q-ENUMERATION OF UP-DOWN WORDS BY NUMBER OF RISES

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This paper is dedicated to Robert F. Tichy on the occasion of his 50^{th} anniversary.

Abstract. In an earlier paper, Carlitz [1] obtained generating functions for the number of up-down permutations counting the number of rises among the "peaks". In the present paper we study different types of random up-down words over the infinite alphabet $\{1, 2, \dots\}$, where the letters have geometric probabilities. We are interested in the probabilities of up-down words of given length to have a given number of rises on the lower level. To get explicit expressions for certain generating functions, it is not necessary to solve differential equations, as opposed to Cartlitz' treatment. Moreover, in the corresponding particular cases we regain the results found by Carlitz.

1. Introduction

Up-down permutations over $\{1, 2, \dots, n\}$ are permutations that satisfy the pattern $<><> \cdots <>$, for odd n, and $<><> \cdots <><$, for even n. Carlitz [1] obtained generating functions for the number of up-down permutations counting the number of rises among the "peaks". In fact, by the natural one-to-one correspondence between them, counting up-down permutations is actually equivalent to counting down-up permutations, i.e., permutations satisfying the pattern $><>< \dots$. Thus, solving the above counting problem for up-down permutations provides also the solution for the corresponding (symmetrical) problem in terms of down-up permutations.

In this paper we consider up-down random words over the infinite alphabet $\{1, 2, 3, \dots\}$, with the assumption that each letter j occurs with (geometric) probability pq^{j-1}, independently, for a $0 < q < 1$ and $p = 1 - q$. We also consider the cases when the inequalities are not strict. When enumerating up-down words we consider the number of rises on the lower level, where by a *rise on the lower level* in a up-down word we mean the occurrence of a (correspondingly) up-down subword abc with $a \le c$, while in the case of a *strict rise* we have $a < c$.

For simplicity, in the following by words we always mean up-down words.

Our aim is to compute the probability that a word of given length satisfies a given up-down pattern and has a given number k of rises on the lower level. Therefore we work with generating functions $F(z, x, u)$, where the coefficient of $z^m x^r u^i$ is the

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probability that an up-down word of length m with last letter i has r rises on the lower level.

Depending on the inequalities that define the up-down pattern and the occurrence of rises, we discuss four cases for words of odd length and then deduce the corresponding results for words of even length.

In the case of words of odd length we use the method of "adding a new slice", i.e., computing an extra up-down-double-step: assuming that a word has length $2n+1$, $n \geq 0$, last letter i and a given number of rises on the lower level, we concatenate it (on the right side) with an arbitrary word kj, and impose the condition that the new word, of length $2n+3$, satisfies the same up-down pattern, by also taking into account whether there has appeared a new rise on the lower level or not. For instance, the sum in (1) occurs, with the mentioned notations, for an up-down word with pattern $<\geq<\geq \cdots \geq$ of odd length.

Analogously as in the case of permutations, one can define down-up random words, but in this case finding the above mentioned probabilities for up-down words does not provide the solution for the corresponding down-up problem. In this paper we restrict ourselves to studying the up-down words, which are nicer to handle. In the limit case when q tends to 1, and hence, all letters tend to occur with the same probability, we regain the formulæ found by Carlitz [1] for the up-down permutations. In fact, in our approach we avoid differential equations, which occur at Carlitz [1], and obtain more general results.

There are also other possible cases corresponding to other ways of combinig these inequalities, which we omit, as they can be treated analogously to the cases studied here.

2. Words of odd length

In this section we consider up-down words of length $2n+1$, for $n \geq 0$. Throughout the paper we use, for $0 < q < 1$ and arbitrary x, the notation $(x)_n = (1-x)\dots(1-xq^{n-1})$.

2.1. **Case 1a. Words of odd length following the pattern $<\geq<\geq \cdots <\geq$ with strict rises on the lower level.** In order to find the probabilities of occurance of up-down words with odd length and given number of strict rises on the lower level, we proceed inductively as it follows.

Assume we have a random word of length $2n+1$ with last letter i, $\omega = \omega_1 \dots \omega_{2n} i$, which has r rises on the lower level, with arbitrarily fixed $i \geq 1$ and $r \geq 0$. Suppose we add two letters, k and j, after the last letter i, and thus get a new word of length $2n+3$, $\omega' = \omega_1 \dots \omega_{2n} ikj$. Then either $i < j$ and the new word ω' has $r+1$ rises, or $i \geq j$ and then ω' has r rises. This is reflected by the following expression

$$\sum_{k>i} pq^{k-1} \sum_{j \leq i} pq^{j-1} u^j + \sum_{k>i} pq^{k-1} \sum_{i<j\leq k} pq^{j-1} u^j \cdot x. \tag{1}$$

This leads to the substitution

$$u^i \longrightarrow \frac{pu}{(uq)_1} q^i + \frac{puT}{(uq)_2} \big(uq^2\big)^i, \tag{2}$$

with $T = T(x,u) = x - 1 - uq(x-q)$. We introduce the generating functions

$$f(z,x) := \sum_{n\geq 0,\ r\geq 1} p(n,r) z^{2n+1} x^r, \tag{3}$$

where $p(n,r)$ is the probability that an up-down word of length $2n+1$ has r strict rises on the lower level, and

$$F(z,x,u) := \sum_{n\geq 0,\ r\geq 0,\ i\geq 1} b(n,r,i)\ z^{2n+1} x^r u^i, \tag{4}$$

where $b(n,r,i)$ is the probability that an up-down word of length $2n+1$ with last letter i has r strict rises on the lower level. We have $f(z,x) = F(z,x,1)$. Our aim is to find the probabilities $p(n,r)$, for $n \geq 0$ and $r \geq 0$. Therefore we compute the generating function $f(z,x)$.

Let $a_n(x,u)$, $n \geq 0$, be the functions with the property

$$F(z,x,u) = \sum_{n\geq 0} a_n(x,u) z^{2n+1}. \tag{5}$$

For given $n \geq 0$, we obtain from (2), that

$$a_{n+1}(x,u) = \frac{pu}{(uq)_1} a_n(x,q) + \frac{puT}{(uq)_2} a_n(x,uq^2), \tag{6}$$

which leads to

$$F(z,x,u) = \frac{puz}{(uq)_1} + \frac{puz^2}{(uq)_1} F(z,x,q) + \frac{puT(x,u)z^2}{(uq)_2} F(z,x,uq^2). \tag{7}$$

This can now be solved by iteration. We have

$$\begin{aligned} F(z,x,u) = &\frac{puz}{(uq)_1} + \frac{puz^2}{(uq)_1} F(z,x,q) + \frac{puT(x,u)z^2}{(uq)_2}\Bigg[\\ &\frac{puq^2z}{(uq^3)_1} + \frac{puq^2z^2}{(uq^3)_1} F(z,x,q) + \frac{puq^2T(x,uq^2)z^2}{(uq^3)_2}\Bigg[\\ &\frac{puq^4z}{(uq^5)_1} + \frac{puq^4z^2}{(uq^5)_1} F(z,x,q) + \frac{puq^4T(x,uq^4)z^2}{(uq^5)_2}\Bigg[\\ &\frac{puq^6z}{(uq^7)_1} + \frac{puq^6z^2}{(uq^7)_1} F(z,x,q) + \frac{puq^6T(x,uq^6)z^2}{(uq^7)_2}\Bigg[\\ &\frac{puq^8z}{(uq^8)_1} + \cdots \Bigg]\Bigg]\Bigg]\Bigg], \end{aligned}$$

which yields

$$F(z,x,u) = \sum_{k\geq 0} \frac{(pu)^{k+1} z^{2k+1} q^{k(k+1)}}{(uq)_{2k+1}} \prod_{j=0}^{k-1} T(x,uq^{2j}) \Big[1 + zF(z,x,q)\Big].$$

For $u = q$ we get

$$F(z,x,q) = \frac{\sum_{k\geq 0} \frac{p^{k+1} z^{2k+1} q^{(k+1)^2}}{(q^2)_{2k+1}} \prod_{j=0}^{k-1} T(x,q^{2j+1})}{1 - \sum_{k\geq 0} \frac{p^{k+1} z^{2k+2} q^{(k+1)^2}}{(q^2)_{2k+1}} \prod_{j=0}^{k-1} T(x,q^{2j+1})},$$

and thus

$$F(z,x,u)=\sum_{k\ge0}\frac{(pu)^{k+1}z^{2k+1}q^{k(k+1)}}{(uq)_{2k+1}}\prod_{j=0}^{k-1}T(x,uq^{2j})\left[1+\frac{\sum_{k\ge0}\frac{p^{k+1}z^{2k+2}q^{(k+1)^2}}{(q^2)_{2k+1}}\prod_{j=0}^{k-1}T(x,q^{2j+1})}{1-\sum_{k\ge0}\frac{p^{k+1}z^{2k+2}q^{(k+1)^2}}{(q^2)_{2k+1}}\prod_{j=0}^{k-1}T(x,q^{2j+1})}\right].$$

By simple computations we obtain

$$\begin{aligned}F(z,&x,u)\\&=\sum_{k\ge0}\frac{(pu)^{k+1}z^{2k+1}q^{k(k+1)}}{(uq)_{2k+1}}\prod_{j=0}^{k-1}T(x,uq^{2j})\Bigg/\left[1-\sum_{k\ge0}\frac{p^{k+1}z^{2k+2}q^{(k+1)^2}}{(q^2)_{2k+1}}\prod_{j=0}^{k-1}T(x,q^{2j+1})\right]\\&=\sum_{k\ge0}\frac{(pu)^{k+1}z^{2k+1}q^{k(k+1)}}{(uq)_{2k+1}}\prod_{j=0}^{k-1}T(x,uq^{2j})\Bigg/\left[1-\sum_{k\ge1}\frac{p^{k}z^{2k}q^{k^2}}{(q^2)_{2k-1}}\prod_{j=1}^{k-1}T(x,q^{2j-1})\right].\end{aligned}$$

As $(q^2)_{2k-1}=(q)_{2k}/p$ and $T(x,1/q)=-p$, we have

$$F(z,x,u)=\frac{\sum_{k\ge0}\frac{(pu)^{k+1}z^{2k+1}q^{k(k+1)}}{(uq)_{2k+1}}\prod_{j=0}^{k-1}T(x,uq^{2j})}{\sum_{k\ge0}\frac{p^kz^{2k}q^{k^2}}{(q)_{2k}}\prod_{j=0}^{k-1}T(x,q^{2j-1})}. \tag{8}$$

By taking $u=1$ we obtain

$$f(z,x)=F(z,x,1)=\frac{\sum_{k\ge0}\frac{p^{k+1}z^{2k+1}q^{k(k+1)}}{(q)_{2k+1}}\prod_{j=0}^{k-1}T(x,q^{2j})}{\sum_{k\ge0}\frac{p^kz^{2k}q^{k^2}}{(q)_{2k}}\prod_{j=0}^{k-1}T(x,q^{2j-1})}. \tag{9}$$

Now we take the limit for $q\to1$:

$$\lim_{q\to1}T(x,q^k)/p=(k+1)x-(k+2),$$

and herefrom,

$$\lim_{q\to1}f(z,x)=\frac{\sum_{k\ge0}\frac{z^{2k+1}}{(2k+1)!}\prod_{j=0}^{k-1}[(2j+1)x-(2j+2)]}{\sum_{k\ge0}\frac{z^{2k}}{(2k)!}\prod_{j=0}^{k-1}[2jx-(2j+1)]}.$$

For $x=1$, we get

$$\lim_{q\to1}f(z,1)=\frac{\sum_{k\ge0}\frac{z^{2k+1}}{(2k+1)!}(-1)^k}{\sum_{k\ge0}\frac{z^{2k}}{(2k)!}(-1)^k}=\tan z,$$

which coincides with the already known results for up-down permutations of odd length (we refer, e.g., to Graham, Knuth and Patashnik [2], and Prodinger [3]), as it should.

2.2. **Case 2a. Words of odd length following the pattern $\le\ge\le\ge\cdots\le\ge$ with rises on the lower level.** As in the previous subsection, we consider a random word of length $2n+1$ with last letter i, which has r rises on the lower level, with arbitrarily fixed $i\ge1$ and $r\ge0$. We add two letters, k and j, after the last letter i and thus get a new word ω' of length $2n+3$. Then either $i\le j$ and the new word ω' has $r+1$ rises, or $i>j$ and then ω' has r rises. This is expressed by the sum

$$\sum_{k\ge i}pq^{k-1}\sum_{j<i}pq^{j-1}u^j+\sum_{k\ge i}pq^{k-1}\sum_{i\le j\le k}pq^{j-1}u^j\cdot x, \tag{10}$$

which leads to the substitution

$$u^i \to \frac{pu}{q(uq)_1} q^i + \frac{pT}{q^2(uq)_2}\big(uq^2\big)^i, \tag{11}$$

with $T = T(x,u) = x - 1 - uq(x-q)$.

Again we consider the generating functions

$$f(z,x) := \sum_{n\ge 0,\ r\ge 1} p(n,r) z^{2n+1} x^r,$$

where $p(n,r)$ is the probability that an up-down word of length $2n+1$ has r rises on the lower level, and

$$F(z,x,u) := \sum_{n\ge 0,\ r\ge 0,\ i\ge 1} b(n,r,i)\ z^{2n+1} x^r u^i,$$

where $b(n,r,i)$ is the probability that an up-down word of length $2n+1$ with last letter i has r rises on the lower level. Again $f(z,x) = F(z,x,1)$ and we compute $f(z,x)$ by first computing $F(z,x,u)$.

Let $a_n(x,u)$, $n \ge 0$, be the functions with the property

$$F(z,x,u) = \sum_{n\ge 0} a_n(x,u) z^{2n+1}.$$

From the relation (11) we obtain, for $n \ge 0$,

$$a_{n+1}(x,u) = \frac{pu}{q(uq)_1} a_n(x,q) + \frac{pT}{q^2(uq)_2} a_n(x,uq^2), \tag{12}$$

and thus

$$F(z,x,u) = \frac{puz}{(uq)_1} + \frac{puz^2}{q(uq)_1} F(z,x,q) + \frac{pT(x,u)z^2}{q^2(uq)_2} F(z,x,uq^2), \tag{13}$$

By iteration, we get

$$\begin{aligned} F(z,x,u) =& \frac{puz}{(uq)_1} + \frac{puz^2}{q(uq)_1} F(z,x,q) + \frac{pT(x,u)z^2}{q^2(uq)_2}\Bigg[\\ & \frac{puq^2z}{(uq^3)_1} + \frac{puq^2z^2}{q(uq^3)_1} F(z,x,q) + \frac{pT(x,uq^2)z^2}{q^2(uq^3)_2}\Bigg[\\ & \frac{puq^4z}{(uq^5)_1} + \frac{puq^4z^2}{q(uq^5)_1} F(z,x,q) + \frac{pT(x,uq^4)z^2}{q^2(uq^5)_2}\Bigg[\\ & \frac{puq^6z}{(uq^7)_1} + \frac{puq^6z^2}{q(uq^7)_1} F(z,x,q) + \frac{pT(x,uq^6)z^2}{q^2(uq^7)_2}\Bigg[\\ & \frac{puq^8z}{(uq^9)_1} + \cdots \Bigg]\Bigg]\Bigg]\Bigg], \end{aligned}$$

which yields

$$F(z,x,u) = pu \sum_{k\ge 0} \frac{p^k z^{2k+1}}{(uq)_{2k+1}} \prod_{j=0}^{k-1} T(x,uq^{2j}) \big(1 + \frac{z}{q} F(z,x,q)\big). \tag{14}$$

Hence for $u=q$ we have

$$F(z,x,q)=\frac{pq\sum_{k\ge0}\frac{p^kz^{2k+1}}{(q^2)_{2k+1}}\prod_{j=0}^{k-1}T(x,q^{2j+1})}{1-\sum_{k\ge0}\frac{p^{k+1}z^{2k+2}}{(q^2)_{2k+1}}\prod_{j=0}^{k-1}T(x,q^{2j+1})},$$

and thus

$$\begin{aligned}F(z,x,u)&=\frac{u\sum_{k\ge0}\frac{p^{k+1}z^{2k+1}}{(uq)_{2k+1}}\prod_{j=0}^{k-1}T(x,uq^{2j})}{1-\sum_{k\ge0}\frac{p^{k+1}z^{2k+2}}{(q^2)_{2k+1}}\prod_{j=0}^{k-1}T(x,q^{2j+1})}\\&=\frac{u\sum_{k\ge0}\frac{p^{k+1}z^{2k+1}}{(uq)_{2k+1}}\prod_{j=0}^{k-1}T(x,uq^{2j})}{1-\sum_{k\ge1}\frac{p^kz^{2k}}{(q^2)_{2k-1}}\prod_{j=1}^{k-1}T(x,q^{2j-1})}\\&=\frac{u\sum_{k\ge0}\frac{p^{k+1}z^{2k+1}}{(uq)_{2k+1}}\prod_{j=0}^{k-1}T(x,uq^{2j})}{\sum_{k\ge0}\frac{p^kz^{2k}}{(q)_{2k}}\prod_{j=0}^{k-1}T(x,q^{2j-1})}.\end{aligned}\tag{15}$$

For $u=1$,

$$f(z,x)=F(z,x,1)=\frac{\sum_{k\ge0}\frac{p^{k+1}z^{2k+1}}{(q)_{2k+1}}\prod_{j=0}^{k-1}T(x,q^{2j})}{\sum_{k\ge0}\frac{p^kz^{2k}}{(q)_{2k}}\prod_{j=0}^{k-1}T(x,q^{2j-1})}.\tag{16}$$

One can check that, as in the previous case, $\lim_{q\to1}f(z,1)=\tan z$, i.e., in the limit case we obtain the tangent numbers, as expected.

2.3. **Case 3a. Words of odd length following the pattern $\le>\le>\cdots\le>$ with rises on the lower level.** We consider a random word of length $2n+1$ with last letter i, which has r rises on the lower level, with arbitrarily fixed $i\ge1$ and $r\ge0$. We add two letters, k and j, after the last letter i and thus get a new word ω' of length $2n+3$. Then either $i\le j$ and the new word ω' has $r+1$ rises, or $i>j$ and then ω' has r rises. This is expressed by the sum

$$\sum_{k\ge i}pq^{k-1}\sum_{j<i}pq^{j-1}u^j+\sum_{k\ge i}pq^{k-1}\sum_{i\le j<k}pq^{j-1}u^j\cdot x,\tag{17}$$

which leads to the substitution

$$u^i\to\frac{pu}{q(uq)_1}q^i+\frac{pT}{q^2(uq)_2}\big(uq^2\big)^i,\tag{18}$$

with $T=T(x,u)=xq-1-uq^2(x-1)$.

Using the notations of the previous subsections, we obtain, proceeding analogously,

$$F(z,x,u)=\frac{pu}{(uq)_1}z+\frac{p}{q}\frac{uz^2}{(uq)_1}F(z,x,q)+\frac{p}{q^2}\cdot\frac{T(x,u)z^2}{(uq)_2}F(z,x,uq^2),\tag{19}$$

i.e., we get the same recursion as in the previous case, but with a different function $T(x,u)$.

Thus,

$$F(z,x,u)=\frac{u\sum_{k\ge0}\frac{p^{k+1}z^{2k+1}}{(uq)_{2k+1}}\prod_{j=0}^{k-1}T(x,uq^{2j})}{\sum_{k\ge0}\frac{p^kz^{2k}}{(q)_{2k}}\prod_{j=0}^{k-1}T(x,q^{2j-1})},\tag{20}$$

and herefrom, for $u = 1$,

$$f(z,x) = F(z,x,1) = \frac{\sum_{k\geq 0} \frac{p^{k+1}z^{2k+1}}{(q)_{2k+1}} \prod_{j=0}^{k-1} T(x,q^{2j})}{\sum_{k\geq 0} \frac{p^k z^{2k}}{(q)_{2k}} \prod_{j=0}^{k-1} T(x,q^{2j-1})}. \tag{21}$$

On the other hand, $\lim_{q\to 1} T(x,q^k)/p = (k+1)x - (k+2)$, which implies

$$\lim_{q\to 1} f(z,x) = \frac{\sum_{k\geq 0} \frac{z^{2k+1}}{(2k+1)!} \prod_{j=0}^{k-1}[(2j+1)x - (2j+2)]}{\sum_{k\geq 0} \frac{z^{2k}}{(2k)!} \prod_{j=0}^{k-1}[2jx - (2j+1)]},$$

and, for $x = 1$,

$$\lim_{q\to 1} f(z,1) = \frac{\sum_{k\geq 0} \frac{z^{2k+1}}{(2k+1)!}(-1)^k}{\sum_{k\geq 0} \frac{z^{2k}}{(2k)!}(-1)^k} = \tan z.$$

2.4. **Case 4a. Words of odd length following the pattern $<><> \cdots <>$ with strict rises on the lower level.** We proceed as in the previous cases. By adding two letters, k and j, after the last letter i of a random word of length $2n+1$ with r strict rises on the lower level, we get a new word ω' of length $2n+3$. Then either $i < j$ and the new word ω' has $r+1$ scrict rises, or $i \geq j$ and then ω' has r strict rises. This is expressed by the sum

$$\sum_{k>i} pq^{k-1} \sum_{j\leq i} pq^{j-1}u^j + \sum_{k>i} pq^{k-1} \sum_{i<j<k} pq^{j-1}u^j \cdot x. \tag{22}$$

The corresponding substitution is

$$u^i \longrightarrow \frac{pu}{(uq)_1} q^i + \frac{puT}{(uq)_2} \big(uq^2\big)^i, \tag{23}$$

with $T = T(x,u) = xq - 1 - uq^2(x-1)$.

Proceeding as in the previous cases, we get

$$F(z,x,u) = \frac{puz}{(uq)_1} + \frac{puz^2}{(uq)_1} F(z,x,q) + \frac{puT(x,u)z^2}{(uq)_2} F(z,x,uq^2). \tag{24}$$

The recursion is the same as in Case 1a, with exception of the expression of $T(x,u)$ which is the same as in Case 3a. Thus,

$$F(z,x,u) = \frac{\sum_{k\geq 0} \frac{(pu)^{k+1}z^{2k+1}q^{k(k+1)}}{(uq)_{2k+1}} \prod_{j=0}^{k-1} T(x,uq^{2j})}{\sum_{k\geq 0} \frac{p^k z^{2k} q^{k^2}}{(q)_{2k}} \prod_{j=0}^{k-1} T(x,q^{2j-1})}, \tag{25}$$

with $T = T(x,u) = xq - 1 - uq^2(x-1)$, and hence, for $u = 1$,

$$f(z,x) = F(z,x,1) = \frac{\sum_{k\geq 0} \frac{p^{k+1}z^{2k+1}q^{k(k+1)}}{(q)_{2k+1}} \prod_{j=0}^{k-1} T(x,q^{2j})}{\sum_{k\geq 0} \frac{p^k z^{2k} q^{k^2}}{(q)_{2k}} \prod_{j=0}^{k-1} T(x,q^{2j-1})}. \tag{26}$$

We have, as in the previous case, $\lim_{q\to 1} f(z,1) = \tan z$.

3. Words of even length

Throughout this section we consider up-down words of length $2n$, $n \geq 1$, where *up* means, as in Section 2, either "$<$" or "$\leq$", and *down* means either "$>$" or "$\geq$", as it will be specified in each case.

It is convenient to approach each word of even length $2n$, with $n \geq 1$, as a word of odd length $2n-1$ concatenated with a letter, i.e. if the word of length $2n-1$ has, e.g., last letter j, then the last two letters of the word of length $2n$ are jl, where l is some integer.

Let αjl be a up-down word with length $2n$, $n \geq 1$, where the prefix α is either the empty word or a word of strictly positive even length $2n-2$, whose last two letters we denote by ik.

Then, in both situations, αj is an up-down word of odd length and the only condition we have to impose in order that αjl is also an up-down order is that the subword jl is "up", i.e., depending on the case, $j < l$ or $j \leq l$.

3.1. Case 1b. Words of even length following the pattern $<\geq<\geq \cdots <\geq<$ with strict rises on the lower level.

We start by concatenating the letter l with the up-down word of odd length αj. The role of the sum (1) occurring in the corresponding odd length case is played in the even length case by the expression

$$\left(\sum_{k>i} pq^{k-1} \sum_{j\leq i} pq^{j-1}u^j + \sum_{k>i} pq^{k-1} \sum_{i<j\leq k} pq^{j-1}u^j \cdot x\right) \sum_{l>j} pq^{l-1}z. \tag{27}$$

Here we use the generating function

$$g(z,x) := \sum_{n\geq 0,\ r\geq 0} p(n,r)z^{2n}x^r, \tag{28}$$

where $p(n,r)$ is the probability that an up-down word of length $2n$ has r strict rises on the lower level, where $n \geq 1$ and $r \geq 0$. Again, the idea is to find $g(z,x)$ and to recuperate herefrom the coefficients $p(n,r)$. For $n=1$ the corresponding term in the above sum is one.

As $\sum_{l>j} pq^{l-1}z = q^j z$, one can obtain the expression (27) by replacing u^j in the sum (1) by $z(uq)^j$.

Consequently, for $u=1$ the role played by $F(z,x,1)$ in (9) it taken here by $zF(z,x,q)$. For the generating function $g(z,x)$ we have in this case

$$\begin{aligned} g(z,x) =& 1 + zF(z,x,q) = 1 + \frac{\sum_{k\geq 0} \frac{p^{k+1}q^{(k+1)^2}z^{2k+2}}{(q^2)_{2k+1}} \prod_{j=0}^{k-1} T(x,q^{2j+1})}{1 - \sum_{k\geq 0} \frac{p^{k+1}q^{(k+1)^2}z^{2k+2}}{(q^2)_{2k+1}} \prod_{j=0}^{k-1} T(x,q^{2j+1})} \\ =& \frac{1}{\sum_{k\geq 0} \frac{p^k q^{k^2} z^{2k}}{(q)_{2k}} \prod_{j=0}^{k-1} T(x,q^{2j-1})}, \end{aligned} \tag{29}$$

with $T = T(x,u) = x - 1 - uq(x-q)$. At the limit we obtain

$$\lim_{q\to 1} \bigl(1 + zF(z,x,q)\bigr) = \frac{1}{\sum_{k\geq 0} \frac{z^{2k}}{(2k)!} \prod_{j=0}^{k-1} [2jx - (2j+1)]}, \tag{30}$$

which leads us to

$$\lim_{q\to 1} g(z,1) = \frac{1}{\sum_{k\geq 0} \frac{z^{2k}}{(2k)!}(-1)^k} = \sec z.$$

3.2. **Case 2b. Words of even length following the pattern $\leq\geq\leq\geq \cdots \leq\geq\leq$ with rises on the lower level.** Again, the starting point is the concatenation of the letter l with an up-down word of odd length αj. Instead of the sum occurring in (10) in the odd length case, here we have

$$\Big(\sum_{k\geq i} pq^{k-1} \sum_{j<i} pq^{j-1}u^j + \sum_{k\geq i} pq^{k-1} \sum_{i\leq j\leq k} pq^{j-1}u^j \cdot x\Big) \sum_{l\geq j} pq^{l-1}z. \tag{31}$$

As $\sum_{l\geq j} pq^{l-1}z = q^{j-1}z$, i.e., the occurrence of u^j in the expression (10) of the corresponding odd length case is replaced by the occurrence of $\frac{z}{q}(uq)^j$ in the even length case. Consequently, for $u = 1$ we replace $F(z, x, 1)$ in (16) by $\frac{z}{q}F(z, x, q)$.

We obtain the generating function

$$\begin{aligned} g(z,x) =& 1 + \frac{z}{q}F(z,x,q) = 1 + \frac{p\sum_{k\geq 0} \frac{p^k z^{2k+2}}{(q^2)_{2k+1}} \prod_{j=0}^{k-1} T(x, q^{2j+1})}{1 - \sum_{k\geq 0} \frac{p^{k+1} z^{2k+2}}{(q^2)_{2k+1}} \prod_{j=0}^{k-1} T(x, q^{2j+1})} \\ =& \frac{1}{\sum_{k\geq 0} \frac{p^k z^{2k}}{(q)_{2k}} \prod_{j=0}^{k-1} T(x, q^{2j-1})}, \end{aligned} \tag{32}$$

where $T = T(x, u) = x - 1 - uq(x - q)$.

We have $\lim_{q\to 1} g(z, 1) = \frac{1}{\sum_{k\geq 0} \frac{z^{2k}}{(2k)!}(-1)^k} = \sec z$.

3.3. **Case 3b. Words of even length following the pattern $\leq>\leq> \cdots \leq>\leq$ with rises on the lower level.** Again, the starting point is the concatenation of the letter l with an up-down word of odd length αj. Instead of the sum occurring in (17) we have, in the even length case,

$$\Big(\sum_{k\geq i} pq^{k-1} \sum_{j<i} pq^{j-1}u^j + \sum_{k\geq i} pq^{k-1} \sum_{i\leq j<k} pq^{j-1}u^j \cdot x\Big) \sum_{l\geq j} pq^{l-1}z. \tag{33}$$

Here the occurrence of u^j in the odd length case is replaced by the occurrence of $\frac{z}{q}(uq)^j$ in the corresponding even length case. Consequently, for $u = 1$ we replace $F(z, x, 1)$ in (21) by $\frac{z}{q}F(z, x, q)$.

We obtain the generating function

$$g(z,x) = 1 + \frac{z}{q}F(z,x,q) = \frac{1}{\sum_{k\geq 0} \frac{p^k z^{2k}}{(q)_{2k}} \prod_{j=0}^{k-1} T(x, q^{2j-1})}, \tag{34}$$

where $T = T(x, u) = xq - 1 - uq^2(x - 1)$.

Again, $\lim_{q\to 1} g(z, 1) = \sec z$ holds.

3.4. **Case 4b. Words of even length following the pattern $<><>\cdots<>$ with strict rises on the lower level.** The concatenation of a letter l to a up-down word of odd length with last letter j is reflected by the expression

$$\Bigg(\sum_{k>i} pq^{k-1} \sum_{j\le i} pq^{j-1}u^j + \sum_{k>i} pq^{k-1} \sum_{i<j<k} pq^{j-1}u^j \cdot x\Bigg) \sum_{l>j} pq^{l-1} z. \tag{35}$$

As in Case 1b, the occurrence of u^j in the odd length case is replaced here by the occurrence of $z(uq)^j$, i.e., we replace $F(z,x,1)$ by $zF(z,x,q)$. We get the generating function

$$g(z,x) = 1 + zF(z,x,q) = \frac{1}{\sum_{k\ge 0} \frac{p^k q^{k^2} z^{2k}}{(q)_{2k}} \prod_{j=0}^{k-1} T(x, q^{2j-1})}, \tag{36}$$

where $T = T(x,u) = xq - 1 - uq^2(x-1)$, and, as in the previous three cases, $\lim_{q\to 1} g(z,1) = \sec z$.

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