Chapter 2

Labelled structures and Exponential Generating Functions

Def: A labelled object has atoms that bear distinct integer labels (canonically taken on interval [1..n] if there are n atoms).

UNLABELLED VERSUS LABELLED OBJECTS

unlabelled "anonymous" atoms ● ●

labelled distinguished atoms ①, ②, ③, ④, ...
("colours")

How many graphs on a set of n distinguishable vertices, say [1..n] ?

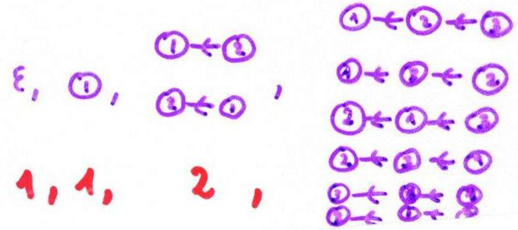
$$G_n = 2^{n(n-1)/2}$$

(Unlabelled problem is harder)

PERMUTATIONS are typical labelled objects

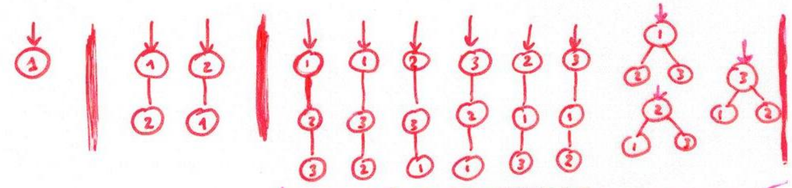
write $\sigma = \begin{pmatrix} 1 & 2 & \cdots & n \\ \sigma_1 & \sigma_2 & \cdots & \sigma_n \end{pmatrix}$ as $\sigma_1 \sigma_2 \cdots \sigma_n$

and view as **a linear digraph** that is **labelled**



1, 1, 2, 6, ..., n!

Rooted trees (graph theoretical) "non-plane"



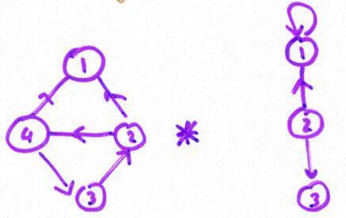
$T_1 = 1$, $T_2 = 2$, $T_3 = 9$, ($T_4 = 64$..)

$U_1 = 1$, $U_2 = 1$, $U_3 = 2$, $U_4 = 4$, ...

Let $\mathcal{B}$ and $\mathcal{C}$ be labelled classes.

Then the Cartesian product $\mathcal{A} = \mathcal{B} \times \mathcal{C}$ is not a labelled class [WHY?]

Given a pair (β, γ) with β and γ well labelled, form all possible relabellings that preserve the order structure within β and γ while giving rise to well labelled objects.



$= \left\{ \text{a collection of } \binom{4+3}{4} \text{ well labelled pairs.} \right\}$

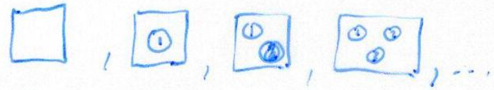
- Labelled product of 2 objects

$$(\alpha * \beta) = \{ \gamma \mid \gamma = (\alpha', \beta');\ \alpha' \underset{\text{order}}{\equiv} \alpha;\ \beta' \underset{\text{order}}{\equiv} \beta;\ \gamma \text{ is well labelled} \}$$

- Labelled product of two classes $\mathcal{C} = \mathcal{A} * \mathcal{B}$

$$\mathcal{C} = \bigcup_{\substack{\alpha \in \mathcal{A} \\ \beta \in \mathcal{B}}} (\alpha * \beta)$$

Example: $\mathcal{U}$ = class of all totally disconnected graphs



1 1 1 1

$\underbrace{\mathcal{U} * \mathcal{U} * \cdots * \mathcal{U}}_{m \text{ times}}$ = Allocations of n elements into m cells; functions from $[1..n] \to [1..m]$

$\mathcal{V} = \mathcal{U} \setminus \{\varepsilon\}$ 0, 1, 1, 1, ...

$\mathcal{V} * \mathcal{V} * \cdots * \mathcal{V}$ = allocations with no empty cell ≡ surjections from $[1..n]$ to $[1..m]$

Define | Labelled Sequences, Labelled Sets, Labelled Cycles |

$\mathcal{S}$ = Set($\mathcal{V}$) = partitions of elements $[1..n]$ into non-empty classes

→ Set partitions

$\mathcal{R}$ = seq($\mathcal{V}$) = ordered partitions / surjections

Dictionary II = theorem 2.1

$$\begin{array}{lcl}
\mathbf{A} \cup \mathbf{B} & \mapsto & A(z)+B(z) \\
\mathbf{A} \ast \mathbf{B} & \mapsto & A(z)\times B(z) \\
\mathrm{Seq}\,\mathbf{A} & \mapsto & \dfrac{1}{1-A(z)} \\
\mathrm{Set}\,\mathbf{A} & \mapsto & \exp(A(z)) \\
\mathrm{Cyc}\,\mathbf{A} & \mapsto & \log \dfrac{1}{1-A(z)}
\end{array}$$

$$\mathcal{C} = \mathcal{A} \ast \mathcal{B}$$

$$C_n = \sum_{k=0}^{n} \binom{n}{k} A_k B_{n-k}$$

$$\frac{C_n}{n!} = \sum_k \frac{A_k}{k!} \cdot \frac{B_{n-k}}{(n-k)!}$$

$$\Rightarrow \boxed{\hat{C}(z) = \hat{A}(z) \cdot \hat{B}(z)} \leftarrow \text{EGF}$$

To a (labelled) class, associate its exponential generating function $\equiv$ EGF

$$(f_n) \longrightarrow f(z) = \sum_{n=0}^{\infty} f_n \frac{z^n}{n!}$$

$$\mathcal{C} \to (C_n) \longrightarrow C(z) = \sum_{n=0}^{\infty} C_n \frac{z^n}{n!}$$

(also written $\hat{C}(z)$ when we want to stress exponential GF)

Exercise: what is the EGF of all permutations?
——— of all binary words?

$$\mathrm{Seq}(A): \quad 1 + A + A^2 + A^3 + \cdots = \frac{1}{1-A}$$

$$\mathrm{Set}(A): \quad 1 + \frac{A}{1!} + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots = \exp(A)$$

$$\mathrm{Cyc}(A): \quad 1 + \frac{A}{1} + \frac{A^2}{2} + \frac{A^3}{3} + \cdots = \log \frac{1}{1-A}$$

EXAMPLE 0: Basic building blocks.

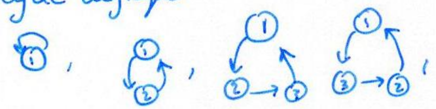
• (Labelled) linear digraphs

ε, ①, ①←②, ②←①, ①←②←③, ①←③←②, ②←①←③, ②←③←①, ③←①←②, ③←②←①

1, 1, 2, 6, ...

$$\begin{cases} \mathcal{P} = \mathrm{Seq}(\mathcal{Z}) \\ P(z) = \dfrac{1}{1-z} \\ P_n = n! \end{cases}$$

• (Labelled) Cycle digraphs



1, 1, 2, ...

$$\begin{cases} \mathcal{K} = \mathrm{Cyc}(\mathcal{Z}) \\ K(z) = \log \dfrac{1}{1-z} \\ K_n = (n-1)! \end{cases}$$

• (Labelled) Disconnected graphs

ε, 

1 1 1 1

$$\begin{cases} \mathcal{U} = \mathrm{Set}(\mathcal{Z}) \\ U(z) = e^z \\ U_n = 1 \end{cases}$$

EXAMPLE 1 $\mathcal{P}erm = \mathrm{Set}(\mathrm{Cycle}(\mathcal{Z}))$

↑ atom (labelled)

$$P(z) = \exp\left(\log \frac{1}{1-z}\right) = \frac{1}{1-z} \qquad \boxed{P_n = n!}$$

Derangements (no fixed point)

$$\mathcal{D} = \mathrm{Set}(\mathrm{Cycle}(\mathcal{Z}) \setminus \mathcal{Z})$$

$$D(z) = \exp\left(\log \frac{1}{1-z} - z\right)$$

$$= \frac{e^{-z}}{1-z} \qquad \frac{D_n}{n!} = 1 - \frac{1}{1!} + \frac{1}{2!} \cdots + \frac{(-1)^n}{n!}$$

$$\sim e^{-1}$$

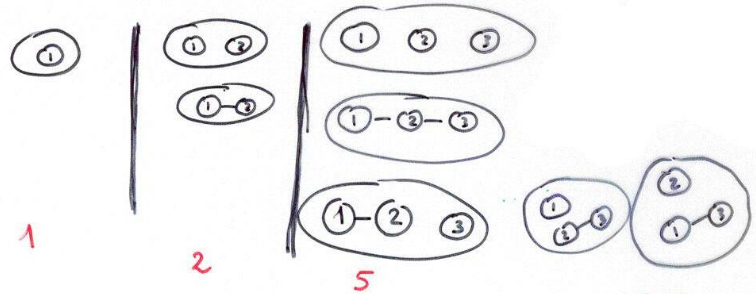
EXAMPLE 2 Labelled trees

$$\mathcal{T} = \mathcal{Z} \star \mathrm{Set}(\mathcal{T})$$

$$T(z) = z e^{T(z)} \qquad \boxed{T_n = n^{n-1}}$$

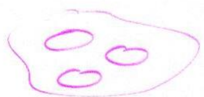
(via Lagrange inversion formula)

■ - Set Partitions



$$B_n = n! \, [z^n] \, e^{e^z - 1} \qquad \mathcal{B} = \mathrm{Set}(\mathrm{Set}_{\geq 1}(\mathcal{Z}))$$

$$= e^{-1} \sum_{k=0}^{\infty} \frac{k^n}{k!}$$



■ - Allocations

$\mathcal{F}_n^{\langle m \rangle}: [1..n] \longrightarrow [1..m]$ ← same as words over m-ary alphabet

all: $\mathcal{U}_{(1)} \star \mathcal{U}_{(2)} \star \cdots \star \mathcal{U}_{(m)}$

$$\boxed{e^{mz}} \Longrightarrow m^n$$

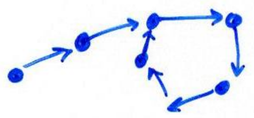
surjective $\boxed{(e^z - 1)^m} \Longrightarrow$ Stirling numbers

$$m! \left\{ {n \atop m} \right\} = \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} k^n$$

injective $\boxed{(1+z)^m} \Longrightarrow \binom{m}{n} \cdot n!$ (arrangements)

Ex: doubly surjective; max occupancy...

EXAMPLE 3 Mappings $= \text{fns } [1..n] \to [1..n]$



$$\begin{cases} \mathcal{M} = \text{Set}(\mathcal{K}) \\ \mathcal{K} = \text{Cycle}(\mathcal{T}) \\ \mathcal{T} = \mathcal{Z} \star \text{Set}(\mathcal{T}) \end{cases}$$

$\boxed{M_n = n^n}$ $\quad K(z) = \log \frac{1}{1-T(z)} \Rightarrow$ proba. connectedness $= O\left(\frac{1}{\sqrt{n}}\right)$

Exercise: A binary functional graph is the graph of a finite function [mapping] such that each value x has either 0 or 2 inverse images (a quadratic polynomial over GF_p).

Q: Find a description of the graphs and a construction

Compute EGF.

Determine number of such beasts!

Exercise: Find the EGF of the labelled connected graphs, call it $K(z)$.

Hint: All graphs: $G(z) = 1 + \sum_{n=1}^{\infty} 2^{\binom{n}{2}} \frac{z^n}{n!}$

Relate $K(z)$ and $G(z)$.

CHAPTER 3

Combinatorial Parameters and Multivariate generating Functions

— RUNS in PERMUTATIONS —

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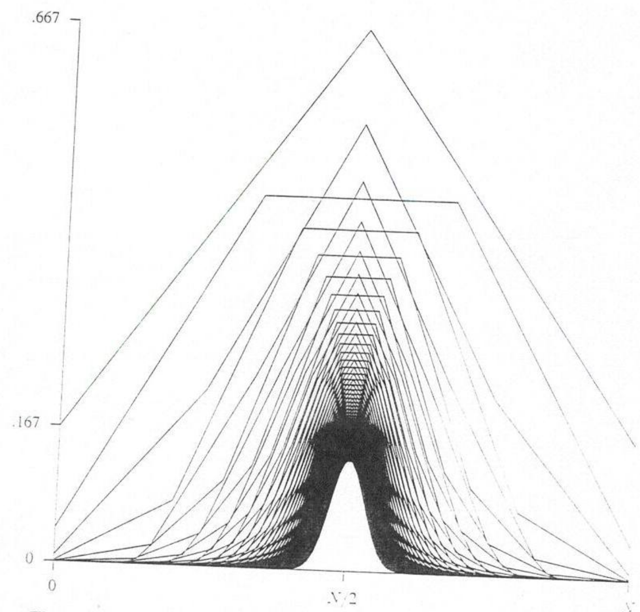
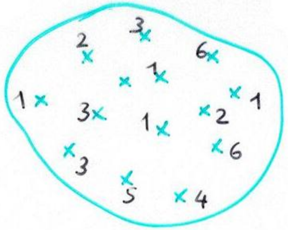
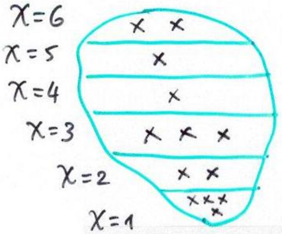


Figure 6.5 Distribution of runs. $3 \le N \le 60$ (k-axes scaled to N) (Eulerian numbers)

Discrete Probabilistic Model





E_N with χ

(eg. Trees_N with path length)

$\{E_{N,k}\}$

$\mathcal{C}$ a class; $C(z,u)$ a bivariate generating function corresp. to parameter χ

Proposition (A). $C(z,1) \equiv C(z)$ the usual counting GF

(B) $[z^n] \frac{\partial}{\partial u} C(z,u)\Big|_{u=1} = \sum_{\gamma \in \mathcal{C}_n} \chi[\gamma]$ $\left(= \sum_k k\, C_{n,k}\right)$

$$\Rightarrow \mathbb{E}_{\mathcal{C}_n}(\chi) = \frac{\text{coeff}[z^n]\, C'_u(z,1)}{\text{coeff}[z^n]\, C(z,1)}.$$

(C) Higher moments correspond to higher derivatives w.r.t. u @ $u=1$.

access to CONCENTRATION

→ Enumeration / Counting

$$\Pr\{\chi = k\} = \frac{E_{N,k}}{E_N},$$

$$E[\chi] = \sum k \cdot \frac{E_{N,k}}{E_N},$$

+ variance, etc.

$\mathcal{C}$: class of comb. structures;

$$\begin{cases} C_n : \text{ \# objects of size } n \\ C_{n,k} : \text{ ———, param.} = k \end{cases}$$

$$\text{(counting)} \begin{cases} C(z) := \sum C_n z^n \\ \widehat{C}(z) := \sum C_n \frac{z^n}{n!} \end{cases}$$

$$\text{(params)} \begin{cases} C(z,u) := \sum C_{n,k} z^n u^k \\ \widehat{C}(z,u) := \sum C_{n,k} u^k \frac{z^n}{n!} \end{cases}$$

Ordinary GF's for unlabelled structures. Exponential GF's for labelled structures.

Also: combinatorial forms. e.g. for ordinary GF's

$$C(z,u) = \sum_{\gamma \in \mathcal{C}} u^{\chi[\gamma]} z^{|\gamma|}.$$

Example 2:

How many cycles does a random permutation of size $\underline{n}$ have? On average; in distribution?

$$\mathcal{P} = \mathrm{Set}(\blacktriangle\, \mathrm{Cycle}(\mathcal{Z}))$$

$$P(\underset{\uparrow\, ,u}{z}) = \exp\left(\underset{\uparrow\, u}{} \log \frac{1}{1-z} \right) \quad \Rightarrow P(z) = \frac{1}{1-z},\ P_n = n!$$

$$P(z,u) = (1-z)^{-u} = \sum_{n,k} \left[{n \atop k} \right] u^k \frac{z^n}{n!}$$

$$= \sum \frac{z^n}{n!} u(u+1)\cdots(u+n-1) \leadsto \text{Stirling's triangle}$$

mean is $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} \sim \log n$ etc...

Definition: A parameter is inherited if

(i). it "passes through" disjoint unions $\mathcal{C} = \mathcal{A} \cup \mathcal{B}$

(ii). it is additive over products $\mathcal{C} = \mathcal{A} \times \mathcal{B}$

(hence also sequences, sets, cycles).

For inherited parameters:

PSEUDO-THEOREM: The previous translation mechanisms still work verbatim!

DICTIONARIES

CONSTRUCTIONS $\longrightarrow$ Generating functions

Example 1: Number of summands in compositions (OPT)

$$\mathcal{C} = \mathrm{Seq}(\mathcal{N}); \quad \mathcal{N} = \mathcal{Z} \times \mathrm{Seq}(\mathcal{Z})$$

$$C(z) = \frac{1}{1 - N(z)}; \quad N(z) = \frac{z}{1-z}$$

$\rightsquigarrow \chi \equiv 1$ on an individual summand

$$C(z,u) = \ ? \quad ; \quad N(z,u) = \frac{zu}{1-z}$$

$\Downarrow$ Pseudo-theorem.

$$C(z,u) = \frac{1}{1 - \frac{zu}{1-z}} = \frac{1-z}{1-z(u+1)} \Longleftarrow$$

$$\Rightarrow C_{n,k} = \binom{n-1}{k-1}$$

Idea of proof: $\mathcal{C} = \mathcal{A} \times \mathcal{B}$ (OGF)

$$\gamma = (\alpha, \beta) \qquad \chi[\gamma] = \chi[\alpha] + \chi[\beta]$$

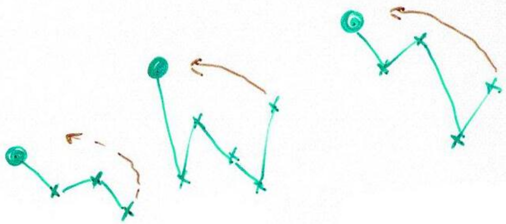
$$C(z,u) = \sum_{\gamma} z^{|\gamma|} u^{\chi[\gamma]}$$

$$= \sum_{(\alpha,\beta)} z^{|\alpha|} u^{\chi[\alpha]} z^{\beta} u^{\chi[\beta]} = A(z,u) \times B(z,u)$$

Exercise: A record in a permutation $\sigma = \sigma_1 \dots \sigma_n$ is an element σ_j which is larger than all preceding σ_k $(k<j)$ (OPT)

Q. Explain why the distribution of the number of records is the same as the distribution of the number of cycles (on $\mathcal{P}_n$).

Hint:



Example 3: Number of leaves on a general plane tree

$$\mathcal{G} = \mathcal{Z} \times \mathrm{Seq}(\mathcal{G}) = \mathcal{Z} + \mathcal{Z} \times \underbrace{\mathcal{G} \times \mathrm{Seq}(\mathcal{G})}_{\mathrm{Seq}_{\geq 1}(\mathcal{G})}$$

$$\Rightarrow G(z) = \frac{z}{1-G(z)} \Rightarrow G_n = \frac{1}{n}\binom{2n-2}{n-1}$$

$$G(z,u) = z\boxed{u} + z\,\frac{G(z,u)}{1-G(z,u)}$$

$\Rightarrow$ quadratic equation $\Rightarrow$ explicit

(also Lagrange).

Exercise: Throw n balls into m $\left\{\begin{smallmatrix}\text{cells}\\ \text{urns}\end{smallmatrix}\right\}$ at random. = allocation $\alpha \in \mathcal{A}_n^{[m]}$. (OPT)

Q1: Justify $A^{[m]}(z) = (e^z)^m \Rightarrow A_n^{[m]} = m^n$.

Q2: Statistics of empty cells? Justify

$$A^{[m]}(z,u) = (e^z - 1 + u)^m$$

and find mean number of empty cells.

Q3: How to generalize to cells containing r elements? Mean? Variance??