[z]$, by definition of α, and so $|g^{-1}[z]| \geq |f^{-1}[z]| = \xi$. Moreover, if $w \in (f \circ \alpha)^{-1}[z]$ is distinct from z, then $w \in X \backslash f[X]$ or $w \in Y^f_{\mu_f}$, and therefore $|g^{-1}[z]| = \xi$. If $z \in Y^f_\xi$ for some $1 < \xi < \aleph_0$, then $(f \circ \alpha)^{-1}[z] \cap f[X] = \{z\}$, so $|g^{-1}[z]| = \xi$. If $z \in Y^f_{\mu_f}$, then $|(\alpha \circ f)^{-1}[w]| \leq \mu_f$ for all $w \in f^{-1}[z]$, so $|g^{-1}[z]| \in \{0, \mu_f\}$; if $z \in S$, then $|g^{-1}[z]| = \mu_f$. Therefore, $s_g(\xi) = s_f(\xi)$ for all $\xi \leq \varepsilon'_f$ and all $\xi \geq \chi_f$, and for all finite ξ. Also, we have seen that $|g^{-1}[z]| = \xi$ with $\varepsilon'_f < \xi < \chi_f$ infinite if and only if $z \in Z_\xi$, and thus $s_g(\xi) = s_p(\xi)$. □

Lemma 46. *Let $f \in \mathscr{O}^{(1)}$ satisfy (μ), (ν), (σ), (s'dec), (n), (ε), (scont), (εreg), and $\varepsilon_f \leq \nu_f$. Let $p \in \mathscr{O}^{(1)}$ be so that $s'_p = s'_f$, $\mu_p = \mu_f$, $\sigma_p = \sigma_f = \kappa$, and $\nu_p = \nu_f$. Assume moreover that $\chi_p = \chi_f$ and $s_p(\geq \xi) = s_f(\geq \xi)$ for all $\varepsilon'_f < \xi < \chi_f$. Then there exists $g \in \langle \{f\} \cup \mathscr{S} \rangle$ such that $s_g \upharpoonright_{[1,\chi_f)} = s_p \upharpoonright_{[1,\chi_f)}$ and such that $s_g \upharpoonright_{[\chi_f,\kappa]} = s_f \upharpoonright_{[\chi_f,\kappa]}$.*

Proof. Observe first that (s'inf) holds for f by Lemma 32 and since f satisfies (ν), (s'dec), and $\varepsilon_f \leq \nu_f$. We have $\varepsilon'_f = \lambda'_f$ since $\varepsilon_f \leq \nu_f$. By Lemma 35, we can find $h \in \langle \{f\} \cup \mathscr{S} \rangle$ such that $s_h \upharpoonright_{[1,\varepsilon'_f)} = s_p \upharpoonright_{[1,\varepsilon'_f)}$ and such that $s_h \upharpoonright_{[\varepsilon'_f,\kappa]} = s_f \upharpoonright_{[\varepsilon'_f,\kappa]}$. Observe that $s_h(\varepsilon'_f) = s_f(\varepsilon'_f) = s_p(\varepsilon'_f)$ since $s'_p = s'_f$ and $\sigma_p = \sigma_f$ and since ε'_f either is an element of $\mathrm{supp}'(s_f)$ or equals μ_f. Clearly, $\chi_h = \chi_f$. Also, because of (ν) and Lemma 11 we have $\nu_h = \nu_f$, and hence $\varepsilon'_h = \varepsilon'_f$. Moreover, $s_h(\geq \xi) = s_f(\geq \xi)$ for all $\xi > \varepsilon'_f$. It is easy to see that h still satisfies the conditions of Lemma 45. Hence, there is $g \in \langle \{h\} \cup \mathscr{S} \rangle$ such that $s_g(\xi) = s_p(\xi)$ for all $\varepsilon'_f < \xi < \chi_f$, and $s_g(\xi) = s_h(\xi)$ for all other $\xi \leq \kappa$. □

2.2.3. *The lion-tail.* We modify functions $f \in \mathscr{O}^{(1)}$ beyond χ_f, thereby completing the case $\varepsilon_f \leq \nu_f$.

Lemma 47. *Let $f \in \mathscr{O}^{(1)}$ satisfy (μ), (σ), (n), (ε), (χ), $\varepsilon_f \leq \nu_f$, and $s_f(\geq \chi_f) > 0$. There exists $g \in \mathscr{O}^{(1)}$ such that $s_g(\xi) = 0$ for all $\chi_f \leq \xi < \varepsilon_f$, such that $s_g(\varepsilon_f) = 1$, and such that $s_g(\xi) = s_f(\xi)$ for all other $\xi \leq \kappa$.*

Proof. Observe that $\mu_f < \chi_f$, for otherwise $\kappa = s_f(\mu_f) = s_f(\geq \chi_f)$ by (σ), contradicting (χ). We can assume $\varepsilon_f > 1$ and therefore that ε_f is infinite by (ε). Because $s_f(\geq \chi_f) > 0$ and (χ) we have that $s_f(\varepsilon_f) > 0$. Fix $y \in Y^f_{\varepsilon_f}$, and let α map $Y^f_{\geq \chi_f}$ injectively into $f^{-1}[y]$. Now let α map every $z \in Y^f_\xi$, where $\mu_f < \xi < \chi_f$, into $f^{-1}[z]$. Extend α by mapping a suitable part of $X \setminus f[X]$ injectively onto the set of those elements of $f^{-1}[Y^f_{\geq \chi_f}] \cup f^{-1}[Y^f_{>1,<\aleph_0}]$ which are not yet in the range of α; we can do that since $\varepsilon_f \leq \nu_f$ implies $|f^{-1}[Y^f_{\geq \chi_f}]| \leq \nu_f$ and (n) implies $|f^{-1}[Y^f_{>1,<\aleph_0}]| \leq \nu_f$. Choose $S \subseteq Y^f_{\mu_f}$ large and such that $Y^f_{\mu_f} \setminus S$ is large and let α map S bijectively onto $f^{-1}[S]$. Extending α to a bijection, we claim that $g = f \circ \alpha \circ f$ has the desired properties and calculate $|g^{-1}[z]|$ for all $z \in X$. If $z \in Y^f_{\mu_f}$, then $|g^{-1}[z]| \in \{0, \mu_f\}$ by construction of α and since μ is either 1 or infinite by (μ); if $z \in S$ then $|g^{-1}[z]| = \mu_f$. If $z \in Y^f_\xi$, where $\mu_f < \xi < \chi_f$, then $|g^{-1}[z]| = |f^{-1}[z]| = \xi$. If $\chi_f \leq \xi \leq \varepsilon_f$ and $z \in Y^f_\xi$, then $|g^{-1}[z]| = 0$ unless $z = y$, in which case $|g^{-1}[z]| = \varepsilon_f$. Therefore, if $\chi_f \leq \xi \leq \varepsilon_f$, then $s_g(\xi) = 0$ unless $\xi = \varepsilon_f$, in which case we have $s_g(\xi) = 1$. If $\mu_f < \xi < \chi_f$, then $|g^{-1}[z]| = \xi$ if and only if $z \in Y^f_\xi$, so $s_g(\xi) = s_f(\xi)$. If $\xi > \varepsilon_f$, then $s_g(\xi) = 0$. Finally, $s_g(\mu_f) = s_f(\mu_f) = \kappa$. □

Lemma 48. *Let $f \in \mathscr{O}^{(1)}$ satisfy (μ), (σ), (n), (ε), (χ), $\varepsilon_f \leq \nu_f$, and $s_f(\geq \chi_f) > 0$. Let $1 \leq n < \aleph_0$. There exists $g \in \mathscr{O}^{(1)}$ such that $s_g(\xi) = 0$ for all $\chi_f \leq \xi < \varepsilon_f$, such that $s_g(\varepsilon_f) = n$, and such that $s_g(\xi) = s_f(\xi)$ for all other $\xi < \kappa$.*

Proof. Using Lemma 47, it is enough to show that assuming $s_f(\xi) = 0$ for all $\chi_f \leq \xi < \varepsilon_f$, and $s_f(\varepsilon_f) = k$, where $1 \leq k < \aleph_0$, we can produce $g \in \langle \{f\} \cup \mathscr{S} \rangle$ such that $s_g(\varepsilon_f) = k+1$ and such that $s_g(\xi) = s_f(\xi)$ for all other $\xi \leq \kappa$; this is legitimate as application of Lemma 47, as well as increasing the value of $s_f(\varepsilon_f)$ by 1, does not change the conditions on f and leaves the values χ_f and ε_f unaltered. To do this, let α map all $z \in Y^f_\xi$, where $\mu_f < \xi < \varepsilon_f$, into $f^{-1}[z]$. Fix $y \in Y^f_{\mu_f}$ and write $Y^f_{\varepsilon_f} = \{z_1, \ldots, z_k\}$. Let α map z_i into $f^{-1}[z_i]$, $2 \leq i \leq k$, and z_1 into $f^{-1}[y]$. Extend α by mapping a suitable part of $X \backslash f[X]$ bijectively onto those elements of $f^{-1}[Y^f_{>1,<\aleph_0}]$ which are not yet in the image of α; we can do that by (n). Choose $S \subseteq Y^f_{\mu_f}$ large and such that $Y^f_{\mu_f} \backslash S$ is still large, and let α map S bijectively onto $f^{-1}[S \cup \{z_1\}]$. Extending α to a bijection, we claim that $g = f \circ \alpha \circ f$ has the desired properties. Indeed, if $\mu_f < \xi < \varepsilon_f$ and $z \in Y^f_\xi$, then $|g^{-1}[z]| = |f^{-1}[z]| = \xi$. If $z \in Y^f_{\varepsilon_f}$, then $|g^{-1}[z]| = |f^{-1}[z]| = \varepsilon_f$ if $z \in \{z_2, \ldots, z_k\}$, and $|g^{-1}[z]| = \mu_f \cdot \varepsilon_f = \varepsilon_f$ if $z = z_1$ since $f^{-1}[z_1] \subseteq \alpha[Y^f_{\mu_f}]$. If $z \in Y^f_{\mu_f}$, then $|g^{-1}[z]| \in \{0, \mu_f\}$ unless $z = y$, in which case $|g^{-1}[z]| = |f^{-1}[z_1]| = \varepsilon_f$; moreover, $|g^{-1}[z]| = \mu_f$ if $z \in S$. Hence, $s_g(\varepsilon_f) = |\{y, z_1, \ldots, z_k\}| = k+1$, $s_g(\xi) = |Y^f_\xi| = s_f(\xi)$ for all $\mu_f < \xi < \varepsilon_f$, and $s_g(\mu_f) = |S| = \kappa = s_f(\mu_f)$. ☐

Lemma 49. *Let $f \in \mathscr{O}^{(1)}$ satisfy (μ), (σ), (n), (ε), (χ), $\varepsilon_f \leq \nu_f$, and $s_f(\geq \chi_f) > 0$. Let $h \in \mathscr{O}^{(1)}$ be so that $\varepsilon_h = \varepsilon_f$ and such that $s_h(\geq \chi_f)$ is finite. Then there exists $g \in \langle \{f\} \cup \mathscr{S} \rangle$ such that $s_g \restriction_{[1,\chi_f)} = s_f \restriction_{[1,\chi_f)}$ and such that $s_g \restriction_{[\chi_f,\kappa]} = s_h \restriction_{[\chi_f,\kappa]}$.*

Proof. Set $n = s_h(\geq \chi_f)$. By Lemma 48, we can assume that $s_f(\xi) = 0$ for all $\chi_f \leq \xi < \varepsilon_f$, and that $s_f(\varepsilon_f) = n$. Write $Y^f_{\varepsilon_f} = \{z_1, \ldots, z_n\}$ and $Y^h_{\geq \chi_f} = \{y_1, \ldots, y_n\}$. Since $\varepsilon_h = \varepsilon_f$ we have $Y^h_{\varepsilon_f} \neq \emptyset$; say without loss of generality that $|h^{-1}[y_1]| = \varepsilon_f$. For all $\mu_f < \xi < \chi_f$, let α map all $y \in Y^f_\xi$ into $f^{-1}[y]$. Now let α map $Y^f_{\varepsilon_f}$ injectively into $f^{-1}[z_1]$. Fix for every $2 \leq i \leq n$ a set $Z_i \subseteq f^{-1}[z_i]$ with $|Z_i| = |h^{-1}[y_i]|$. Let α map a suitable part of $X \backslash f[X]$ bijectively onto the union of $\bigcup_{2 \leq i \leq n} f^{-1}[z_i] \backslash Z_i$ with the set of those elements of $f^{-1}[Y^f_{>1,<\aleph_0}]$ which are not yet in the range of α; this can be done as $\varepsilon_f \leq \nu_f$ and as f satisfies (n). Take $S \subseteq Y^f_{\mu_f}$ large and so that $Y^f_{\mu_f} \backslash S$ is still large, and let α map S bijectively onto $f^{-1}[S] \cup \bigcup_{2 \leq i \leq n} Z_i$. Extend α to a bijection and set $g = f \circ \alpha \circ f$. If $y \in Y^f_{\mu_f}$, then $|g^{-1}[y]| \in \{0, \mu_f\}$, and if $y \in S$ then $|g^{-1}[y]| = \mu_f$. If $y \in Y^f_\xi$, where $\mu_f < \xi < \chi_f$, then $|g^{-1}[y]| = |f^{-1}[y]| = \xi$. Now assume $y \in Y^f_{\varepsilon_f}$, and say first that $y = z_1$. Then $|g^{-1}[y]| = \varepsilon_f = |h^{-1}[y_1]|$. If $y = z_i$, where $2 \leq i \leq n$, then $|g^{-1}[y]| = |Z_i| \cdot \mu_f = |Z_i| = |h^{-1}[y_i]|$. Thus, we have that $s_g(\xi) = s_f(\xi)$ for all $\xi < \chi_f$, and $s_g(\xi) = s_h(\xi)$ for all $\xi \geq \chi_f$. □

Lemma 50. *Let $f \in \mathscr{O}^{(1)}$ satisfy (μ), (ν), (σ), (s'dec), (n), (ε), (scont), (χ), (εreg), $\varepsilon_f \leq \nu_f$ and $s_f(\geq \chi_f) > 0$. Let $p \in \mathscr{O}^{(1)}$ be so that $s'_p = s'_f$, $\mu_p = \mu_f$, $\sigma_p = \sigma_f = \kappa$, and $\nu_p = \nu_f$. Assume moreover that $\chi_p = \chi_f$, that $s_p(\geq \xi) = s_f(\geq \xi)$ for all $\varepsilon'_f < \xi < \chi_f$, that $\varepsilon_p = \varepsilon_f$, and that $s_p(\geq \chi_f)$ is finite. Then there exists $g \in \langle \{f\} \cup \mathscr{S} \rangle$ such that $s_g = s_p$.*

Proof. By Lemma 46, there exists $q \in \langle \{f\} \cup \mathscr{S} \rangle$ such that $s_q \restriction_{[1,\chi_f)} = s_p \restriction_{[1,\chi_f)}$ and $s_q \restriction_{[\chi_f,\kappa]} = s_f \restriction_{[\chi_f,\kappa]}$. This function q obviously still satisfies the conditions of Lemma 49; also, $\varepsilon_p = \varepsilon_f = \varepsilon_q$ and $\chi_q = \chi_f$. Therefore, that lemma implies that q together with $\mathscr{S}$ generates a function g such that $s_g \restriction_{[1,\chi_f)} = s_q \restriction_{[1,\chi_f)} = s_p \restriction_{[1,\chi_f)}$ and such that $s_g \restriction_{[\chi_f,\varepsilon_f]} = s_p \restriction_{[\chi_f,\varepsilon_f]}$. Hence, $s_g = s_p$. □

Proposition 51. *Let $f \in \mathscr{O}^{(1)}$ be so that $\varepsilon_f \leq \nu_f$. If f moreover satisfies (μ), (ν), (σ), (ρ), (s'dec), (n), (ε), (scont) and (χ), then it is $\mathscr{S}$-minimal.*

Remark 52. In this situation f automatically satisfies (#ε) and (λ'), as $\varepsilon_f \leq \nu_f$.

Proof. Let $g \in \langle \{f\} \cup \mathscr{S} \rangle \setminus \mathscr{S}$. By Lemmas 27, 29 and 32, f satisfies (εreg), (κ), and (s'inf). We have $\varepsilon_g = \varepsilon_f$, $\nu_g = \nu_f$, $\mu_g = \mu_f$, $\chi_g = \chi_f$, and $s'_g = s'_f$, by Lemmas 30, 42, and 33, respectively. By Lemma 40, $s_g(\geq \xi) = s_f(\geq \xi)$ for all $\xi < \chi_f$. The latter fact, together with the fact that χ_g is either infinite and regular or not greater than 2 provided by Lemma 41, implies that g satisfies (scont). By Lemma 30, g satisfies (μ), (ν), (σ), (ε), and (εreg). Because $s'_g = s'_f$, g satisfies (s'dec) and (s'inf), and by Lemma 14 it satisfies (n).
Now if $s_f(\geq \chi_f) = 0$, then by Lemma 46 we find $h \in \langle \{g\} \cup \mathscr{S} \rangle$ such that $s_h \restriction_{[1,\chi_f)} = s_f \restriction_{[1,\chi_f)}$ and such that $s_h \restriction_{[\chi_f,\kappa]} = s_g \restriction_{[\chi_f,\kappa]}$. But by Lemma 38, we have $s_h(\geq \chi_f) = s_g(\geq \chi_f) = s_f(\geq \chi_f) = 0$. Hence, $s_h = s_f$ so that since also $\nu_h = \nu_g = \nu_f$ by Lemma 11, we conclude $f \in \langle \{g\} \cup \mathscr{S} \rangle$.
If on the other hand $s_f(\geq \chi_f) > 0$, then also $s_g(\geq \chi_f) > 0$; $s_g(\geq \chi_f)$ is finite by Lemma 38, so g satisfies (χ). Therefore by Lemma 50 there exists $h \in \langle \{g\} \cup \mathscr{S} \rangle$ such that $s_h = s_f$; since $\nu_h = \nu_g = \nu_f$ we infer $f \in \langle \{g\} \cup \mathscr{S} \rangle$. □

3. Necessities for $\mathscr{S}$-minimality

We prove that the conditions of Theorem 5 are necessary for a function to be $\mathscr{S}$-minimal.

3.1. **Farmers.**

Lemma 53. *If f is $\mathscr{S}$-minimal, then it is constant or has large range.*

Proof. If f has small range, then there exists $y \in X$ with $|f^{-1}[y]| > |f[X]|$, for $\kappa = \sum_{z \in f[X]} |f^{-1}[z]|$. Let α map $f[X]$ injectively into $f^{-1}[y]$. Since both domain and range of the partial function α are co-large, we can extend it to a bijection on X. The function $g = f \circ \alpha \circ f$ is constant and an element of $\langle \{f\} \cup \mathscr{S} \rangle$. Since f is $\mathscr{S}$-minimal, we must have $f \in \langle \{g\} \cup \mathscr{S} \rangle$, which is only possible if f is constant itself. □

Lemma 54. *If f is $\mathscr{S}$-minimal, then it satisfies (ρ), i.e. $\rho_f < \kappa$.*

Proof. Assume to the contrary that $Y^f_{>\mu_f}$ is large. Let $Z \subseteq Y^f_{>\mu_f}$ be large and so that $Y^f_{>\mu_f} \setminus Z$ is large as well, and let α map $Y^f_{\mu_f} \cup Z$ bijectively onto $f^{-1}[Z]$. Both range and domain of α are co-large, so we can extend it to a permutation on X. Now all kernel classes of $g = f \circ \alpha \circ f$ are strictly larger than μ_f. Indeed, assume $|g^{-1}[y]| = \mu_f$ for some $y \in X$. Then there exists $z \in f^{-1}[y] \cap (\alpha \circ f)[X]$; for this z we must have $|(\alpha \circ f)^{-1}[z]| = \mu_f$. By construction of α we conclude that $z \in f^{-1}[Z]$, and so $y \in Z$. But then $|g^{-1}[y]| \geq |f^{-1}[y] \cap (\alpha \circ f)[X]| = |f^{-1}[y]| > \mu_f$, a contradiction. So indeed $\mu_g > \mu_f$ and we cannot get back f from g and $\mathscr{S}$, contradicting that f be $\mathscr{S}$-minimal. □

Lemma 55. *If f is $\mathscr{S}$-minimal and nonconstant, then it satisfies (σ), i.e. $\sigma_f = \kappa$.*

Proof. We know from Lemma 53 that f has large range. Therefore, $\sigma_f + \rho_f = |f[X]| = \kappa$. Since $\rho_f < \kappa$ by Lemma 54, we infer $\sigma_f = \kappa$. □

Lemma 56. *If f is $\mathscr{S}$-minimal, then it satisfies (μ), i.e. $\mu_f = 1$ or μ_f is infinite.*

Proof. Assume that μ_f is finite but not equal to 1. Then f is nonconstant, and therefore $Y^f_{\mu_f}$ is large by Lemma 55. Let $S \subseteq Y^f_{\mu_f}$ be large and such that $Y^f_{\mu_f} \setminus S$ is still large and let α map S bijectively onto $f^{-1}[S]$. Both domain and range of α are co-large, so we can extend it to a bijection on X. Set $g = f \circ \alpha \circ f$. Then for all $y \in S$, $|g^{-1}[y]| = \mu_f^2 > \mu_f$. Thus, $s_g(> \mu_f) = \kappa$. Now if $\mu_g > \mu_f$, then obviously $f \notin \langle \{g\} \cup \mathscr{S} \rangle$, so f is not $\mathscr{S}$-minimal; if on the other hand $\mu_g = \mu_f$, then Lemma 54 implies that g is not $\mathscr{S}$-minimal as $\rho_g = s_g(> \mu_g) = \kappa$, hence in that case f is not $\mathscr{S}$-minimal either, a contradiction. □

3.2. **The return of the man who wasn't there.**

Lemma 57. *If f is $\mathscr{S}$-minimal, then it satisfies (ν), i.e. if μ_f is finite, then ν_f is infinite or zero.*

Proof. If μ_f is finite, then $\mu_f = 1$ by Lemma 56. Assume that in this situation, $0 < \nu_f < \aleph_0$. Fix $y \in Y^f_1$, and choose $\alpha \in \mathscr{S}$ so that $f^{-1}[y] \cap (\alpha \circ f[X]) = \emptyset$. Set $g = f \circ \alpha \circ f$; then $\nu_g \geq |(X \setminus f[X]) \cup \{y\}| = \nu_f + 1$, in contradiction with the obvious fact that if f is $\mathscr{S}$-minimal, then $\nu_g = \nu_f$ for all $g \in \langle \{f\} \cup \mathscr{S} \rangle \setminus \mathscr{S}$. □

3.3. **The revenge of the dwarf-box.**

Lemma 58. *Let f be $\mathscr{S}$-minimal. If ν_f is finite, then $s_f(n) = 0$ for all $1 < n < \aleph_0$. If ν_f is infinite, then $s_f(n) \leq \nu_f$ for all $1 < n < \aleph_0$. In particular, there exist no finite cardinals greater than 1 in the strong support of s_f and f satisfies (n).*

Proof. The lemma is trivial if μ_f is infinite, so we can assume $\mu_f = 1$ by Lemma 56. Then Lemma 57 implies that ν_f is zero or infinite. So all we have to show is that $s_f(n) \leq \nu_f$ for all $1 < n < \aleph_0$. Suppose there is $1 < n < \aleph_0$ with $s_f(n) > \nu_f$, and let n be minimal with this property. Choose $Z \subseteq Y_1^f$ such that $|Z| = |Y_n^f|$ and such that $Y_1^f \backslash Z$ is large; we can do this since Y_1^f is large by Lemma 55. For every $y \in Y_n^f$, let α map y into $f^{-1}[y]$, and let it moreover map exactly one element of Z into $f^{-1}[y]$ in such a way that it stays injective. Extend the mapping to a bijection on X and set $g = f \circ \alpha \circ f$. Observe that we must have $\nu_g = \nu_f$ since f is $\mathscr{S}$-minimal. We claim that for all $1 < k \leq n$, $s_g(k) \leq \nu_f$. Indeed, for $1 < k < n$ this follows from Lemma 13, since $s_f(i) \leq \nu_f$ for all $1 < i \leq k$ and since ν_f is zero or infinite. Now assume $|g^{-1}[y]| = n$ for some $y \in X$. If $|f^{-1}[y]| > n$, then $f^{-1}[y] \backslash (\alpha \circ f[X])$ must be non-empty, which happens at most ν_f times. If $|f^{-1}[y]| = n$, then $\alpha(y) \in f^{-1}[y]$, so $g^{-1}[y]$ contains $f^{-1}[y]$; moreover, by construction of α, $g^{-1}[y]$ contains $f^{-1}[z]$ for some $z \in Z$. Hence, $|g^{-1}[y]| > n$. Finally, if $|f^{-1}[y]| < n$, then there must exist $z \in f^{-1}[y]$ with $1 < |(\alpha \circ f)^{-1}[z]| \leq n$. By construction of α, $|(\alpha \circ f)^{-1}[z]| = n$ is impossible, so this can occur at most $s_f(> 1, < n) \leq \nu_f$ times. Therefore, $s_g(n) \leq \nu_f$. Now Lemma 13 implies that $s_h(n) \leq \nu_f$ for all $h \in \langle \{g\} \cup \mathscr{S} \rangle$; whence, $f \notin \langle \{g\} \cup \mathscr{S} \rangle$, contradicting its $\mathscr{S}$-minimality. $\square$

3.4. **The decline of the valley of giants.**

Lemma 59. *If f is $\mathscr{S}$-minimal, then it satisfies (s'dec), i.e. s'_f is strictly decreasing.*

Proof. Assume there exist $\psi_1 < \psi_2$ in the strong support of s_f with $s_f(\psi_1) \leq s_f(\psi_2)$, and let ψ_1 be minimal with this property. $\sigma_f = \kappa$ by Lemma 55, and $\rho_f < \kappa$ by Lemma 54, so in particular $\mu_f < \psi_1$. Lemma 58 tells us that ψ_1 cannot be finite.

Fix $Z \subseteq Y_{\psi_2}^f$ such that $|Z| = |Y_{\psi_1}^f|$. This is possible since $s_f(\psi_1) \leq s_f(\psi_2)$. Choose $S \subseteq Y_{\mu_f}^f$ such that $Y_{\mu_f}^f \backslash S$ is large and such that $|S| = |f^{-1}[Z]|$. Let α map $Y_{\psi_1}^f \cup S$ bijectively onto $f^{-1}[Z]$, and Z injectively into $f^{-1}[Y_{\psi_1}^f]$ in such a way that for all $y \in Y_{\psi_1}^f$ there exists $z \in Z$ with $\alpha(z) \in f^{-1}[y]$. We can do that since $|Z| = |Y_{\psi_1}^f|$. Let α moreover map all $y \in Y_{>\mu_f}^f \backslash (Y_{\psi_1}^f \cup Z)$ into $f^{-1}[y]$. Domain and range of α are co-large as they are disjoint from $Y_{\mu_f}^f \backslash S$ and $f^{-1}[Y_{\mu_f}^f \backslash S]$, respectively, so the function can be extended to a bijection on X; set $g = f \circ \alpha \circ f$.

We claim that $s_g(\psi_1) = 0$ and calculate $|g^{-1}[y]|$ for all $y \in X$. If $y \in Y^f_\xi$, where $\xi < \psi_1$ is infinite, then $|g^{-1}[y]| \leq \xi < \psi_1$ since $(f \circ \alpha)^{-1}[y] \subseteq Y^f_\xi \cup Y^f_{\mu_f} \cup (X \backslash f[X])$ by construction of α. If $\xi < \psi_1$ is finite, then for the same reason we have that $|g^{-1}[y]|$ is finite, so again $|g^{-1}[y]| < \psi_1$. If $y \in Y^f_{>\psi_1} \backslash Z$, then $\alpha(y) \in f^{-1}[y]$ and so $|g^{-1}[y]| \geq |f^{-1}[y]| > \psi_1$, and if $y \in Z$, then $f^{-1}[y] \cap (\alpha \circ f)[X] = f^{-1}[y]$ by construction of α, so again $|g^{-1}[y]| \geq |f^{-1}[y]| > \psi_1$. Finally, consider $y \in Y^f_{\psi_1}$. Then by construction of α there exists $z \in Z$ with $\alpha(z) \in f^{-1}[y]$. But then $|g^{-1}[y]| \geq |f^{-1}[z]| = \psi_2 > \psi_1$, and we have shown $s_g(\psi_1) = 0$.

Because s'_f is strictly decreasing below ψ_1 by the choice of ψ_1, $\operatorname{supp}'(s_f)$ is finite below ψ_1; therefore, there exists $\lambda < \psi_1$ such that $\operatorname{supp}'(s_f) \cap (\lambda, \psi_1)$ is empty. Moreover, if $\psi_1 > \nu_f$, we can certainly choose λ so that $\lambda \geq \nu_f$.

Consider the case where $\psi_1 > \nu_f$; in that case, s_f vanishes on the interval (λ, ψ_1). Because f is $\mathscr{S}$-minimal we must have $\nu_g = \nu_f < \psi_1$. Since $s_g(\xi) \leq s_f(> \lambda, \leq \xi) + \min(\nu_f, s_f(> \xi, \leq \nu_f)) = s_f(> \lambda, \leq \xi) = 0$ for all $\lambda < \xi < \psi_1$ by Lemma 20, the same lemma implies that $s_h(\psi_1) \leq s_g(> \lambda, \leq \psi_1) + \min(\nu_g, s_g(> \psi_1, \leq \nu_g)) = s_g(> \lambda, \leq \psi_1) = 0$ for all $h \in \langle \{g\} \cup \mathscr{S} \rangle$, so $f \notin \langle \{g\} \cup \mathscr{S} \rangle$, in contradiction with the $\mathscr{S}$-minimality of f.

If $\psi_1 \leq \nu_f$, then $s_f(\psi_1) > \nu_f$ as $\psi_1 \in \operatorname{supp}'(s_f)$; also, $\nu_g = \nu_f$ by the $\mathscr{S}$-minimality of f. By Lemma 20, we have $s_g(\xi) \leq s_f(> \lambda, \leq \xi) + \nu_f \leq \nu_f$ for all $\lambda < \xi < \psi_1$. Therefore by the same lemma, if $h \in \langle \{g\} \cup \mathscr{S} \rangle$, then $s_h(\psi_1) \leq s_g(> \lambda, \leq \psi_1) + \nu_g \leq \nu_g$. But $\nu_g = \nu_f < s_f(\psi_1)$, so $h \neq f$, implying $f \notin \langle \{g\} \cup \mathscr{S} \rangle$ and again contradicting that f be $\mathscr{S}$-minimal.

□

3.5. **The king.**

Lemma 60. *If f is $\mathscr{S}$-minimal, then it satisfies (ε), i.e. $\varepsilon_f = 1$ or ε_f is infinite.*

Proof. Assume not, and fix $y \in Y^f_{\varepsilon_f}$ and any $z \neq y$. Let $\alpha \in \mathscr{S}$ be so that it maps $\{y, z\}$ injectively into $f^{-1}[y]$. Then setting $g = f \circ \alpha \circ f$ we have that $|g^{-1}[y]| \geq |f^{-1}[y] \cup f^{-1}[z]| > \varepsilon_f$. All functions generated by g with $\mathscr{S}$ have a class larger than ε_f, which implies $f \notin \langle \{g\} \cup \mathscr{S} \rangle$ and contradicts that f is $\mathscr{S}$-minimal. □

Lemma 61. *Let f be $\mathscr{S}$-minimal. Then it satisfies ($\#\varepsilon$), i.e. if $\nu_f < \varepsilon_f$, then $s_f(\varepsilon_f)$ is infinite.*

Proof. By Lemma 59, the restriction of s_f to its support beyond ν_f is strictly decreasing, so the support beyond ν_f is finite and thus $s_f(\varepsilon_f) > 0$. Assume $s_f(\varepsilon_f) < \aleph_0$. By Lemma 60, ε_f is one or infinite; $\varepsilon_f = 1$, however, is clearly impossible since it would mean that f is injective but has only finitely many kernel classes. Fix $S \subseteq Y^f_{\mu_f}$ such that $|S| = |f^{-1}[Y^f_{\varepsilon_f}]| \geq \varepsilon_f$ and such that $Y^f_{\mu_f} \backslash S$ is still large. Let α map S bijectively onto $f^{-1}[Y^f_{\varepsilon_f}]$, as well as $Y^f_{\varepsilon_f}$ injectively into $f^{-1}[S]$. The domain of α is disjoint from $Y^f_{\mu_f} \backslash S$ and hence co-large, and so is its range as it is disjoint from $f^{-1}[Y^f_{\mu_f}\setminus S]$, so α can be extended to a permutation on X. The function $g = f \circ \alpha \circ f$ satisfies $s_g(\geq \varepsilon_f) \geq s_f(\varepsilon_f) + 1$. Indeed, if $y \in Y^f_{\varepsilon_f}$, then $|g^{-1}[y]| \geq |f^{-1}[y] \cap (\alpha \circ f)[X]| = |f^{-1}[y]| = \varepsilon_f$ since $f^{-1}[y] \cap (\alpha \circ f)[X] = f^{-1}[y]$ by construction of α. Also, taking an arbitrary $z \in Y^f_{\varepsilon_f}$ and setting $w = f \circ \alpha(z) \in S$, we have $|g^{-1}[w]| \geq \varepsilon_f$. Thus indeed, $s_g(\geq \varepsilon_f) \geq |Y^f_{\varepsilon_f} \cup \{w\}| = s_f(\varepsilon_f) + 1$. However, Lemma 15 gives us $s_g(> \varepsilon_f) = 0$, and so $s_g(\varepsilon_f) \geq s_f(\varepsilon_f) + 1$. Since $\nu_g = \nu_f < \varepsilon_f$ by the $\mathscr{S}$-minimality of f, we have that Lemma 22 yields $s_h(\varepsilon_f) \geq s_g(\varepsilon_f) > s_f(\varepsilon_f)$ for all $h \in \langle \{g\} \cup \mathscr{S}\rangle \setminus \mathscr{S}$, so $f \notin \langle \{g\} \cup \mathscr{S}\rangle$, contradicting the assumption that f is $\mathscr{S}$-minimal. □

Lemma 62. *Let f be $\mathscr{S}$-minimal. Then it satisfies (εreg), i.e., either $s_f(\varepsilon_f) > 0$ or ε_f is regular.*

Proof. Assume ε_f is singular and that $s_f(\varepsilon_f) = 0$. Let $\eta < \varepsilon_f$ be the cofinality of ε_f, let $\vartheta \geq \eta$ be in the support of s_f, and fix $y \in Y^f_\vartheta$. Let $(\zeta_\tau)_{\tau<\eta}$ be a strictly increasing sequence of cardinalities in the support of s_f which is cofinal in ε_f and larger than μ_f, and fix $y_\tau \in Y^f_{\zeta_\tau}$ for all $\tau < \eta$. Set $Y = \{y_\tau : \tau < \eta\}$. Let α map Y injectively into $f^{-1}[y]$, and extend it to a bijection on X. This is possible since α is not defined on $Y^f_{\mu_f}$ and its range is disjoint from $f^{-1}[Y^f_{\mu_f}]$ and since the two sets are large by Lemma 55. Set $g = f \circ \alpha \circ f$. Then $|g^{-1}[y]| \geq |\bigcup_{\tau<\eta} f^{-1}[y_\tau]| = \sum_{\tau<\eta} \zeta_\tau = \varepsilon_f$. Therefore, g has a kernel class larger than all kernel classes of f, so that it cannot generate f together with $\mathscr{S}$, a contradiction. □

3.6. **Continuity.**

Lemma 63. *Let f be $\mathscr{S}$-minimal. Then $s_f(\geq \xi) = \min\{s_f(\geq \zeta) : \zeta < \xi\}$ for all singular $\xi \leq \chi_f$.*

Proof. Assume there is $\xi \leq \chi_f$ singular with $s_f(\geq \xi) < \vartheta = \min\{s_f(\geq \zeta) : \zeta < \xi\}$, and let $\eta < \xi$ be the cofinality of ξ. Clearly, $\xi > \mu_f$. Let $\max\{\eta, \mu_f\} < \zeta < \xi$ be so that $s_f(\geq \zeta) = \vartheta$.

Observe next that for all $\zeta \leq \psi < \xi$ and all $\lambda < \vartheta$ there exists $\psi \leq \psi' < \xi$ with $s_f(\psi') > \lambda$, for otherwise $\vartheta = s_f(\geq \psi) \leq \lambda \cdot \xi$, implying $\vartheta = \xi$, and thus $\varepsilon_f \leq \vartheta = \xi \leq \varepsilon_f$. However, $\vartheta = \varepsilon_f$ implies $s_f(\varepsilon_f) = 0$ since $\zeta < \chi_f$ and $s_f(\geq \zeta) = \vartheta$, contradicting Lemma 62 as $\varepsilon_f = \xi$ is singular. By our observation we can thin out the interval (ζ, ξ) and find a strictly increasing sequence of cardinals $(\zeta_\tau)_{\tau<\eta}$ greater than ζ and cofinal in ξ, such that the sequence $(\delta_\tau)_{\tau<\eta} = (s_f(\zeta_\tau))_{\tau<\eta}$ has the property that for all $\lambda < \vartheta$ and all $\tau < \eta$ there exists $\tau < \tau' < \eta$ such that $\delta_{\tau'} > \lambda$. Write $Y^f_{\zeta_\tau} = \{y^i_{\zeta_\tau} : i < \delta_\tau\}$ for all $\tau < \eta$ (with the variable i referring to all ordinals below ϑ). Set $S^i = \{y^i_{\zeta_\tau} : \tau < \eta \wedge i < \delta_\tau\}$, for all $i < \vartheta$.

Write $Y^f_{\geq\zeta,<\xi} = \{z_i : i < \vartheta\}$ (again with i referring to ordinals). Let α map S^i injectively into $f^{-1}[z_i]$, for all $i < \vartheta$, extend α to a bijection, and set $g = f \circ \alpha \circ f$. Then $|g^{-1}[z_i]| \geq |f^{-1}[S^i]| = |\bigcup\{f^{-1}[y^i_{\zeta_\tau}] : \tau < \eta \wedge i < \delta_\tau\}| = \sum_{\tau<\eta\wedge i<\delta_\tau} \zeta_\tau = \xi$, the latter equality holding since we chose the sequence $(\zeta_\tau)_{\tau<\eta}$ so that for all $\tau < \eta$ there exists $\tau < \tau' < \eta$ with $i < \delta_{\tau'}$ and since $(\zeta_\tau)_{\tau<\eta}$ is cofinal in ξ. Thus, $s_g(\geq \xi) \geq \vartheta$. Now g does not have any kernel class larger than all kernel classes of f, because f is $\mathscr{S}$-minimal; hence, $s_g(\geq \xi)$ is larger that all cardinals in $\mathrm{supp}(s_g)$, and thus $\xi < \chi_g$. Moreover, g satisfies (ε) and (εreg), by Lemmas 60 and 62. Therefore, $s_h(\geq \xi) \geq s_g(\geq \xi) = \vartheta > s_f(\geq \xi)$ for all $h \in \langle\{g\} \cup \mathscr{S}\rangle\backslash\mathscr{S}$ by Lemma 39, contradicting that f is $\mathscr{S}$-minimal. □

Lemma 64. *Let f be $\mathscr{S}$-minimal. Then $s_f(\geq n) = s_f(\geq 2)$ for all finite $2 \leq n \leq \chi_f$.*

Proof. It suffices to show that $s_f(\geq n) = s_f(\geq n+1)$ for all finite $2 \leq n < \chi_f$. Assume to the contrary that $s_f(\geq n) > s_f(\geq n+1)$ for some finite $2 \leq n < \chi_f$. By Lemma 60, f satisfies (ε). This, together with the fact that there is no $\zeta \in \mathrm{supp}(s_f)$ with $\zeta \geq s_f(\geq n)$, implies that $s_f(\geq n)$ must be infinite, and hence $s_f(\geq n) = s_f(n)$ as $s_f(> n) < s_f(\geq n)$. Let α map Y^f_n injectively into $f^{-1}[Y^f_n]$ in such a way that $|f^{-1}[y] \cap \alpha[Y^f_n]| = 2$ for all $y \in Y^f_n$. Because α satisfies (μ) and (σ) by Lemmas 56 and 55, we have that $Y^f_1 \neq Y^f_n$ is large, so we can extend α to a permutation of X and set $g = f \circ \alpha \circ f$. Then for all $y \in Y^f_n$ we have that $|g^{-1}[y]| \geq 2 \cdot n > n$. Hence, $s_g(\geq n+1) \geq s_f(\geq n) > s_f(\geq n+1)$. We clearly have $\varepsilon_g = \varepsilon_f$ and $s_g(\varepsilon_f) = 0$ iff $s_f(\varepsilon_f) = 0$, as f is $\mathscr{S}$-minimal, so $s_g(\geq n+1) \geq s_f(\geq n)$ and $\chi_f > n$ imply $\chi_g > n+1$. Also, g satisfies (εreg) by Lemma 62. Thus, Lemma 39 implies that $s_h(\geq n+1) \geq s_g(\geq n+1) > s_f(\geq n+1)$ for all $h \in \langle\{g\} \cup \mathscr{S}\rangle\backslash\mathscr{S}$, so f is not $\mathscr{S}$-minimal, a contradiction. □

Lemma 65. *Let f be $\mathscr{S}$-minimal. Then it satisfies (scont), i.e. $s_f(\geq \xi) = \min\{s_f(\geq \zeta) : \zeta < \xi\}$ for all singular $\xi \leq \chi_f$ and $s_f(\geq n) = s_f(\geq 2)$ for all finite $2 \leq n \leq \chi_f$.*

Proof. This is the consequence of Lemmas 63 and 64. □

3.7. **The rage of the lion-tail.**

Lemma 66. *If f is $\mathscr{S}$-minimal and nonconstant, then it satisfies (κ), i.e. if $\nu_f = \kappa$, then $s_f(\kappa) = 0$.*

Proof. Assume $\nu_f = \kappa$ and $s_f(\kappa) > 0$, and let $y \in Y^f_\kappa$. Let α map $f[X]$ injectively into a co-large part of $f^{-1}[y]$. Since both domain and range of α are co-large, we can extend it to a function in $\mathscr{S}$. Then $g = f \circ \alpha \circ f$ is constant and generates together with $\mathscr{S}$ a proper subclone of $\langle\{f\} \cup \mathscr{S}\rangle$, contradicting that f is $\mathscr{S}$-minimal. □

Lemma 67. *If f is $\mathscr{S}$-minimal, then it satisfies (χ), i.e. if $\varepsilon_f \leq \nu_f$, then $s_f(\geq \chi_f)$ is finite.*

Proof. We can assume that f is nonconstant and that $\varepsilon_f > 1$; then ε_f is infinite by Lemma 60. Also, we may assume that $\mu_f < \chi_f$, for otherwise $s_f(\kappa) > 0$ as $s_f(\mu_f) = \sigma_f = \kappa$ by Lemma 55 and thus $\nu_f \geq \varepsilon_f = \kappa$, contradicting that f satisfies (κ) by Lemma 66. Suppose $s_f(\geq \chi_f)$ is infinite; we want to derive a contradiction. By Lemma 65, f satisfies (scont), and therefore $\chi_f \leq 2$ or χ_f is infinite and regular by Lemma 41. Fix $y \in X$ with $|f^{-1}[y]| \geq s_f(\geq \chi_f)$. Let α map $Y^f_{\geq\chi_f}$ injectively into $f^{-1}[y]$, and a suitable part of $X\backslash f[X]$ bijectively onto $f^{-1}[Y^f_{\geq\chi_f}\backslash\{y\}]$; this is possible as $|f^{-1}[Y^f_{\geq\chi_f}]| \leq \varepsilon_f \cdot |Y^f_{\geq\chi_f}| \leq \varepsilon_f \leq \nu_f$. Extend α to a bijection on X. The function $g = f \circ \alpha \circ f$ satisfies $s_g(\geq \chi_f) \leq 1$. Indeed, if $|g^{-1}[z]| \geq \chi_f$, then either $|f^{-1}[z]| \geq \chi_f$ or there exists $w \in f^{-1}[z]$ with $|(\alpha \circ f)^{-1}[w]| \geq \chi_f$, because $\chi_f \leq 2$ or χ_f is infinite and regular. But if $|f^{-1}[z]| \geq \chi_f$, then for $z \neq y$ we have that $f^{-1}[z] \cap \alpha \circ f[X] = \emptyset$, by definition of α, so $|g^{-1}[z]| = 0$; the other possibility does not occur unless $z = y$, and we have shown $s_g(\geq \chi_f) \leq 1$. Therefore, $s_h(\geq \chi_f)$ is finite for all $h \in \langle\{g\} \cup \mathscr{S}\rangle$ by Lemma 38, and hence $f \notin \langle\{g\} \cup \mathscr{S}\rangle$, a contradiction. □

3.8. **Existence of the hole.**

Lemma 68. *Let $f \in \mathscr{O}^{(1)}$ satisfy (μ), (ν), (σ), (s'dec), and (n). Then there exists $g \in \langle\{f\} \cup \mathscr{S}\rangle$ such that $s_g(\xi) = 0$ for all $\xi < \varepsilon'_f$ with $\xi \notin \mathrm{supp}'(s_f)$, and $s_g(\xi) = s_f(\xi)$ for all other $\xi \leq \kappa$. In particular, there are no elements below ε'_f in the weak support of s_g.*

Proof. This can be proven exactly like Lemma 34, replacing λ'_f by ε'_f. □

Lemma 69. *Let f be $\mathscr{S}$-minimal. Then f satisfies (λ'), i.e. if $\varepsilon_f > \nu_f$, then $s_f(\xi) = 0$ for all $\xi \in (\lambda'_f, \nu_f]$.*

Proof. Assume $\varepsilon_f = \varepsilon'_f > \nu_f$. By Lemmas 56, 57, 55, 59 and 58, f satisfies the conditions of Lemma 68. Therefore there exists $g \in \langle\{f\} \cup \mathscr{S}\rangle\backslash\mathscr{S}$ such that $s_g(\xi) = 0$ for all $\xi \in (\lambda'_f, \nu_f]$. Now if $\xi \in (\lambda'_f, \nu_f]$ is infinite, then Lemma 20 implies $s_h(\xi) \leq s_g(> \lambda'_f, \leq \xi) + \min(\nu_g, s_g(> \xi, \leq \nu_g)) = 0 + \min(\nu_g, 0) = 0$ for all all $h \in \langle\{g\} \cup \mathscr{S}\rangle$, so in particular this holds for f. If $\xi \in (\lambda'_f, \nu_f]$ is finite, then $\lambda'_f = 1$ as g satisfies (n) by Lemma 58; moreover, g satisfies (ν) by Lemma 57, and so Lemma 13 yields $s_h(\xi) \leq s_g(> 1, \leq \xi) + \min(\nu_g, s_g(> \xi, \leq \nu_g)) = 0 + \min(\nu_g, 0) = 0$ for all $h \in \langle\{g\} \cup \mathscr{S}\rangle$. Hence, $s_f(\xi) = 0$. □

4. Proofs of the corollaries

Proof of Theorem 6. Assume first that $\langle\{f\} \cup \mathscr{S}\rangle = \langle\{g\} \cup \mathscr{S}\rangle$. By Lemma 30 we have $\mu_g = \mu_f$, $\nu_g = \nu_f$, and $\varepsilon_g = \varepsilon_f$. Lemma 33 implies $s'_g = s'_f$. We have $\chi_g = \chi_f$ by Lemma 42, and by Lemma 40 we have $s_g(\geq \xi) = s_f(\geq \xi)$ for all $\xi < \chi_f$. Obviously $s_f(\varepsilon_f) > 0$ implies $s_g(\varepsilon_f) > 0$.

For the other direction, assume first that $\varepsilon_f > \nu_f$. By Lemma 35, there exists $h \in \langle\{f\} \cup \mathscr{S}\rangle$ such that $s_h \restriction_{[1,\lambda'_f)} = s_g \restriction_{[1,\lambda'_f)}$ and $s_h \restriction_{[\lambda'_f,\kappa]} = s_f \restriction_{[\lambda'_f,\kappa]}$; thus,

$s_h \restriction_{(\nu_f,\kappa]} = s_g \restriction_{(\nu_f,\kappa]}$ as $s'_g = s'_f$. Also, we have that $\mathrm{supp}(s_h) \cap (\lambda'_f, \nu_f] = \mathrm{supp}(s_f) \cap (\lambda'_f, \nu_f]$ is empty, by (λ'); for the same reason, s_g vanishes in that interval, too. Therefore, $s_h = s_g$ so that since by Lemma 11 also $\nu_h = \nu_f = \nu_g$ we conclude $g \in \langle \{f\} \cup \mathscr{S} \rangle$.
Next assume $\varepsilon_f \leq \nu_f$ and $s_f(\geq \chi_f) = 0$; then also $s_g(\geq \chi_f) = 0$ as $\varepsilon_g = \varepsilon_f$ and since $s_g(\varepsilon_g) = 0$ iff $s_f(\varepsilon_f) = 0$. By Lemma 46 there exists $h \in \langle \{f\} \cup \mathscr{S} \rangle$ such that $s_h \restriction_{[1,\chi_f)} = s_g \restriction_{[1,\chi_f)}$, and such that $s_h(\xi) = s_f(\xi) = 0$ for all $\xi \geq \chi_f$; hence, $s_h = s_g$ and we are done.
Finally, if $\varepsilon_f \leq \nu_f$ and $s_f(\geq \chi_f) > 0$, then $s_f(\geq \chi_f)$ is finite by (χ), and so is $s_g(\geq \chi_g)$ for the same reason. With the help of Lemma 50 we can construct $h \in \langle \{f\} \cup \mathscr{S} \rangle$ such that $s_h = s_g$. □

Proof of Corollary 7. By Theorem 6, the clone an $\mathscr{S}$-minimal function f generates is fully determined by the decreasing sequences $s'_f(\xi)$ and $s_f(\geq \xi)$, as well as by the values μ_f, ν_f, χ_f, ε_f, and $s_f(\varepsilon_f)$. Since $s'_f(\xi)$ and $s_f(\geq \xi)$ are decreasing, they are determined by the finitely many points where they decrease, together with their values at those points. Therefore, for all determining parameters we have at most as many possibilities as there are cardinals below $\kappa = \aleph_\alpha$, which is $\max\{|\alpha|, \aleph_0\}$, so the number of clones minimal in $[\langle \mathscr{S} \rangle, \mathscr{O}]$ is not more than that.
On the other hand, using Theorem 5 one sees that the functions $f \in \mathscr{O}^{(1)}$ with $\mu_f = \kappa$, $s_f(\kappa) = \kappa$ and $\nu_f = \nu < \kappa$ are $\mathscr{S}$-minimal for all $\nu < \kappa$, and by Theorem 6 they generate distinct clones. Therefore, the number of clones minimal in $[\langle \mathscr{S} \rangle, \mathscr{O}]$ is at least $\max\{|\alpha|, \aleph_0\}$. □

Proof of Corollary 8. The $\mathscr{S}$-minimality of the functions which generate those monoids can easily be verified by Theorem 5.
To see that the mentioned monoids are the only monoids minimal in $[\mathscr{S}, \mathscr{O}^{(1)}]$, let f be $\mathscr{S}$-minimal and non-constant. If $\mu_f = \aleph_0$ and $\nu_f < \aleph_0$, then f with $\mathscr{S}$ generates $\mathscr{I}_{\nu_f}$. We cannot have $\mu_f = \aleph_0$ and $\nu_f = \aleph_0$, because this would contradict (χ) or (σ). So let $\mu_f = 1$; then ν_f is zero or infinite by (ν). We distinguish two cases. Assume first that $\varepsilon_f = \mu_f = 1$. Then $\nu_f > 0$ since $f \notin \mathscr{S}$, so ν_f is infinite and it is easily seen that in this case, f generates $\mathscr{H}$. Now consider the case where $\varepsilon_f > 1$; we claim that this cannot happen. Indeed, we would have to have $\varepsilon_f = \aleph_0$ by (ε). By (ρ), $s_f(> 1)$ is finite and therefore $s_f(\varepsilon_f) > 0$. But then $\chi_f = 1$ by definition, contradicting (χ) or (σ). □

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