PROBABILISTIC ANALYSIS OF AN EXHAUSTIVE SEARCH ALGORITHM IN RANDOM GRAPHS

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(joint work with Hsien-Kuei Hwang, Cyril Banderier and Vlady Ravelomanana)

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MAXIMUM INDEPENDENT SET

Independent set

An independent (or stable) set in a graph is a set of vertices no two of which share the same edge.

MIS $= \{1, 3, 5, 7\}$

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NP hard!!

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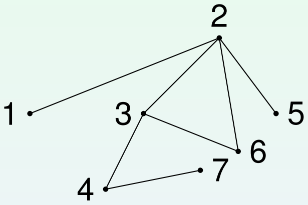

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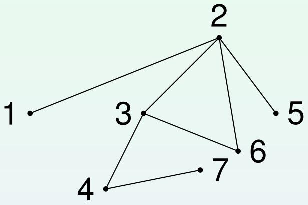

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Equivalent version

The same problem as MAXIMUM CLIQUE on the complementary graph (clique $=$ complete subgraph).

THEORETICAL RESULTS

Random models: Erdős-Rényi's $G_{n,p}$

Vertex set $= \{1, 2, \ldots, n\}$ **and all edges occur independently with the same probability** p**.**

The cardinality of an MIS in $G_{n,p}$

Matula (1970), Grimmett and McDiarmid (1975), Bollobas and Erdős (1976), Frieze (1990): If $pn \to \infty$**, then (**$q := 1 - p$**)**

$$|\mathbf{MIS}_n| \sim 2\log_{1/q} pn \quad \textbf{whp},$$

where $q = 1 - p$**.**

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DESCRIPTION OF THE GREEDY ALGORITHM

Problem

Find such an independent set A in G that no other node from G can be added to A without destroying the independence of A.

Solution

Initially $A = \emptyset$.

- **Chose $v \in G$.**
- **$A := A \cup \{v\}$, $G := G * v$, where $G * v$ is the graph obtained from G by deleting node v together with all its neighboring nodes and their edges.**
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A GREEDY ALGORITHM

Recurrence

The size of the resulting independent set S_n satisfies recurrence relation:

$$S_n \stackrel{d}{=} 1 + S_{n-1-\textbf{Binom}(n-1;p)} \qquad (n \geqslant 1),$$

with $S_0 \equiv 0$.

ANALYSIS OF THE GREEDY ALGORITHM

Relatively easy

- **Mean:** $\mathbb{E}(S_n) \sim \log_{1/q} n$ + **a bounded periodic function.**
- **Variance:** $\mathbb{V}(S_n) \sim$ **a bounded periodic function.**
- **Limit distribution does not exist:** $\mathbb{E}\left(e^{(X_n - \log_{1/q} n)y}\right) \sim F(\log_{1/q} n; y)$, **where**

$$F(u;y) := \frac{1-e^y}{\log(1/q)} \left(\prod_{\ell \geqslant 1} \frac{1-e^y q^\ell}{1-q^\ell}\right) \sum_{j \in \mathbb{Z}} \Gamma\left(-\frac{y+2j\pi i}{\log(1/q)}\right) e^{2j\pi i u}.$$

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A BETTER ALGORITHM?

Goodness of GREEDY IS

Grimmett and McDiarmid (1975), Karp (1976), Fernandez de la Vega (1984), Gazmuri (1984), McDiarmid (1984):

Asymptotically, the GREEDY IS is half optimal.

Can we do better?

Frieze and McDiarmid (1997, *RSA*), Algorithmic theory of random graphs, Research Problem 15:

Construct a polynomial time algorithm that finds an independent set of size at least $(\frac{1}{2}+\varepsilon)|MIS_n|$ ***whp or show that such an algorithm does not exist modulo some reasonable conjecture in the theory of computational complexity such as, e.g.,*** $P \neq NP$.

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POSITIVE RESULTS

Exact algorithms

A huge number of algorithms proposed in the literature; see Bomze et al.'s survey (in *Handbook of Combinatorial Optimization*, 1999).

Special algorithms

- **Chvátal (1977) proposes *exhaustive* algorithms where almost all $G_{n,1/2}$ creates at most $n^{2(1+\log_2 n)}$ subproblems.**
- **Pittel (1982):**

$$\mathbb{P}\left(n^{\frac{1-\varepsilon}{4}\log_{1/q} n} \leqslant \text{Time}^{\text{used by}}_{\text{Chvátal's algo}} \leqslant n^{\frac{1+\varepsilon}{2}\log_{1/q} n}\right) \geqslant 1 - e^{-c\log^2 n}$$

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Problem

Suppose we want to compute the stability number of the graph G, that is $|MIS(G)|$.

- **Chose a node $v \in G$.**
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- **Delete from G the node v together with all its neighboring nodes and their edges. The obtained graph denote by $G * v$. Compute $|MIS(G * v)|$.**
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The time needed to complete the algorithm X_n is a random variable satisfying recurrence relation

$$X_n \stackrel{d}{=} X_{n-1} + X^*_{n-1-\textbf{Binom}(n-1;p)} \qquad (n \geqslant 2),$$

with $X_0 = 0$ **and** $X_1 = 1$**.**

Special cases

- **If** p **is close to 1, then the second term is small, so we expect a** ***polynomial*** **time bound.**
- If p is sufficiently small, then the second term is large, and we expect an *exponential* time bound.
- What happens for p in between?

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The expected value $\mu_n := \mathbb{E}(X_n)$ satisfies

$$\mu_n = \mu_{n-1} + \sum_{0 \leqslant j < n} \binom{n-1}{j} p^j q^{n-1-j} \mu_{n-1-j}.$$

with $\mu_0 = 0$ **and** $\mu_1 = 1$**.**

Poisson generating function

Let $\tilde{f}(z) := e^{-z} \sum_{n \geqslant 0} \mu_n z^n / n!$**. Then**

$$\tilde{f}'(z) = \tilde{f}(qz) + e^{-z}.$$

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RESOLUTION OF THE RECURRENCE

Laplace transform

The Laplace transform of $\tilde{f}$

$$\mathscr{L}(s) = \int_0^{\infty} e^{-xs}\tilde{f}(x)\,\mathbf{d}x$$

satisfies

$$s\mathscr{L}(s) = \frac{1}{q}\mathscr{L}\left(\frac{s}{q}\right) + \frac{1}{s+1}.$$

Exact solutions

$$\mathscr{L}(s) = \sum_{j\geqslant 0} \frac{q^{\binom{j+1}{2}}}{s^{j+1}(s+q^j)}.$$

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Inverting gives $\tilde{f}(z) = \sum_{j \geqslant 0} \frac{q^{\binom{j+1}{2}}}{j!} z^{j+1} \int_0^1 e^{-q^j uz}(1-u)^j \, \mathrm{d}u$**.**

Thus $\mu_n = \sum_{1 \leqslant j \leqslant n} \binom{n}{j} (-1)^j \sum_{1 \leqslant \ell \leqslant j} (-1)^\ell q^{j(\ell-1)-\binom{\ell}{2}}$**, or**

$$\mu_n = n \sum_{0 \leqslant j < n} \binom{n-1}{j} q^{\binom{j+1}{2}} \sum_{0 \leqslant \ell < n-j} \binom{n-1-j}{\ell} \frac{q^{j\ell}(1-q^j)^{n-1-j-\ell}}{j+\ell+1}.$$

Neither is useful for numerical purposes for large n***.***

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ASYMPTOTICS OF μ_n

Poisson heuristic (de-Poissonization, saddle-point method)

$$\begin{aligned}\mu_n &= \frac{n!}{2\pi i}\oint_{|z|=n} z^{-n-1}e^z\tilde{f}(z)\,\mathbf{d}z\\ &\approx \sum_{j\geqslant 0}\frac{\tilde{f}^{(j)}(n)}{j!}\frac{n!}{2\pi i}\oint_{|z|=n} z^{-n-1}e^z(z-n)^j\,\mathbf{d}z\\ &= \tilde{f}(n)+\sum_{j\geqslant 2}\frac{\tilde{f}^{(j)}(n)}{j!}\,\tau_j(n),\end{aligned}$$

where $\tau_j(n) := n![z^n]e^z(z-n)^j = j![z^j](1+z)^n e^{-nz}$ **(Charlier polynomials). In particular,** $\tau_0(n) = 1$, $\tau_1(n) = 0$, $\tau_2(n) = -n$, $\tau_3(n) = 2n$, **and** $\tau_4(n) = 3n^2 - 6n$.

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A MORE PRECISE EXPANSION FOR $\tilde{f}(x)$

Asymptotics of $\tilde{f}(x)$

Let $\rho = 1/\log(1/q)$ and $R\log R = x/\rho$. Then

$$\tilde{f}(x) \sim \frac{R^{\rho+1/2} e^{(\rho/2)(\log R)^2} G(\rho \log R)}{\sqrt{2\pi\rho\log R}} \left(1 + \sum_{j\geqslant 1} \frac{\phi_j(\rho\log R)}{(\rho\log R)^j}\right),$$

as $x \to \infty$, where $G(u) := q^{(\{u\}^2+\{u\})/2} F(q^{-\{u\}})$,

$$F(s) = \sum_{-\infty<j<\infty} \frac{q^{j(j+1)/2}}{1+q^j s} s^{j+1},$$

and the $\phi_j(u)$'s are bounded, 1-periodic functions of u involving the derivatives $F^{(j)}(q^{-\{u\}})$.

A MORE EXPLICIT ASYMPTOTIC APPROXIMATION

$R = x/\rho/W(x/\rho)$, Lambert's W-function

$$W(x) = \log x - \log\log x + \frac{\log\log x}{\log x} + \frac{(\log\log x)^2 - 2\log\log x}{2(\log x)^2} + \cdots .$$

So that

$$\tilde{f}(x) \sim \frac{x^{\rho+1/2} G\left(\rho \log \frac{x/\rho}{\log(x/\rho)}\right)}{\sqrt{2\pi}\rho^{\rho+1/2}\log x} \exp\left(\frac{\rho}{2}\left(\log \frac{x/\rho}{\log(x/\rho)}\right)^2\right).$$

Method of proof: a variant of the saddle-point method

$$\tilde{f}(x) = \frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty} e^{xs}\mathscr{L}(s)\,ds$$

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JUSTIFICATION OF THE POISSON HEURISTIC

Four properties are sufficient

The following four properties are enough to justify the Poisson-Charlier expansion.

- $\tilde{f}'(z) = \tilde{f}(qz) + e^{-z}$;
- $F(s) = sF(qs)$ ($F(s) = \sum_{i\in\mathbb{Z}} q^{j(j+1)/2} s^{j+1}/(1+q^j s)$);
- $\dfrac{\tilde{f}^{(j)}(x)}{\tilde{f}(x)} \sim \left(\dfrac{\rho \log x}{x}\right)^j$;
- $|f(z)| \leqslant f(|z|)$ **where** $f(z) := e^z \tilde{f}(z)$.

Thus ($\rho = 1/\log(1/q)$)

$$\mu_n \sim \frac{n^{\rho+1/2} G\left(\rho \log \frac{n/\rho}{\log(n/\rho)}\right)}{\sqrt{2\pi}\rho^{\rho+1/2}\log n} \exp\left(\frac{\rho}{2}\left(\log\frac{n/\rho}{\log(n/\rho)}\right)^2\right).$$

JUSTIFICATION OF THE POISSON HEURISTIC

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Rough estimates

Corollary

Thus we have

$$\mathbb{E}X_n \asymp n^{\rho+1/2} \exp\left(\frac{\rho}{2}\left(\log\frac{n/\rho}{\log(n/\rho)}\right)^2\right),$$

where

$$\rho = \rho(p) = \frac{1}{\log\frac{1}{1-p}}.$$

Compare with the result of Pittel (1982)

$$\mathbb{P}\left(n^{\frac{1-\varepsilon}{4}\log_{1/q} n} \leqslant \textbf{Time}^{\textbf{used by}}_{\textbf{Chvátal's algo}} \leqslant n^{\frac{1+\varepsilon}{2}\log_{1/q} n}\right) \geqslant 1 - e^{-c\log^2 n}$$

Numerical example

n=300

If we take $n = 300$ **then according to our result for** $p = 0.4$

$$\mathbb{E}X_n \approx 1.12 \cdot 10^{11}$$

while for $p = 0.6$

$$\mathbb{E}X_n \approx 3.38 \cdot 10^{7}$$

This means that our algorithm for $p = 0.6$ **runs almost** 3300 **times faster than for** $p = 0.4$**.**

Idealized model

Dependence of X_n

Unfortunately X_n in the recurrence

$$X_n \stackrel{d}{=} X_{n-1} + X^*_{n-1-\mathbf{Binom}(n-1;p)} \qquad (n \geqslant 2),$$

with $X_0 = 0$ and $X_1 = 1$, are *not* independent!

Idealized model

What will happen if we assume that X_n *are* independent?

VARIANCE OF X_n under the assumption of independence

$$\sigma_n := \sqrt{\mathbb{V}(X_n)}$$

$$\sigma_n^2 = \sigma_{n-1}^2 + \sum_{0\leqslant j<n} \pi_{n,j}\sigma_{n-1-j}^2 + T_n, \quad \pi_{n,j} := \binom{n-1}{j} p^j q^{n-1-j},$$

where $T_n := \sum_{0\leqslant j<n} \pi_{n,j}\Delta_{n,j}^2$**,** $\Delta_{n,j} := \mu_j + \mu_{n-1} - \mu_n$**.**

Asymptotic transfer: $a_n = a_{n-1} + \sum_{0\leqslant j<n} \pi_{n,j} a_{n-1-j} + b_n$

If $b_n \sim n^\beta (\log n)^\kappa \tilde{f}(n)^\alpha$**, where** $\alpha > 1$**,** $\beta, \kappa \in \mathbb{R}$**, then**

$$a_n \sim \sum_{j\leqslant n} b_j \sim \frac{n}{\alpha\rho \log n} b_n.$$

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ASYMPTOTICS OF THE VARIANCE

Asymptotics of T_n: by elementary means

$$T_n \sim q^{-1} p \rho^4 n^{-3} (\log n)^4 \tilde{f}(n)^2.$$

Applying the asymptotic transfer

$$\sigma_n^2 \sim C n^{-2} (\log n)^3 \tilde{f}(n)^2.$$

where $C := p\rho^3/(2q)$**.**

$$\frac{\textbf{Variance}}{\textbf{Mean}^2} \sim C \frac{(\log n)^3}{n^2}$$

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ASYMPTOTIC NORMALITY OF X_n

Convergence in distribution

The distribution of X_n is asymptotically normal

$$\frac{X_n - \mu_n}{\sigma_n} \xrightarrow{d} \mathscr{N}(0,1),$$

with convergence of all moments.

Proof by the method of moments

- **Derive recurrence for $\mathbb{E}(X_n - \mu_n)^m$.**
- **Prove by induction (using the asymptotic transfer) that**

$$\mathbb{E}(X_n - \mu_n)^m \begin{cases} \sim \dfrac{(m)!}{(m/2)!2^{m/2}}\,\sigma_n^m, & \textbf{if } 2 \mid m, \\ = o(\sigma_n^m), & \textbf{if } 2 \nmid m, \end{cases}$$

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A STRAIGHTFORWARD EXTENSION

$b = 1, 2, \ldots$

$$X_n \stackrel{d}{=} X_{n-b} + X^*_{n-b-\mathbf{Binom}(n-b;p)},$$

with $X_n = 0$ **for** $n < b$ **and** $X_b = 1$**.**

A NATURAL VARIANT

What happens if $X_n \stackrel{d}{=} X_{n-1} + X^*_{\text{uniform[0,n-1]}}$?

$$\mu_n = \mu_{n-1} + \frac{1}{n} \sum_{0 \leqslant j < n} \mu_j,$$

satisfies $\mu_n \sim cn^{-1/4} e^{2\sqrt{n}}$**.**

Limit law not Gaussian (by method of moments)

$$\frac{X_n}{\mu_n} \xrightarrow{d} X,$$

where $g(z) := \sum_{m \geqslant 1} \mathbb{E}(X^m) z^m / (m \cdot m!)$ **satisfies**

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CONCLUSION

Random graph algorithms:
a rich source of interesting recurrences