Cayley maps on tori

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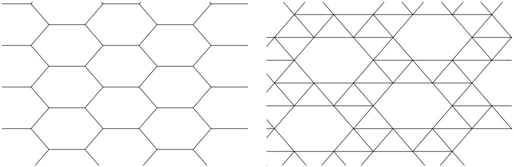
Tilings of the plane



(a) a regular tiling   (b) a semi-regular tiling

Objects of interest

$$
\begin{array}{ccc}
\text{plane } \mathbf{R}^2 & \longleftarrow & \begin{array}{c}\text{semiregular tiling } \mathcal{T} \text{ with} \\ \text{Cayley/vertex transitive action of} \\ \text{wallpaper group } \Gamma\end{array} \\
\downarrow & & \downarrow \\
\text{torus } \mathbf{R}^2/\Lambda & \longleftarrow & \begin{array}{c}\text{a map } \mathcal{T}/\Lambda \text{ with Cayley/vertex} \\ \text{transitive action of finite group } G\end{array}
\end{array}
$$

Outline

Review of history

New results

Isomorphism problem for quotients of extensions of **p3**

Open problems

Group theory

- Burnside (1911)
- ...
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- Hayakawa, Kuribayashi, Kuribayashi(1999)

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- Hayakawa, Kuribayashi, Kuribayashi(1999)
- Coxeter-Moser
  - list of toric groups
  - worked with regular tilings of $\mathbf{R}^2$

Graph theory

- Thomassen, Babai (1991)
  - proof of Babai's conjecture
  - topological description of vertex transitive graphs on torus and Klein bottle
  - almost all vertex transitive graphs are vertex transitive maps
- Gross, Tucker
  - an indirect description of groups
  - e.g. $\langle x, y : x, y : x^3 = y^2 = [x, y]^3 = 1, \ldots \rangle$ where the subgroup $\langle x, yxy \rangle$ has index 2

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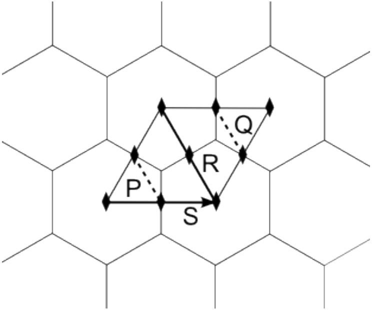
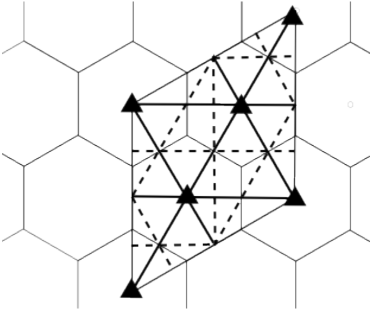
Actions of wallpaper groups on semiregular tilings

We obtain the complete list of pairs $(\Gamma, \mathcal{T})$ where Γ is a wallpaper group that acts on a semiregular tiling $\mathcal{T}$.



(c) action of **cm**



(d) action of **p3m1**

Lists of groups

There are 29 parametric families of toric groups

- 5 arise from 5 wallpaper groups preserving orientation (**p1**,**p2**, **p3**, **p4**, **p6**)
- 2 families arise from each of the remaining 12 wallpaper groups

Vertex transitive implies Cayley

Theorem If M is a vertex transitive map on torus then it admits a Cayley map structure.

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NOT true for other surfaces (of negative Euler characteristic).

Summary

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- 17 wallpaper groups
- 42 different pairings of wallpaper group and semiregular tiling

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- 17 parametrized families of Cayley maps on torus

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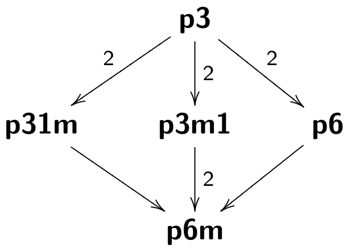
Subgroup structure



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- **p3m1**, **p31m**, **p6m** yield two families each

$$\Gamma/(X^a)$$
$$\Gamma/(X^{2a} Y^a)$$

Isomorphism problem

What is the intersection of families of groups:

- $\mathbf{p3}(b, c)$, $\mathbf{p6}(b, c)$
- $\mathbf{p31m}_1(a)$, $\mathbf{p31m}_3(a)$
- $\mathbf{p3m1}_1(a)$, $\mathbf{p3m1}_3(a)$
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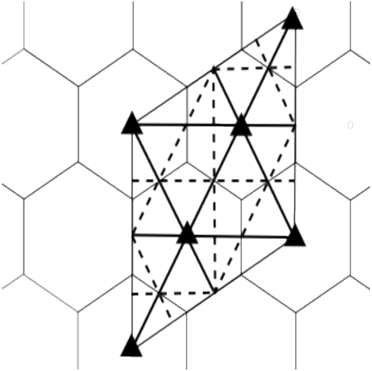
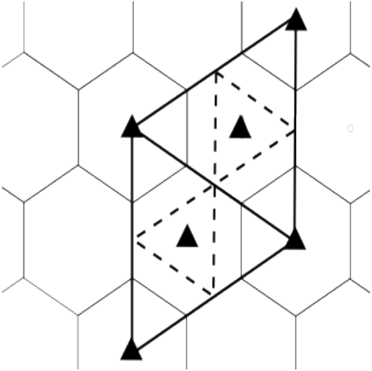
- $\mathbf{p3m1}_1(a) = \mathbf{p31m}_1(a)$ for a not divisible by 3.

Actions on hexagonal tiling

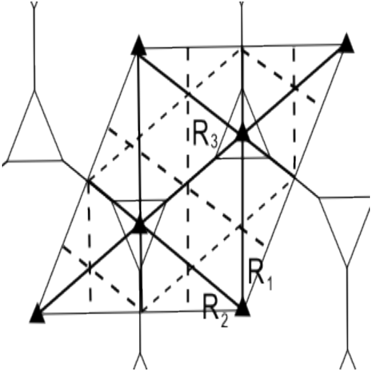


(e) action of **p3m1**

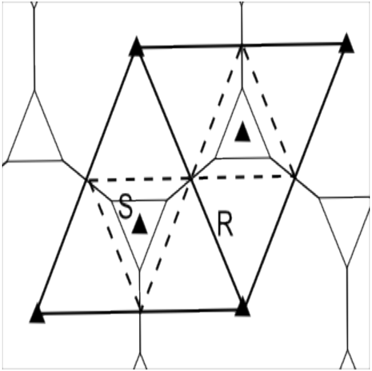


(f) action of **p31m**

Actions on truncated hexagonal tiling

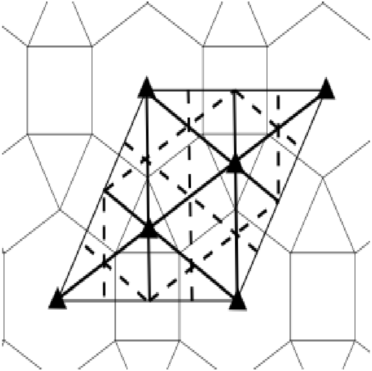


(g) action of **p3m1**

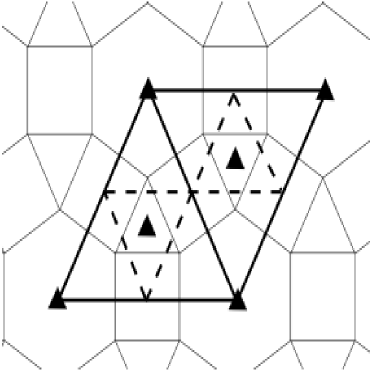


(h) action of **p31m**

Actions on small rhombitrihexagonal tiling



(i) action of **p3m1**



(j) action of **p31m**

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Group isomorphisms

Open question to find intersection of various families:

$$\begin{aligned}
\mathbf{p2}(2, 2k, 2) &= \mathbf{pmg}_1(k, 2),\\
\mathbf{pgg}_1(2k+1, 2l) &= \mathbf{pmg}_1(2l, 2k+1),\\
\mathbf{p3m1}_1(a) &= \mathbf{p31m}_1(a), \quad \text{if } 3 \not| a\\
&\dots
\end{aligned}$$

Graphs vs. maps

Find all vertex transitive graphs on torus and Klein bottle that are not vertex transitive maps. Do they admit Cayley structure?