Antibandwidth of Hamming graphs

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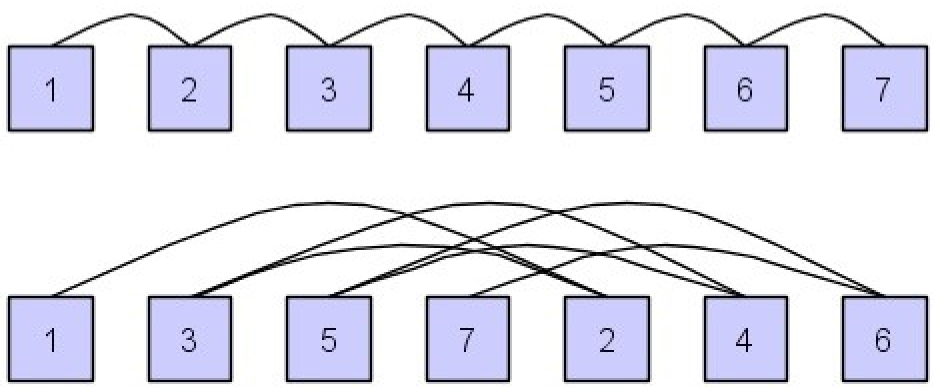
Antibandwidth problem

- Consists of placing the vertices of a graph on a line in consecutive integer points in such a way that the minimum difference of adjacent vertices is maximized.
- Just another labeling problem (from graph theory point of view)

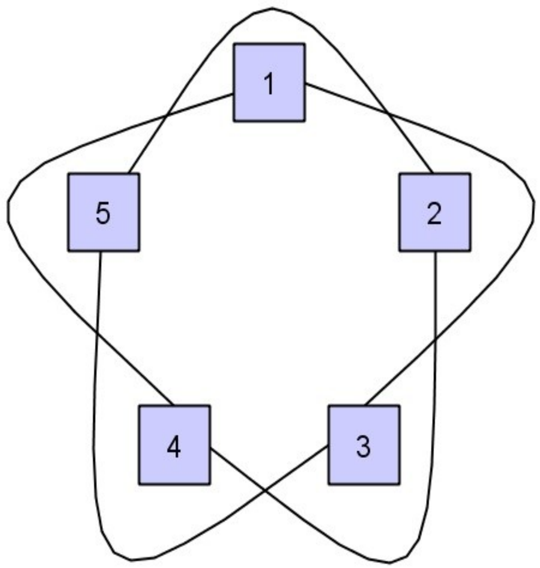
Antibandwidth problem

- Example (linear)



Cyclic antibandwidth

- Example (cyclic)



Motivation

- Introduced in connection with multiprocessor scheduling problems
- Dual problem to bandwidth problem

Bandwidth = to find vertex ordering of a graph G s.t. the length of the longest edge is minimized.

Antibandwidth = to find vertex ordering of a graph G s.t. the length of the shortest edge is maximized.

Motivation

- Radio frequency assignment problem

  N transmitters, N frequencies.

  Graph representation: transmitters = vertices, egdes = possible interference

  Goal: to assign all frequencies to transmitters in a one-to-one manner s.t. interferencing transmitters have as different frequencies as possible.

Motivation

- Enemy facility location problem

  Graph representation: vertices = facilities, edges = hostility

  Goal: to arrange facilities in a line (cycle, mesh,...) s.t. the critical (smallest) distance between two enemies is maximized

Motivation

- Coding theory

  Host graph: n-dimensional hypercube

  Guest graph: complete graph K_p

  Then $ab(K_p, Q_n) \geq A$ is equivalent to an existence of a code with minimal Hamming distance greater or equal to A.

Previous results

- NP-complete (Leung, Vornberger)
- Polynomially solvable for the complements of
  - Interval
  - Arborescent comparability
  - Threshold

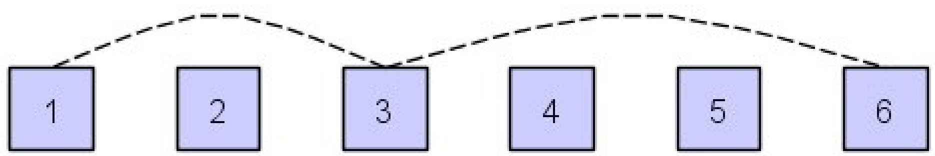
  graphs.

  (Donnely, Isaak, *Hamiltonian powers in … graphs)*

Previous results

- Trivial upper bound for connected graphs:

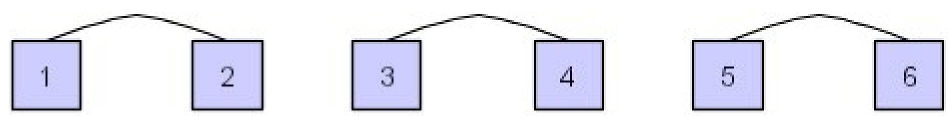
$$ab(G) \leq \lfloor \frac{n}{2} \rfloor$$



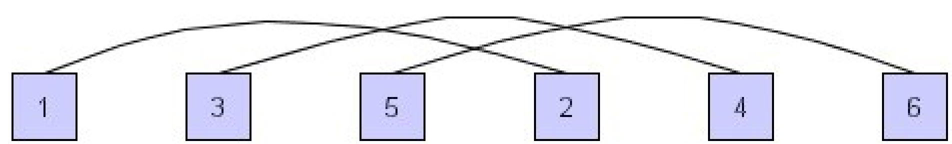
Previous results

- Also interesting for disconnected graphs.
  - Exact values for graphs consisting of copies of simple graphs.

Bandwidth:



Antibandwidth:



Previous results

- Known results for:
  - Paths, cycles, special trees
  - complete and complete bipartite graphs
  - 2D, 3D and n-dimensional meshes
  - tori, hypercubes
- Cyclic antibandwidth
  - Embedding into cycle

Previous results

Raspaud, Schröder, Sýkora, Török, Vrťo:

Antibandwidth and Cyclic Antibandwidth of Meshes and Hypercubes

Calamoneri, Massini, Török, Vrťo:

Antibandwidth of complete k-ary trees

Török, Vrťo:

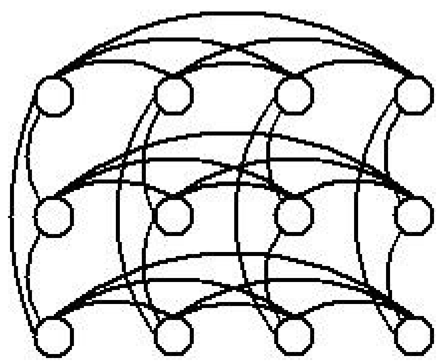
Antibandwidth of three-dimensional meshes

Antibandwidth of d-dimensional meshes

Hamming graph

- Definition
  - Cartesian product of d complete graphs

    $K_{n_k}, k=1,2,\ldots,d$, i.e. $\Pi_{k=1}^{d} K_{n_k}$



Hamming graph $K_3 \times K_4$

Hamming graph

- Definition
  - Vertices are d-tuples

$$(i_{1,} i_{2,} \ldots, i_d), i_k \in 0,1,2, \ldots, n_k - 1$$

  - Two vertices $(i_{1,} i_{2,} \ldots, i_d)$ and $(j_{1,} j_{2,} \ldots, j_d)$ are adjacent iff the two d-tuples differ in precisely one coordinate

Results

- Work in progress
- Theorem

$$\begin{aligned}&For\ d \geq 2,\\&2 \leq n_1 \leq n_2 \leq \ldots \leq n_d\end{aligned}$$

$$ab(\Pi_{k=1}^{d} K_{n_k}) = n_1 n_2 \ldots n_{d-1},\ if\ n_{d-1} \neq n_d$$

$$ab(\Pi_{k=1}^{d} K_{n_k}) = n_1 n_2 \ldots n_{d-1} - 1,\ if\ n_{d-1} = n_d,\ n_{d-2} \neq n_{d-1}$$

$$n_1 n_2 \ldots n_{d-1} - min(n_1 n_2 \ldots n_{d-2},\ n_{q+1} \ldots n_{d-1}) \leq ab(\Pi_{k=1}^{d} K_{n_k}) \leq n_1 n_2 \ldots n_{d-1} - 1$$

where q is minimal index such that $q \leq d-2$ and $n_q = n_d$

Results

- Labelling
  - Case I. $n_{d-1} \neq n_d$

$$f(i_1, i_2, \ldots, i_d) = ((N_{d-1} + N_{d-1}/N_1) i_1 + (N_{d-1} + N_{d-1}/N_2) i_2 + (N_{d-1} + N_{d-1}/N_3) i_3 + \ldots + (N_{d-1} + N_{d-1}/N_{d-1}) i_{d-1} + N_{d-1} i_d) \, mod \, N_d$$

$N_j = \left| \Pi_{k=1}^{j} K_{n_k} \right|$, number of vertices of the product

  - Example $K_2 \times K_3 \times K_4$

$$f(i_1, i_2, i_3) = (9\mathrm{i}_1 + 7\mathrm{i}_2 + 6\mathrm{i}_3) \, mod \, 24$$

Results

- Example of labelling of $K_2 \times K_3 \times K_4$

0	7	14
9	16	23

6	13	20
15	22	5

12	19	2
21	4	11

18	1	8
3	10	17

Results

- Labelling
  - Case II. $n_{d-1} = n_d$, let q be minimal s.t.

    $q \le d - 1 \, and \, n_q = n_d$

$$
\begin{aligned}
f(i_1, i_2, \ldots, i_d) = (&(N_{d-1} + N_{d-1}/N_1) i_1 \\
&+ (N_{d-1} + N_{d-1}/N_2) i_2 + \ldots + (N_{d-1} + N_{d-1}/N_{q-1}) i_{q-1} \\
&+ (N_{d-1} - N_{d-1}/N_q) i_q + \ldots \\
&+ (N_{d-1} - N_{d-1}/N_{d-1}) i_{d-1} + N_{d-1} i_d) \, mod \, N_d
\end{aligned}
$$

Results

- Labelling
  - Case III. $n_{d-1}=n_d$, define the alternative labelling

$$f(i_1, i_2, \ldots, i_d) = ((N_{d-1} - N_0)i_1 + (N_{d-1} - N_1)i_2 + \ldots$$
$$+ (N_{d-1} - N_{d-2})i_{d-1} + N_{d-1} i_d) \, mod \, N_d$$

Results

- Antibandwidth and cyclic antibandwidth have the same value
- Open problem

$$n^2 - n \leq ab(K_n \times K_n \times K_n) \leq n^2 - 1$$

- Conjecture

$$ab(K_n \times K_n \times K_n) = n^2 - n$$

The End

Thank You