A Removal Lemma for linear equations

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Structure

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Statement: Removal Lemma for Groups

Theorem (Removal Lemma for Groups, Green 2005)

Let **G** *be a finite Abelian group of order* N*.*
Let $m \geq 3$ *be an integer, and let* $X_1, \ldots, X_m$ *be subsets of* G*.*
If there is $o(N^{m-1})$ *solutions to the equation* $x_1 + \cdots + x_m = 0$ *with* $x_i \in X_i$*,*
then we may remove $o(N)$ *elements from each* X_i *so as to leave sets* X_i'*, such that there is*
no solutions to $x_1' + \cdots + x_m' = 0$ *with* $x_i' \in X_i'$ *for all* i*.*

Green's proof uses a Szemerédi Regularity Lemma–like for Abelian groups.

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Statement: Removal Lemma for Graphs

Lemma (Removal Lemma for Graphs, Füredi/Rödl observation on the Rusza-Szemerédi 1976)

Let H be a graph on h vertices.
Let G be a graph on n vertices.
If the number of copies of $H \subset G$ is $o(n^h)$
then there exists a set $E' \subset E(G)$ with $|E'| = o(n^2)$ such that $G \setminus E'$ is H–free.

The proof uses the Szemerédi Regularity Lemma.

Statement: edge-colored Removal Lemma

Lemma (Removal Lemma for edge-colored graphs)

Let H a graph with its edges colored with c colors.
If G contains less than $o(n^h)$ copies of H (the colors of edges in the copy and H must be the same),
then there exists a set E' of at most $o(n^2)$ edges such that $G \setminus E'$ is H-free.

Removal La. edge–colored graphs $\Longrightarrow$ Green's result any group.

Structure

Systems of equations over finite fields

Using similar ideas:
Hypergraph Removal Lemma $\Longrightarrow$ Removal Lemma Linear Systems

Theorem (removal lemma for systems of equations)

Let $\mathbf{G} = \mathbb{F}_q$ be a finite field of order $q = p^n$, p prime.
Let $X_1, \ldots, X_m$ be subsets of $\mathbb{F}_q$.
Let S be a system of k equations over m variables $Ax = b$.
Suppose that there are $o(q^{m-k})$ solutions to S with $x_i \in X_i$ for all i.
Then, there exist sets $X'_1, \ldots, X'_m$ with $|X_i \setminus X'_i| = o(q)$ such that there is no solution to the system S with $x_i \in X'_i$ for all i.

Result independently proved by Shapira.

Removal Lemma for Hypergraphs, edge–colored version

Lemma (Removal Lemma for edge-colored hypergraphs, Austin & Tao, 2008+)

Let H be a k–uniform hypergraph on h vertices, edges colored with c colors.
If a k–uniform hypergraph G, edges colored with c colors, contains less than $o(n^h)$ copies of H (the colors of edges in the copy and H must be the same),
then there exists a set E' of at most $o(n^k)$ hyperedges such that $G \setminus E'$ is H–free.

Other **sufficient versions** can be proved using the **Hypergraph Regularity Method** proved by Nagel, Rödl, Schacht, Skokan and Gowers.

Proof Removal La.: one equation

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 0$$

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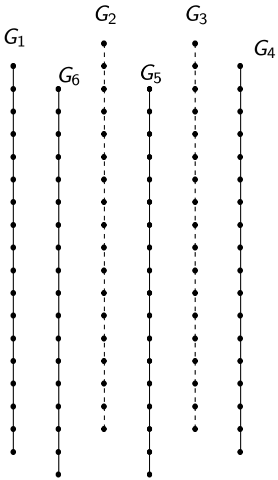
Build a convenient graph

Proof Removal La.: one equation



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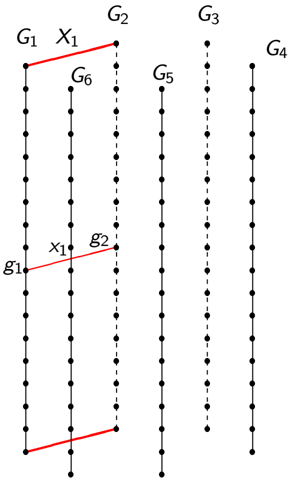

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$$-g_1 + g_2 \qquad = x_1$$

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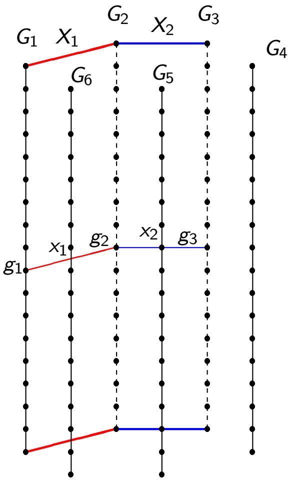

$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 0$

Build a convenient graph

$$
\begin{aligned}
-g_1 + g_2 \quad &= x_1 \\
- g_2 + g_3 \quad &= x_2
\end{aligned}
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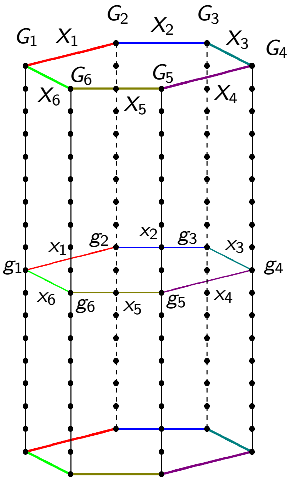
Proof Removal La.: one equation



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-g_4 + g_5 &= x_4 \\
-g_5 + g_6 &= x_5 \\
g_1 - g_6 &= x_6
\end{aligned}
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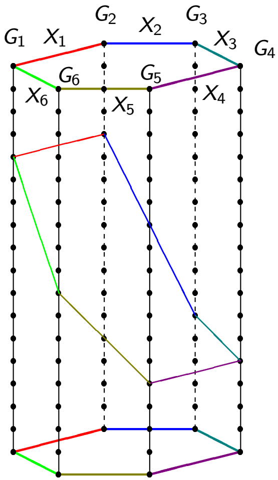

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Equivalence solutions ↔ subgraphs

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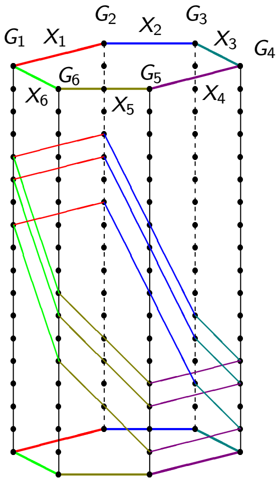

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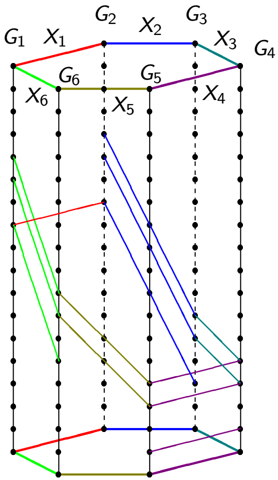

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Equivalence solutions $\leftrightarrow$ subgraphs

Apply Rem. La. edge-colored graphs

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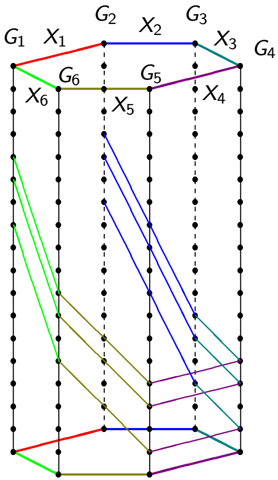

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Apply Rem. La. edge-colored graphs

Use the Pidgeonhole principle

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g_1 \quad - g_6 &= x_6
\end{aligned}
$$

Proof Removal La.: one equation

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 0$$

$$\begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} g_1 \\ g_2 \\ g_3 \\ g_4 \\ g_5 \\ g_6 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix}$$

Proof Removal La.: various equations

System:

$$\begin{pmatrix} 1 & 0 & -1 & 0 & 1 & -1 \\ 0 & 1 & 2 & 1 & 0 & 2 \end{pmatrix} \overrightarrow{x} = 0$$

Representation:

$$\begin{pmatrix} -1 & 1/2 & -1 & 0 & 0 & 0 \\ 0 & -1 & -2 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & 2 \\ 1 & 0 & 0 & 0 & -1 & -1 \\ 0 & -1/2 & 0 & 0 & 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} g_1 \\ g_2 \\ g_3 \\ g_4 \\ g_5 \\ g_6 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix}$$

General systems of equations: requires some adaptation.

Differences with Shapira's proof

Shapira's construction:

- $O(m^2)$–uniform hypergraphs.

Our proof:

- $k+1$–uniform hypergraphs.

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- "Structured": some edges comes from the elements in $x_i \in X_i$ for $i > k$. The apparition of the other edges in the subgraph means that some equations are fullilled.

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- All edges treated equally: the sewing are made thanks to the structure of the solution space.

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Shapira's construction:

- $O(m^2)$–uniform hypergraphs.
- "Structured": some edges comes from the elements in $x_i \in X_i$ for $i > k$. The apparition of the other edges in the subgraph means that some equations are fullilled.
- Relies more on the matrix of the system.

Our proof:

- $k+1$–uniform hypergraphs.
- All edges treated equally: the sewing are made thanks to the structure of the solution space.
- Relies on the relations between the columns of A.

Related results

Candela:

- Same result if any $k \times k$ submatrix of A is non–singular. Similar construction.

Szegedy:

- Proved a general framework. Symmetry Preserving Removal Lemma.

Consequences: Szemerédi Theorem

Let A be a $k \times m$.
Let $Ax = 0$ be a system of equations with $x_i \in X \subset \mathbb{F}_q$.
Denote by A_i a column of A.

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Corollary

If $\sum_{i=1}^{m} A_i = 0$ *and*
$|X| = \Omega(q)$,
then the number of solutions is $\Omega(q^{m-k})$.

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Corollary (Szemerédi Thm. finite fields, Varnavides version)

Let $k \geq exponent(\mathbb{F}_q)$.
If $|X| = \Omega(q)$
then it contains $\Omega(q^2)$ arithmetic progressions of length k.

Thanks for your attention!