Classification of regular maps on a given surface

J. Širáň

Introduction

Introduction

Regular maps are generalizations of Platonic solids (viewed as tessellations of the sphere) to surfaces of higher genus.

Introduction

Regular maps are generalizations of Platonic solids (viewed as tessellations of the sphere) to surfaces of higher genus.

Their formal study was initiated by Brahana in the 1920s and continued by Coxeter and others decades later.

Introduction

Regular maps are generalizations of Platonic solids (viewed as tessellations of the sphere) to surfaces of higher genus.

Their formal study was initiated by Brahana in the 1920s and continued by Coxeter and others decades later.

Regular maps on the sphere and the torus and other orientable surfaces of small genus are now quite well understood.

Introduction

Regular maps are generalizations of Platonic solids (viewed as tessellations of the sphere) to surfaces of higher genus.

Their formal study was initiated by Brahana in the 1920s and continued by Coxeter and others decades later.

Regular maps on the sphere and the torus and other orientable surfaces of small genus are now quite well understood. However, until recently, the situation for surfaces of higher genus has been something of a mystery.

Introduction

Regular maps are generalizations of Platonic solids (viewed as tessellations of the sphere) to surfaces of higher genus.

Their formal study was initiated by Brahana in the 1920s and continued by Coxeter and others decades later.

Regular maps on the sphere and the torus and other orientable surfaces of small genus are now quite well understood. However, until recently, the situation for surfaces of higher genus has been something of a mystery.

In particular, some long-standing questions have remained open, about the genera of orientable surfaces carrying a regular map having no multiple edges, or a regular map that is chiral (admitting no reflectional symmetry).

Introduction

Regular maps are generalizations of Platonic solids (viewed as tessellations of the sphere) to surfaces of higher genus.

Their formal study was initiated by Brahana in the 1920s and continued by Coxeter and others decades later.

Regular maps on the sphere and the torus and other orientable surfaces of small genus are now quite well understood. However, until recently, the situation for surfaces of higher genus has been something of a mystery.

In particular, some long-standing questions have remained open, about the genera of orientable surfaces carrying a regular map having no multiple edges, or a regular map that is chiral (admitting no reflectional symmetry).

We will survey the recent substantial progress towards classification of regular maps on a given surface.

Surfaces:

Surfaces: orientable, genus $g \geq 0$;

Surfaces: orientable, genus $g \geq 0$; non-orientable, genus $h \geq 1$.

Surfaces: orientable, genus $g \geq 0$; non-orientable, genus $h \geq 1$.

Map:

Surfaces: orientable, genus $g \geq 0$; non-orientable, genus $h \geq 1$.

Map: Cellular embedding of a graph on a surface.

Surfaces: orientable, genus $g \geq 0$; non-orientable, genus $h \geq 1$.

Map: Cellular embedding of a graph on a surface.

Map of type $\{m, k\}$:

Surfaces: orientable, genus $g \geq 0$; non-orientable, genus $h \geq 1$.

Map: Cellular embedding of a graph on a surface.

Map of type $\{m, k\}$: Vertex valence k, face length m.

Surfaces: orientable, genus $g \geq 0$; non-orientable, genus $h \geq 1$.

Map: Cellular embedding of a graph on a surface.

Map of type $\{m, k\}$: Vertex valence k, face length m.

Map automorphisms:

Surfaces: orientable, genus $g \geq 0$; non-orientable, genus $h \geq 1$.

Map: Cellular embedding of a graph on a surface.

Map of type $\{m, k\}$: Vertex valence k, face length m.

Map automorphisms: viewed as permutations of darts, that is, edges with direction.

Surfaces: orientable, genus $g \geq 0$; non-orientable, genus $h \geq 1$.

Map: Cellular embedding of a graph on a surface.

Map of type $\{m, k\}$: Vertex valence k, face length m.

Map automorphisms: viewed as permutations of darts, that is, edges with direction.

A map M of type $\{m, k\}$ is regular

Surfaces: orientable, genus $g \geq 0$; non-orientable, genus $h \geq 1$.

Map: Cellular embedding of a graph on a surface.

Map of type $\{m, k\}$: Vertex valence k, face length m.

Map automorphisms: viewed as permutations of darts, that is, edges with direction.

A map M of type $\{m, k\}$ is regular if and only if there is a k-fold rotation r about a vertex

Surfaces: orientable, genus $g \geq 0$; non-orientable, genus $h \geq 1$.

Map: Cellular embedding of a graph on a surface.

Map of type $\{m, k\}$: Vertex valence k, face length m.

Map automorphisms: viewed as permutations of darts, that is, edges with direction.

A map M of type $\{m, k\}$ is regular if and only if there is a k-fold rotation r about a vertex and an m-fold rotation s about the centre of an incident face,

Surfaces: orientable, genus $g \geq 0$; non-orientable, genus $h \geq 1$.

Map: Cellular embedding of a graph on a surface.

Map of type $\{m, k\}$: Vertex valence k, face length m.

Map automorphisms: viewed as permutations of darts, that is, edges with direction.

A map M of type $\{m, k\}$ is regular if and only if there is a k-fold rotation r about a vertex and an m-fold rotation s about the centre of an incident face, with the product rs being an involutory rotation around the midpoint of an edge incident with the vertex and the face.

Surfaces: orientable, genus $g \geq 0$; non-orientable, genus $h \geq 1$.

Map: Cellular embedding of a graph on a surface.

Map of type $\{m, k\}$: Vertex valence k, face length m.

Map automorphisms: viewed as permutations of darts, that is, edges with direction.

A map M of type $\{m, k\}$ is regular if and only if there is a k-fold rotation r about a vertex and an m-fold rotation s about the centre of an incident face, with the product rs being an involutory rotation around the midpoint of an edge incident with the vertex and the face.

A regular map M is reflexible

Surfaces: orientable, genus $g \geq 0$; non-orientable, genus $h \geq 1$.

Map: Cellular embedding of a graph on a surface.

Map of type $\{m, k\}$: Vertex valence k, face length m.

Map automorphisms: viewed as permutations of darts, that is, edges with direction.

A map M of type $\{m, k\}$ is regular if and only if there is a k-fold rotation r about a vertex and an m-fold rotation s about the centre of an incident face, with the product rs being an involutory rotation around the midpoint of an edge incident with the vertex and the face.

A regular map M is reflexible if it contains a non-trivial automorphism fixing a dart.

Surfaces: orientable, genus $g \geq 0$; non-orientable, genus $h \geq 1$.

Map: Cellular embedding of a graph on a surface.

Map of type $\{m, k\}$: Vertex valence k, face length m.

Map automorphisms: viewed as permutations of darts, that is, edges with direction.

A map M of type $\{m, k\}$ is regular if and only if there is a k-fold rotation r about a vertex and an m-fold rotation s about the centre of an incident face, with the product rs being an involutory rotation around the midpoint of an edge incident with the vertex and the face.

A regular map M is reflexible if it contains a non-trivial automorphism fixing a dart.

Regular maps on non-orientable surfaces are automatically reflexible.

Surfaces: orientable, genus $g \geq 0$; non-orientable, genus $h \geq 1$.

Map: Cellular embedding of a graph on a surface.

Map of type $\{m, k\}$: Vertex valence k, face length m.

Map automorphisms: viewed as permutations of darts, that is, edges with direction.

A map M of type $\{m, k\}$ is regular if and only if there is a k-fold rotation r about a vertex and an m-fold rotation s about the centre of an incident face, with the product rs being an involutory rotation around the midpoint of an edge incident with the vertex and the face.

A regular map M is reflexible if it contains a non-trivial automorphism fixing a dart.

Regular maps on non-orientable surfaces are automatically reflexible.

A regular map that is not reflexible is called chiral.

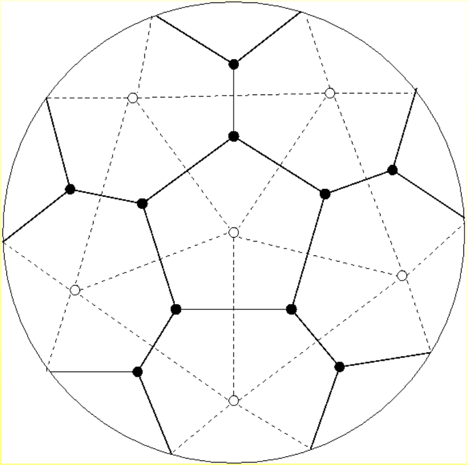
Example of a non-spherical regular map

Example of a non-spherical regular map

The Petersen Graph on the projective plane, with its dual – K_6:

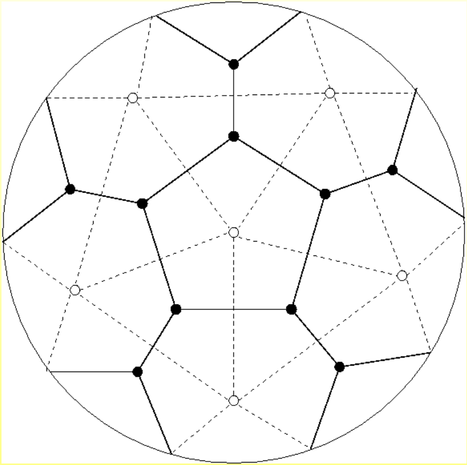
Example of a non-spherical regular map

The Petersen Graph on the projective plane, with its dual – K_6:



Example of a non-spherical regular map

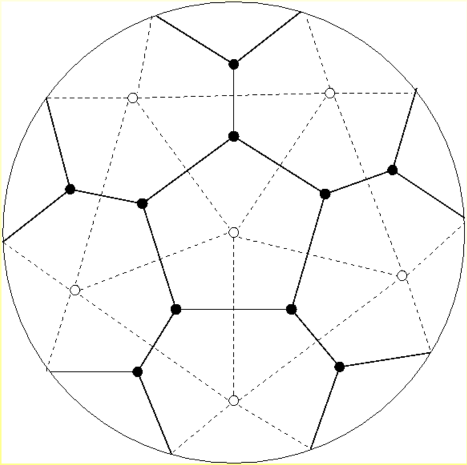
The Petersen Graph on the projective plane, with its dual – K_6:



Map elements:

Example of a non-spherical regular map

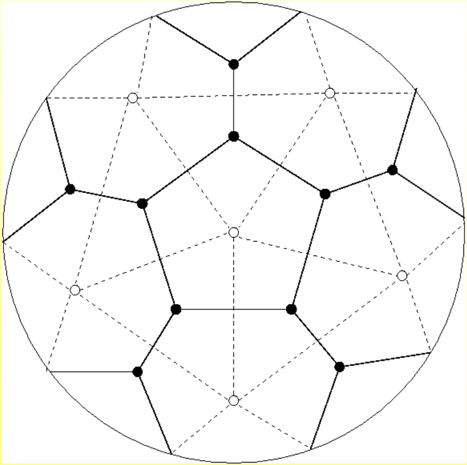
The Petersen Graph on the projective plane, with its dual – K_6:



Map elements:
faces,

Example of a non-spherical regular map

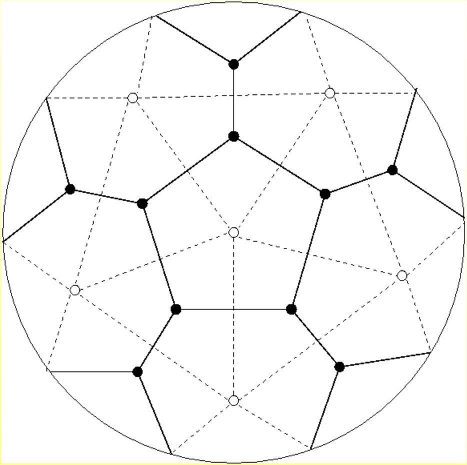
The Petersen Graph on the projective plane, with its dual – K_6:



Map elements:
faces, vertices,

Example of a non-spherical regular map

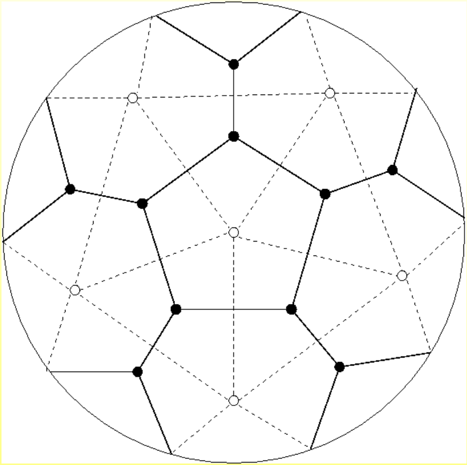
The Petersen Graph on the projective plane, with its dual – K_6:



Map elements:
faces, vertices,
edges,

Example of a non-spherical regular map

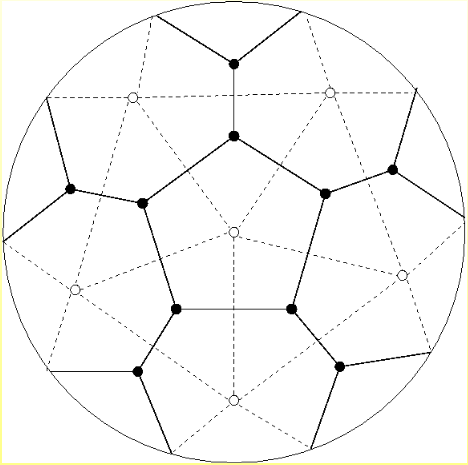
The Petersen Graph on the projective plane, with its dual – K_6:



Map elements:
faces, vertices,
edges, flags

Example of a non-spherical regular map

The Petersen Graph on the projective plane, with its dual – K_6:

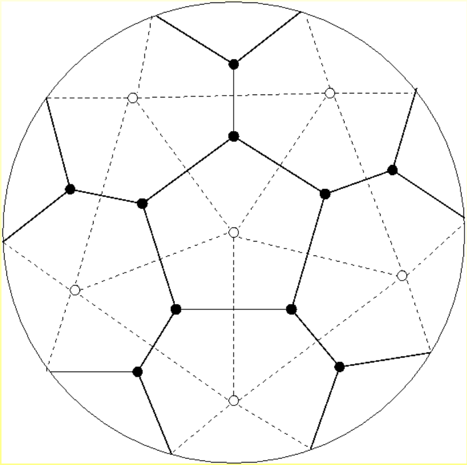


Map elements:
faces, vertices,
edges, flags

Automorphisms:

Example of a non-spherical regular map

The Petersen Graph on the projective plane, with its dual – K_6:

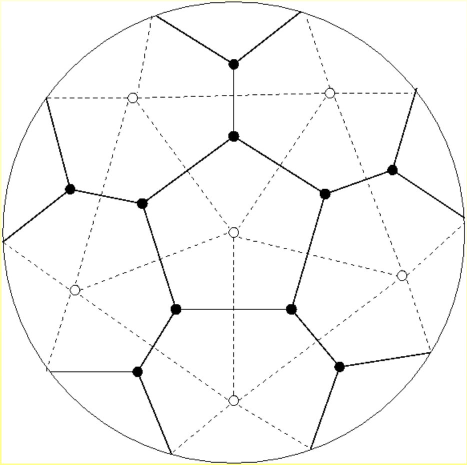


Map elements:
faces, vertices,
edges, flags

Automorphisms:
- 10 visible

Example of a non-spherical regular map

The Petersen Graph on the projective plane, with its dual – K_6:

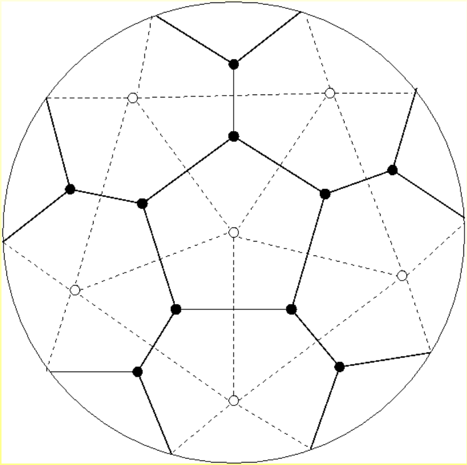


Map elements:
faces, vertices,
edges, flags

Automorphisms:
- 10 visible
- 60 in total

Example of a non-spherical regular map

The Petersen Graph on the projective plane, with its dual – K_6:



Map elements:
faces, vertices,
edges, flags

Automorphisms:
- 10 visible
- 60 in total

regular on flags

M - a regular map on an orientable surface;

M - a regular map on an orientable surface; rotations r and s

M - a regular map on an orientable surface; rotations r and s

By connectedness, r and s generate $G = \mathrm{Aut}^o(M)$, and

M - a regular map on an orientable surface; rotations r and s

By connectedness, r and s generate $G = \mathrm{Aut}^o(M)$, and

$$Aut^o(M) = \langle r, s |\ r^k = s^m = (rs)^2 = \ldots = 1\rangle$$

M - a regular map on an orientable surface; rotations r and s

By connectedness, r and s generate $G = \mathrm{Aut}^o(M)$, and

$$Aut^o(M) = \langle r, s |\ r^k = s^m = (rs)^2 = \ldots = 1\rangle$$

[Additional relators – non-contractible closed curves on the surface.]

Therefore, G is a quotient of the ordinary $(k, m, 2)$-triangle group

M - a regular map on an orientable surface; rotations r and s

By connectedness, r and s generate $G = \mathrm{Aut}^o(M)$, and

$$Aut^o(M) = \langle r, s |\ r^k = s^m = (rs)^2 = \ldots = 1 \rangle$$

[Additional relators – non-contractible closed curves on the surface.]

Therefore, G is a quotient of the ordinary $(k, m, 2)$-triangle group

$$\Delta^o(k, m, 2) = \left\langle\, a, b, c \mid a^k = b^m = c^2 = abc = 1 \,\right\rangle$$

M - a regular map on an orientable surface; rotations r and s

By connectedness, r and s generate $G = \mathrm{Aut}^o(M)$, and

$$Aut^o(M) = \langle r, s |\ r^k = s^m = (rs)^2 = \ldots = 1 \rangle$$

[Additional relators – non-contractible closed curves on the surface.]

Therefore, G is a quotient of the ordinary $(k, m, 2)$-triangle group

$$\Delta^o(k, m, 2) = \left\langle\, a, b, c \mid a^k = b^m = c^2 = abc = 1 \,\right\rangle$$

under an epimorphism taking a to r and b to s.

M - a regular map on an orientable surface; rotations r and s

By connectedness, r and s generate $G = \mathrm{Aut}^o(M)$, and

$$Aut^o(M) = \langle r, s |\ r^k = s^m = (rs)^2 = \ldots = 1 \rangle$$

[Additional relators – non-contractible closed curves on the surface.]

Therefore, G is a quotient of the ordinary $(k, m, 2)$-triangle group

$$\Delta^o(k, m, 2) = \left\langle\, a, b, c \mid a^k = b^m = c^2 = abc = 1 \,\right\rangle$$

under an epimorphism taking a to r and b to s.

The map M is reflexible

M - a regular map on an orientable surface; rotations r and s

By connectedness, r and s generate $G = \mathrm{Aut}^o(M)$, and

$$Aut^o(M) = \langle r, s \mid r^k = s^m = (rs)^2 = \ldots = 1 \rangle$$

[Additional relators – non-contractible closed curves on the surface.]

Therefore, G is a quotient of the ordinary $(k, m, 2)$-triangle group

$$\Delta^o(k, m, 2) = \left\langle\, a, b, c \mid a^k = b^m = c^2 = abc = 1 \,\right\rangle$$

under an epimorphism taking a to r and b to s.

The map M is reflexible if and only if the group $G = \mathrm{Aut}^o(M)$ admits an automorphism of order two taking r to r^{-1} and s to s^{-1}.

M - a regular map on an orientable surface; rotations r and s

By connectedness, r and s generate $G = \mathrm{Aut}^o(M)$, and

$$Aut^o(M) = \langle r, s |\ r^k = s^m = (rs)^2 = \ldots = 1 \rangle$$

[Additional relators – non-contractible closed curves on the surface.]

Therefore, G is a quotient of the ordinary $(k, m, 2)$-triangle group

$$\Delta^o(k, m, 2) = \left\langle\ a, b, c \mid a^k = b^m = c^2 = abc = 1 \ \right\rangle$$

under an epimorphism taking a to r and b to s.

The map M is reflexible if and only if the group $G = \mathrm{Aut}^o(M)$ admits an automorphism of order two taking r to r^{-1} and s to s^{-1}.

Conversely, given any epimorphism from $\Delta^o(k, m, 2)$ to a finite group G with torsion-free kernel,

M - a regular map on an orientable surface; rotations r and s

By connectedness, r and s generate $G = \mathrm{Aut}^o(M)$, and

$$Aut^o(M) = \langle r, s |\ r^k = s^m = (rs)^2 = \ldots = 1\rangle$$

[Additional relators – non-contractible closed curves on the surface.]

Therefore, G is a quotient of the ordinary $(k, m, 2)$-triangle group

$$\Delta^o(k, m, 2) = \left\langle\, a, b, c \mid a^k = b^m = c^2 = abc = 1 \,\right\rangle$$

under an epimorphism taking a to r and b to s.

The map M is reflexible if and only if the group $G = \mathrm{Aut}^o(M)$ admits an automorphism of order two taking r to r^{-1} and s to s^{-1}.

Conversely, given any epimorphism from $\Delta^o(k, m, 2)$ to a finite group G with torsion-free kernel, a regular map M on an orientable surface can be constructed

M - a regular map on an orientable surface; rotations r and s

By connectedness, r and s generate $G = \mathrm{Aut}^o(M)$, and

$$Aut^o(M) = \langle r, s |\ r^k = s^m = (rs)^2 = \ldots = 1 \rangle$$

[Additional relators – non-contractible closed curves on the surface.]

Therefore, G is a quotient of the ordinary $(k, m, 2)$-triangle group

$$\Delta^o(k, m, 2) = \left\langle\ a, b, c \mid a^k = b^m = c^2 = abc = 1\ \right\rangle$$

under an epimorphism taking a to r and b to s.

The map M is reflexible if and only if the group $G = \mathrm{Aut}^o(M)$ admits an automorphism of order two taking r to r^{-1} and s to s^{-1}.

Conversely, given any epimorphism from $\Delta^o(k, m, 2)$ to a finite group G with torsion-free kernel, a regular map M on an orientable surface can be constructed using (right) cosets of the images of $\langle a \rangle$, $\langle b \rangle$ and $\langle c \rangle$ as vertices, faces and edges,

M - a regular map on an orientable surface; rotations r and s

By connectedness, r and s generate $G = \mathrm{Aut}^o(M)$, and

$$Aut^o(M) = \langle r, s |\ r^k = s^m = (rs)^2 = \ldots = 1 \rangle$$

[Additional relators – non-contractible closed curves on the surface.]

Therefore, G is a quotient of the ordinary $(k, m, 2)$-triangle group

$$\Delta^o(k, m, 2) = \left\langle\, a, b, c \mid a^k = b^m = c^2 = abc = 1 \,\right\rangle$$

under an epimorphism taking a to r and b to s.

The map M is reflexible if and only if the group $G = \mathrm{Aut}^o(M)$ admits an automorphism of order two taking r to r^{-1} and s to s^{-1}.

Conversely, given any epimorphism from $\Delta^o(k, m, 2)$ to a finite group G with torsion-free kernel, a regular map M on an orientable surface can be constructed using (right) cosets of the images of $\langle a \rangle$, $\langle b \rangle$ and $\langle c \rangle$ as vertices, faces and edges, with incidence given by non-empty intersection,

M - a regular map on an orientable surface; rotations r and s

By connectedness, r and s generate $G = \mathrm{Aut}^o(M)$, and

$$Aut^o(M) = \langle r, s |\ r^k = s^m = (rs)^2 = \ldots = 1 \rangle$$

[Additional relators – non-contractible closed curves on the surface.]

Therefore, G is a quotient of the ordinary $(k, m, 2)$-triangle group

$$\Delta^o(k, m, 2) = \left\langle\ a, b, c \mid a^k = b^m = c^2 = abc = 1\ \right\rangle$$

under an epimorphism taking a to r and b to s.

The map M is reflexible if and only if the group $G = \mathrm{Aut}^o(M)$ admits an automorphism of order two taking r to r^{-1} and s to s^{-1}.

Conversely, given any epimorphism from $\Delta^o(k, m, 2)$ to a finite group G with torsion-free kernel, a regular map M on an orientable surface can be constructed using (right) cosets of the images of $\langle a \rangle$, $\langle b \rangle$ and $\langle c \rangle$ as vertices, faces and edges, with incidence given by non-empty intersection, and then G acts regularly on the darts of M by (right) multiplication.

Regular maps in mathematics

Regular maps in mathematics

Up to isomorphism and duality, 1-1 correspondence between:

Regular maps in mathematics

Up to isomorphism and duality, 1-1 correspondence between:

- regular orientable maps of type $\{m, k\}$ with $k \geq m$

Regular maps in mathematics

Up to isomorphism and duality, 1-1 correspondence between:

- regular orientable maps of type $\{m, k\}$ with $k \geq m$
- groups $\langle r, s | \ r^k = s^m = (rs)^2 = \ldots = 1\rangle$

Regular maps in mathematics

Up to isomorphism and duality, 1-1 correspondence between:

- regular orientable maps of type $\{m, k\}$ with $k \geq m$
- groups $\langle r, s |\ r^k = s^m = (rs)^2 = \ldots = 1 \rangle$
- torsion-free normal subgroups of triangle groups $\Delta^o(k, m, 2)$

Regular maps in mathematics

Up to isomorphism and duality, 1-1 correspondence between:

- regular orientable maps of type $\{m, k\}$ with $k \geq m$
- groups $\langle r, s |\ r^k = s^m = (rs)^2 = \ldots = 1\rangle$
- torsion-free normal subgroups of triangle groups $\Delta^o(k, m, 2)$
- images M of smooth coverings $\mathcal{U}(m, k) \to M$

Regular maps in mathematics

Up to isomorphism and duality, 1-1 correspondence between:

- regular orientable maps of type $\{m, k\}$ with $k \geq m$
- groups $\langle r, s | \; r^k = s^m = (rs)^2 = \ldots = 1 \rangle$
- torsion-free normal subgroups of triangle groups $\Delta^o(k, m, 2)$
- images M of smooth coverings $\mathcal{U}(m, k) \to M$

Maps, Riemann surfaces, and Galois theory:

Regular maps in mathematics

Up to isomorphism and duality, 1-1 correspondence between:

- regular orientable maps of type $\{m, k\}$ with $k \geq m$
- groups $\langle r, s |\ r^k = s^m = (rs)^2 = \ldots = 1\rangle$
- torsion-free normal subgroups of triangle groups $\Delta^o(k, m, 2)$
- images M of smooth coverings $\mathcal{U}(m, k) \to M$

Maps, Riemann surfaces, and Galois theory:

A Riemann surface $\mathcal{F}$ is definable via a complex polynomial equation $F(x, y) = 0$ with algebraic coefficients if and only if $\mathcal{F} = \mathcal{U}/H$, for some normal subgroup H of a triangle group $\Delta^o(k, m, 2)$.
[Weil 1950 – Belyj 1972]

Regular maps in mathematics

Up to isomorphism and duality, 1-1 correspondence between:

- regular orientable maps of type $\{m, k\}$ with $k \geq m$
- groups $\langle r, s |\ r^k = s^m = (rs)^2 = \ldots = 1 \rangle$
- torsion-free normal subgroups of triangle groups $\Delta^o(k, m, 2)$
- images M of smooth coverings $\mathcal{U}(m, k) \to M$

Maps, Riemann surfaces, and Galois theory:

A Riemann surface $\mathcal{F}$ is definable via a complex polynomial equation $F(x, y) = 0$ with algebraic coefficients if and only if $\mathcal{F} = \mathcal{U}/H$, for some normal subgroup H of a triangle group $\Delta^o(k, m, 2)$.
[Weil 1950 – Belyj 1972]

The absolute Galois group can be studied
via its action on (regular) maps! [Grothendieck 1981]

Further motivation

Further motivation

Classification of regular maps on a given surface would therefore have consequences in numerous branches of mathematics.

Further motivation

Classification of regular maps on a given surface would therefore have consequences in numerous branches of mathematics.

One more piece of motivation:

Further motivation

Classification of regular maps on a given surface would therefore have consequences in numerous branches of mathematics.

One more piece of motivation:

By a celebrated theorem of Hurwitz, for any $g \geq 2$ the order of a finite group acting as a group of conformal automorphisms of the Riemann surface of genus g is bounded above by $84(g-1)$.

Further motivation

Classification of regular maps on a given surface would therefore have consequences in numerous branches of mathematics.

One more piece of motivation:

By a celebrated theorem of Hurwitz, for any $g \geq 2$ the order of a finite group acting as a group of conformal automorphisms of the Riemann surface of genus g is bounded above by $84(g-1)$.

A classical problem here is classification of the largest possible group of automorphisms for any given orientable genus $g \geq 2$.

Further motivation

Classification of regular maps on a given surface would therefore have consequences in numerous branches of mathematics.

One more piece of motivation:

By a celebrated theorem of Hurwitz, for any $g \geq 2$ the order of a finite group acting as a group of conformal automorphisms of the Riemann surface of genus g is bounded above by $84(g-1)$.

A classical problem here is classification of the largest possible group of automorphisms for any given orientable genus $g \geq 2$. Accola showed that this problem reduces to a large extent, for infinitely many genera,

Further motivation

Classification of regular maps on a given surface would therefore have consequences in numerous branches of mathematics.

One more piece of motivation:

By a celebrated theorem of Hurwitz, for any $g \geq 2$ the order of a finite group acting as a group of conformal automorphisms of the Riemann surface of genus g is bounded above by $84(g-1)$.

A classical problem here is classification of the largest possible group of automorphisms for any given orientable genus $g \geq 2$. Accola showed that this problem reduces to a large extent, for infinitely many genera, to the classification of all regular maps on a surface of given genus.

Classification of regular maps

Classification of regular maps

Classification of regular maps has been approached from three main directions:

Classification of regular maps

Classification of regular maps has been approached from three main directions:

- by automorphism groups

Classification of regular maps

Classification of regular maps has been approached from three main directions:

- by automorphism groups
- by underlying graphs

Classification of regular maps

Classification of regular maps has been approached from three main directions:

- by automorphism groups
- by underlying graphs
- by supporting surfaces

Classification of regular maps

Classification of regular maps has been approached from three main directions:

- by automorphism groups
- by underlying graphs
- by supporting surfaces

Other approaches to the study of regular maps by a combination of graph-theoretic, algebraic, and topological means:

Classification of regular maps

Classification of regular maps has been approached from three main directions:

- by automorphism groups
- by underlying graphs
- by supporting surfaces

Other approaches to the study of regular maps by a combination of graph-theoretic, algebraic, and topological means:

- constructions using suitable graphs, groups, or tools (coverings)

Classification of regular maps

Classification of regular maps has been approached from three main directions:

- by automorphism groups
- by underlying graphs
- by supporting surfaces

Other approaches to the study of regular maps by a combination of graph-theoretic, algebraic, and topological means:

- constructions using suitable graphs, groups, or tools (coverings)
- structural investigation (short cycles, representativity – planar width)

Classification of regular maps

Classification of regular maps has been approached from three main directions:

- by automorphism groups
- by underlying graphs
- by supporting surfaces

Other approaches to the study of regular maps by a combination of graph-theoretic, algebraic, and topological means:

- constructions using suitable graphs, groups, or tools (coverings)
- structural investigation (short cycles, representativity – planar width)
- imposing additional algebraic structure – regular Cayley maps

Classification of regular maps

Classification of regular maps has been approached from three main directions:

- by automorphism groups
- by underlying graphs
- by supporting surfaces

Other approaches to the study of regular maps by a combination of graph-theoretic, algebraic, and topological means:

- constructions using suitable graphs, groups, or tools (coverings)
- structural investigation (short cycles, representativity – planar width)
- imposing additional algebraic structure – regular Cayley maps
- research motivated by computer-aided results

Classification by automorphism groups:

Classification by automorphism groups:

- cyclic or dihedral automorphism groups – easy exercises

Classification by automorphism groups:

- cyclic or dihedral automorphism groups – easy exercises
- very hard for non-trivial classes of groups

Classification by automorphism groups:

- cyclic or dihedral automorphism groups – easy exercises
- very hard for non-trivial classes of groups
- orientaby regular maps with automorphism groups isomorphic to $PSL(2, q)$ and $PGL(2, q)$ – Sah (1969)

Classification by automorphism groups:

- cyclic or dihedral automorphism groups – easy exercises
- very hard for non-trivial classes of groups
- orientaby regular maps with automorphism groups isomorphic to $PSL(2, q)$ and $PGL(2, q)$ – Sah (1969)
- Suzuki simple groups for maps of type $\{4, 5\}$ – Jones (1993)

Classification by automorphism groups:

- cyclic or dihedral automorphism groups – easy exercises
- very hard for non-trivial classes of groups
- orientaby regular maps with automorphism groups isomorphic to $PSL(2, q)$ and $PGL(2, q)$ – Sah (1969)
- Suzuki simple groups for maps of type $\{4, 5\}$ – Jones (1993)
- Ree simple groups for maps of type $\{3, 7\}$ – Jones (1994)

Classification by automorphism groups:

- cyclic or dihedral automorphism groups – easy exercises
- very hard for non-trivial classes of groups
- orientaby regular maps with automorphism groups isomorphic to $PSL(2, q)$ and $PGL(2, q)$ – Sah (1969)
- Suzuki simple groups for maps of type $\{4, 5\}$ – Jones (1993)
- Ree simple groups for maps of type $\{3, 7\}$ – Jones (1994)
- Sah's classification recently extended to nonorientable regular maps and hypermaps with automorphism groups isomorphic to $PSL(2, q)$ and $PGL(2, q)$ – Conder, Potočnik and JŠ (to appear)

Classification by automorphism groups:

- cyclic or dihedral automorphism groups – easy exercises
- very hard for non-trivial classes of groups
- orientaby regular maps with automorphism groups isomorphic to $PSL(2, q)$ and $PGL(2, q)$ – Sah (1969)
- Suzuki simple groups for maps of type $\{4, 5\}$ – Jones (1993)
- Ree simple groups for maps of type $\{3, 7\}$ – Jones (1994)
- Sah's classification recently extended to nonorientable regular maps and hypermaps with automorphism groups isomorphic to $PSL(2, q)$ and $PGL(2, q)$ – Conder, Potočnik and JŠ (to appear)
- clasification of nonorientable regular maps with almost-Sylow-cyclic automorphism groups (groups in which every odd-order Sylow subgroup is cyclic and every even-order Sylow subgroup has a cyclic subgroup of index 2) – Conder, Potočnik and JŠ (submitted)

Regular maps on surfaces of low genus

Sphere:

Regular maps on surfaces of low genus

Sphere: Platonic maps (and ∞ of trivial maps)

Regular maps on surfaces of low genus

Sphere: Platonic maps (and ∞ of trivial maps)

Projective plane:

Regular maps on surfaces of low genus

Sphere:	Platonic maps (and ∞ of trivial maps)
Projective plane:	Petersen, K_4, duals (and ∞ of trivial maps)

Regular maps on surfaces of low genus

Sphere:	Platonic maps (and ∞ of trivial maps)
Projective plane:	Petersen, K_4, duals (and ∞ of trivial maps)
Torus:	

Regular maps on surfaces of low genus

Sphere:	Platonic maps (and ∞ of trivial maps)
Projective plane:	Petersen, K_4, duals (and ∞ of trivial maps)
Torus:	Infinitely many *non*trivial regular maps

Regular maps on surfaces of low genus

Sphere:	Platonic maps (and ∞ of trivial maps)
Projective plane:	Petersen, K_4, duals (and ∞ of trivial maps)
Torus:	Infinitely many *non*trivial regular maps
Klein bottle:	

Regular maps on surfaces of low genus

Sphere:	Platonic maps (and ∞ of trivial maps)
Projective plane:	Petersen, K_4, duals (and ∞ of trivial maps)
Torus:	Infinitely many *non*trivial regular maps
Klein bottle:	No regular map!

Regular maps on surfaces of low genus

Sphere:	Platonic maps (and ∞ of trivial maps)
Projective plane:	Petersen, K_4, duals (and ∞ of trivial maps)
Torus:	Infinitely many *non*trivial regular maps
Klein bottle:	No regular map!

Hurwitz Theorem - A consequence:

Regular maps on surfaces of low genus

Sphere:	Platonic maps (and ∞ of trivial maps)
Projective plane:	Petersen, K_4, duals (and ∞ of trivial maps)
Torus:	Infinitely many *non*trivial regular maps
Klein bottle:	No regular map!

Hurwitz Theorem - A consequence:

Every surface with a negative Euler characteristic supports just a finite number of regular maps.

Regular maps on surfaces of low genus

Sphere:	Platonic maps (and ∞ of trivial maps)
Projective plane:	Petersen, K_4, duals (and ∞ of trivial maps)
Torus:	Infinitely many *non*trivial regular maps
Klein bottle:	No regular map!

Hurwitz Theorem - A consequence:

Every surface with a negative Euler characteristic
supports just a finite number of regular maps.

Beginning of a systematic treatment of the classification: Brahana 1922

Regular maps on surfaces of low genus

Sphere:	Platonic maps (and ∞ of trivial maps)
Projective plane:	Petersen, K_4, duals (and ∞ of trivial maps)
Torus:	Infinitely many *non*trivial regular maps
Klein bottle:	No regular map!

Hurwitz Theorem - A consequence:

Every surface with a negative Euler characteristic supports just a finite number of regular maps.

Beginning of a systematic treatment of the classification: Brahana 1922

- orientable (nonorientable) surfaces up to genus 7 (8) – Brahana (1922), Sherk (1959), Grek (1963,66), Garbe (1969,78), Coxeter and Moser (1984), Scherwa (1985), Bergau and Garbe (1978,89)

Regular maps on surfaces of low genus

Sphere:	Platonic maps (and ∞ of trivial maps)
Projective plane:	Petersen, K_4, duals (and ∞ of trivial maps)
Torus:	Infinitely many *non*trivial regular maps
Klein bottle:	No regular map!

Hurwitz Theorem - A consequence:

Every surface with a negative Euler characteristic supports just a finite number of regular maps.

Beginning of a systematic treatment of the classification: Brahana 1922

- orientable (nonorientable) surfaces up to genus 7 (8) – Brahana (1922), Sherk (1959), Grek (1963,66), Garbe (1969,78), Coxeter and Moser (1984), Scherwa (1985), Bergau and Garbe (1978,89)
- computer-aided extension up to orientable genus 15 and nonorientable genus 30 – Conder and Dobcsányi (2001)

Breakthrough in the classification problem

Breakthrough in the classification problem

By (roughly) end of 20th century: Classification of regular maps was available for a finite number of surfaces only.

Breakthrough in the classification problem

By (roughly) end of 20th century: Classification of regular maps was available for a finite number of surfaces only.

Regular maps on nonorientable surfaces with $\chi = -p$ for p prime

Breakthrough in the classification problem

By (roughly) end of 20th century: Classification of regular maps was available for a finite number of surfaces only.

Regular maps on nonorientable surfaces with $\chi = -p$ for p prime

Let $\nu(p)$ be the number of pairs (j, l) such that j and l are odd, coprime, $j > l \geq 3$, and $(j-1)(l-1) = p+1$.

Breakthrough in the classification problem

By (roughly) end of 20th century: Classification of regular maps was available for a finite number of surfaces only.

Regular maps on nonorientable surfaces with $\chi = -p$ for p prime

Let $\nu(p)$ be the number of pairs (j, l) such that j and l are odd, coprime, $j > l \geq 3$, and $(j-1)(l-1) = p+1$.

Theorem. [A. Breda, R. Nedela, J. Širáň, Trans. Amer. Math. Soc. 2005]

Breakthrough in the classification problem

By (roughly) end of 20th century: Classification of regular maps was available for a finite number of surfaces only.

Regular maps on nonorientable surfaces with $\chi = -p$ for p prime

Let $\nu(p)$ be the number of pairs (j, l) such that j and l are odd, coprime, $j > l \geq 3$, and $(j-1)(l-1) = p+1$.

Theorem. [A. Breda, R. Nedela, J. Širáň, Trans. Amer. Math. Soc. 2005] *Let p be an odd prime and let $n(p)$ be the number of regular maps with $\chi = -p$, up to isomorphism and duality. Then:*

Breakthrough in the classification problem

By (roughly) end of 20th century: Classification of regular maps was available for a finite number of surfaces only.

Regular maps on nonorientable surfaces with $\chi = -p$ for p prime

Let $\nu(p)$ be the number of pairs (j, l) such that j and l are odd, coprime, $j > l \geq 3$, and $(j-1)(l-1) = p+1$.

Theorem. [A. Breda, R. Nedela, J. Širáň, Trans. Amer. Math. Soc. 2005] *Let p be an odd prime and let $n(p)$ be the number of regular maps with $\chi = -p$, up to isomorphism and duality. Then: $n(3) = 4$,*

Breakthrough in the classification problem

By (roughly) end of 20th century: Classification of regular maps was available for a finite number of surfaces only.

Regular maps on nonorientable surfaces with $\chi = -p$ for p prime

Let $\nu(p)$ be the number of pairs (j, l) such that j and l are odd, coprime, $j > l \geq 3$, and $(j-1)(l-1) = p+1$.

Theorem. [A. Breda, R. Nedela, J. Širáň, Trans. Amer. Math. Soc. 2005]
Let p be an odd prime and let $n(p)$ be the number of regular maps with $\chi = -p$, up to isomorphism and duality. Then: $n(3) = 4$, $n(7) = 3$,

Breakthrough in the classification problem

By (roughly) end of 20th century: Classification of regular maps was available for a finite number of surfaces only.

Regular maps on nonorientable surfaces with $\chi = -p$ for p prime

Let $\nu(p)$ be the number of pairs (j, l) such that j and l are odd, coprime, $j > l \geq 3$, and $(j-1)(l-1) = p+1$.

Theorem. [A. Breda, R. Nedela, J. Širáň, Trans. Amer. Math. Soc. 2005]
Let p be an odd prime and let $n(p)$ be the number of regular maps with $\chi = -p$, up to isomorphism and duality. Then: $n(3) = 4$, $n(7) = 3$, $n(13) = 1$,

Breakthrough in the classification problem

By (roughly) end of 20th century: Classification of regular maps was available for a finite number of surfaces only.

Regular maps on nonorientable surfaces with $\chi = -p$ for p prime

Let $\nu(p)$ be the number of pairs (j, l) such that j and l are odd, coprime, $j > l \geq 3$, and $(j-1)(l-1) = p+1$.

Theorem. [A. Breda, R. Nedela, J. Širáň, Trans. Amer. Math. Soc. 2005]
Let p be an odd prime and let $n(p)$ be the number of regular maps with $\chi = -p$, up to isomorphism and duality. Then: $n(3) = 4$, $n(7) = 3$, $n(13) = 1$, and for $p \neq 7, 13$, $n(p)$ is equal to

Breakthrough in the classification problem

By (roughly) end of 20th century: Classification of regular maps was available for a finite number of surfaces only.

Regular maps on nonorientable surfaces with $\chi = -p$ for p prime

Let $\nu(p)$ be the number of pairs (j, l) such that j and l are odd, coprime, $j > l \geq 3$, and $(j-1)(l-1) = p+1$.

Theorem. [A. Breda, R. Nedela, J. Širáň, Trans. Amer. Math. Soc. 2005]
Let p be an odd prime and let $n(p)$ be the number of regular maps with $\chi = -p$, up to isomorphism and duality. Then: $n(3) = 4$, $n(7) = 3$, $n(13) = 1$, and for $p \neq 7, 13$, $n(p)$ is equal to

$$0 \quad \text{if} \quad p \equiv 1 \pmod{12}$$

Breakthrough in the classification problem

By (roughly) end of 20th century: Classification of regular maps was available for a finite number of surfaces only.

Regular maps on nonorientable surfaces with $\chi = -p$ for p prime

Let $\nu(p)$ be the number of pairs (j, l) such that j and l are odd, coprime, $j > l \geq 3$, and $(j-1)(l-1) = p+1$.

Theorem. [A. Breda, R. Nedela, J. Širáň, Trans. Amer. Math. Soc. 2005]
Let p be an odd prime and let $n(p)$ be the number of regular maps with $\chi = -p$, up to isomorphism and duality. Then: $n(3) = 4$, $n(7) = 3$, $n(13) = 1$, and for $p \neq 7, 13$, $n(p)$ is equal to

0	*if*	$p \equiv 1 \pmod{12}$
1	*if*	$p \equiv 5 \pmod{12}$

Breakthrough in the classification problem

By (roughly) end of 20th century: Classification of regular maps was available for a finite number of surfaces only.

Regular maps on nonorientable surfaces with $\chi = -p$ for p prime

Let $\nu(p)$ be the number of pairs (j, l) such that j and l are odd, coprime, $j > l \geq 3$, and $(j-1)(l-1) = p+1$.

Theorem. [A. Breda, R. Nedela, J. Širáň, Trans. Amer. Math. Soc. 2005]
Let p be an odd prime and let $n(p)$ be the number of regular maps with $\chi = -p$, up to isomorphism and duality. Then: $n(3) = 4$, $n(7) = 3$, $n(13) = 1$, and for $p \neq 7, 13$, $n(p)$ is equal to

$$\begin{array}{lll} 0 & \textit{if} & p \equiv 1 \pmod{12} \\ 1 & \textit{if} & p \equiv 5 \pmod{12} \\ \nu(p) & \textit{if} & p \equiv -5 \pmod{12} \end{array}$$

Breakthrough in the classification problem

By (roughly) end of 20th century: Classification of regular maps was available for a finite number of surfaces only.

Regular maps on nonorientable surfaces with $\chi = -p$ for p prime

Let $\nu(p)$ be the number of pairs (j, l) such that j and l are odd, coprime, $j > l \geq 3$, and $(j-1)(l-1) = p+1$.

Theorem. [A. Breda, R. Nedela, J. Širáň, Trans. Amer. Math. Soc. 2005]
Let p be an odd prime and let $n(p)$ be the number of regular maps with $\chi = -p$, up to isomorphism and duality. Then: $n(3) = 4$, $n(7) = 3$, $n(13) = 1$, and for $p \neq 7, 13$, $n(p)$ is equal to

$$\begin{array}{ccc} 0 & \textit{if} & p \equiv 1 \pmod{12} \\ 1 & \textit{if} & p \equiv 5 \pmod{12} \\ \nu(p) & \textit{if} & p \equiv -5 \pmod{12} \\ \nu(p)+1 & \textit{if} & p \equiv -1 \pmod{12}. \end{array}$$

Proof strategy

Proof strategy

A number of steps in the proof of the result of Breda-Nedela-Š depend on the primality of $-\chi$.

Proof strategy

A number of steps in the proof of the result of Breda-Nedela-Š depend on the primality of $-\chi$.

Oddness of $-\chi$ implies that the Sylow 2-subgroups of G are dihedral. This enables one to use the powerful result of Gorenstein and Walter:

Proof strategy

A number of steps in the proof of the result of Breda-Nedela-Š depend on the primality of $-\chi$.

Oddness of $-\chi$ implies that the Sylow 2-subgroups of G are dihedral. This enables one to use the powerful result of Gorenstein and Walter:

If G is a group with a dihedral Sylow 2-subgroup and if $O(G)$ is the (unique) maximal normal subgroup of G of odd order, then $G/O(G)$ is isomorphic to either

Proof strategy

A number of steps in the proof of the result of Breda-Nedela-Š depend on the primality of $-\chi$.

Oddness of $-\chi$ implies that the Sylow 2-subgroups of G are dihedral. This enables one to use the powerful result of Gorenstein and Walter:

If G is a group with a dihedral Sylow 2-subgroup and if $O(G)$ is the (unique) maximal normal subgroup of G of odd order, then $G/O(G)$ is isomorphic to either

(a) a Sylow 2-subgroup of G, or

Proof strategy

A number of steps in the proof of the result of Breda-Nedela-Š depend on the primality of $-\chi$.

Oddness of $-\chi$ implies that the Sylow 2-subgroups of G are dihedral. This enables one to use the powerful result of Gorenstein and Walter:

If G is a group with a dihedral Sylow 2-subgroup and if $O(G)$ is the (unique) maximal normal subgroup of G of odd order, then $G/O(G)$ is isomorphic to either

(a) a Sylow 2-subgroup of G, or

(b) the alternating group A_7, or

Proof strategy

A number of steps in the proof of the result of Breda-Nedela-Š depend on the primality of $-\chi$.

Oddness of $-\chi$ implies that the Sylow 2-subgroups of G are dihedral. This enables one to use the powerful result of Gorenstein and Walter:

If G is a group with a dihedral Sylow 2-subgroup and if $O(G)$ is the (unique) maximal normal subgroup of G of odd order, then $G/O(G)$ is isomorphic to either

(a) a Sylow 2-subgroup of G, or

(b) the alternating group A_7, or

(c) a subgroup of $Aut(PSL(2,q))$ containing $PSL(2,q)$, q odd.

Proof strategy

A number of steps in the proof of the result of Breda-Nedela-Š depend on the primality of $-\chi$.

Oddness of $-\chi$ implies that the Sylow 2-subgroups of G are dihedral. This enables one to use the powerful result of Gorenstein and Walter:

If G is a group with a dihedral Sylow 2-subgroup and if $O(G)$ is the (unique) maximal normal subgroup of G of odd order, then $G/O(G)$ is isomorphic to either

(a) a Sylow 2-subgroup of G, or

(b) the alternating group A_7, or

(c) a subgroup of $Aut(PSL(2, q))$ containing $PSL(2, q)$, q odd.

We then derive enough information to identify $O(G)$ and then G itself.

State-of-the-art around 2006

- Classification of regular maps of non-orientable genus $p+2$, p prime:

State-of-the-art around 2006

- Classification of regular maps of non-orientable genus $p+2$, p prime: Breda, Nedela and Š 05

State-of-the-art around 2006

- Classification of regular maps of non-orientable genus $p+2$, p prime: Breda, Nedela and Š 05
- An independent proof, extended to cover classification of regular hypermaps of genus $p+2$:

State-of-the-art around 2006

- Classification of regular maps of non-orientable genus $p+2$, p prime:
  Breda, Nedela and Š 05
- An independent proof, extended to cover classification of regular hypermaps of genus $p+2$:
  Jones 03

State-of-the-art around 2006

- Classification of regular maps of non-orientable genus $p+2$, p prime:
  Breda, Nedela and Š 05
- An independent proof, extended to cover classification of regular hypermaps of genus $p+2$:
  Jones 03
- Classification of regular maps of orientable genus $p+1$ with 'large' automorphism groups (of order greater than $6(g-1)$):

State-of-the-art around 2006

- Classification of regular maps of non-orientable genus $p+2$, p prime:
  Breda, Nedela and Š 05
- An independent proof, extended to cover classification of regular hypermaps of genus $p+2$:
  Jones 03
- Classification of regular maps of orientable genus $p+1$ with 'large' automorphism groups (of order greater than $6(g-1)$):
  Belolipetsky and Jones 05

State-of-the-art around 2006

- Classification of regular maps of non-orientable genus $p+2$, p prime:
  Breda, Nedela and Š 05
- An independent proof, extended to cover classification of regular hypermaps of genus $p+2$:
  Jones 03
- Classification of regular maps of orientable genus $p+1$
  with 'large' automorphism groups (of order greater than $6(g-1)$):
  Belolipetsky and Jones 05
  involves three families of chiral maps

State-of-the-art around 2006

- Classification of regular maps of non-orientable genus $p+2$, p prime:
  Breda, Nedela and Š 05
- An independent proof, extended to cover classification of regular hypermaps of genus $p+2$:
  Jones 03
- Classification of regular maps of orientable genus $p+1$ with 'large' automorphism groups (of order greater than $6(g-1)$):
  Belolipetsky and Jones 05
  involves three families of chiral maps
- Classification of small cases carried over to $g \leq 100$ $h \leq 200$ with the help of more powerful computational methods:

State-of-the-art around 2006

- Classification of regular maps of non-orientable genus $p+2$, p prime:
  Breda, Nedela and Š 05
- An independent proof, extended to cover classification of regular hypermaps of genus $p+2$:
  Jones 03
- Classification of regular maps of orientable genus $p+1$
  with 'large' automorphism groups (of order greater than $6(g-1)$):
  Belolipetsky and Jones 05
  involves three families of chiral maps
- Classification of small cases carried over to $g \leq 100$ $h \leq 200$ with the help of more powerful computational methods:
  Conder 07

The new list of M. Conder has generated a number of open questions.

Gaps in the genus spectra and degeneracy

Gaps in the genus spectra and degeneracy

Well known: For every $g > 0$ there exists a reflexible regular map on an orientable surface of genus g

Gaps in the genus spectra and degeneracy

Well known: For every $g > 0$ there exists a reflexible regular map on an orientable surface of genus g – e.g., of type $\{4g, 4g\}$.

Gaps in the genus spectra and degeneracy

Well known: For every $g > 0$ there exists a reflexible regular map on an orientable surface of genus g – e.g., of type $\{4g, 4g\}$.

No ‘gaps’ in the genus spectrum of orientable surfaces carrying reflexible regular maps.

Gaps in the genus spectra and degeneracy

Well known: For every $g > 0$ there exists a reflexible regular map on an orientable surface of genus g – e.g., of type $\{4g, 4g\}$.

No 'gaps' in the genus spectrum of orientable surfaces carrying reflexible regular maps. However, their underlying graphs are highly degenerate.

Gaps in the genus spectra and degeneracy

Well known: For every $g > 0$ there exists a reflexible regular map on an orientable surface of genus g – e.g., of type $\{4g, 4g\}$.

No 'gaps' in the genus spectrum of orientable surfaces carrying reflexible regular maps. However, their underlying graphs are highly degenerate.

There is no analogous family for non-orientable surfaces.

Gaps in the genus spectra and degeneracy

Well known: For every $g > 0$ there exists a reflexible regular map on an orientable surface of genus g – e.g., of type $\{4g, 4g\}$.

No 'gaps' in the genus spectrum of orientable surfaces carrying reflexible regular maps. However, their underlying graphs are highly degenerate.

There is no analogous family for non-orientable surfaces. In fact, by the result of Breda, Nedela, Š there are no regular maps on surfaces of genus $p + 2$ for all primes $p \equiv 1 \bmod 12$ and $p \neq 13$.

Gaps in the genus spectra and degeneracy

Well known: For every $g > 0$ there exists a reflexible regular map on an orientable surface of genus g – e.g., of type $\{4g, 4g\}$.

No 'gaps' in the genus spectrum of orientable surfaces carrying reflexible regular maps. However, their underlying graphs are highly degenerate.

There is no analogous family for non-orientable surfaces. In fact, by the result of Breda, Nedela, Š there are no regular maps on surfaces of genus $p + 2$ for all primes $p \equiv 1 \bmod 12$ and $p \neq 13$.

On the other hand, Conder and Everitt proved that non-orientable surfaces of more than 75 per cent of all genera carry some regular map.)

Gaps in the genus spectra and degeneracy

Well known: For every $g > 0$ there exists a reflexible regular map on an orientable surface of genus g – e.g., of type $\{4g, 4g\}$.

No 'gaps' in the genus spectrum of orientable surfaces carrying reflexible regular maps. However, their underlying graphs are highly degenerate.

There is no analogous family for non-orientable surfaces. In fact, by the result of Breda, Nedela, Š there are no regular maps on surfaces of genus $p + 2$ for all primes $p \equiv 1 \bmod 12$ and $p \neq 13$.

On the other hand, Conder and Everitt proved that non-orientable surfaces of more than 75 per cent of all genera carry some regular map.)

Similar questions about the genus spectra of:

Gaps in the genus spectra and degeneracy

Well known: For every $g > 0$ there exists a reflexible regular map on an orientable surface of genus g – e.g., of type $\{4g, 4g\}$.

No 'gaps' in the genus spectrum of orientable surfaces carrying reflexible regular maps. However, their underlying graphs are highly degenerate.

There is no analogous family for non-orientable surfaces. In fact, by the result of Breda, Nedela, Š there are no regular maps on surfaces of genus $p + 2$ for all primes $p \equiv 1 \bmod 12$ and $p \neq 13$.

On the other hand, Conder and Everitt proved that non-orientable surfaces of more than 75 per cent of all genera carry some regular map.)

Similar questions about the genus spectra of:

- regular but chiral orientable maps,

Gaps in the genus spectra and degeneracy

Well known: For every $g > 0$ there exists a reflexible regular map on an orientable surface of genus g – e.g., of type $\{4g, 4g\}$.

No 'gaps' in the genus spectrum of orientable surfaces carrying reflexible regular maps. However, their underlying graphs are highly degenerate.

There is no analogous family for non-orientable surfaces. In fact, by the result of Breda, Nedela, Š there are no regular maps on surfaces of genus $p+2$ for all primes $p \equiv 1 \bmod 12$ and $p \neq 13$.

On the other hand, Conder and Everitt proved that non-orientable surfaces of more than 75 per cent of all genera carry some regular map.)

Similar questions about the genus spectra of:

- regular but chiral orientable maps,
- surfaces supporting only degenerate regular maps.

Regular maps on a given surface – two extreme cases

Regular maps on a given surface – two extreme cases

Let G be the automorphism groups of a regular orientable map of type $\{m, k\}$ of genus g.

Regular maps on a given surface – two extreme cases

Let G be the automorphism groups of a regular orientable map of type $\{m, k\}$ of genus g. Euler's formula gives:

Regular maps on a given surface – two extreme cases

Let G be the automorphism groups of a regular orientable map of type $\{m, k\}$ of genus g. Euler's formula gives:

$$|G|(km - 2k - 2m) = 4km(g - 1)$$

Regular maps on a given surface – two extreme cases

Let G be the automorphism groups of a regular orientable map of type $\{m, k\}$ of genus g. Euler's formula gives:

$$|G|(km - 2k - 2m) = 4km(g - 1)$$

Two extreme cases:

Regular maps on a given surface – two extreme cases

Let G be the automorphism groups of a regular orientable map of type $\{m, k\}$ of genus g. Euler's formula gives:

$$|G|(km - 2k - 2m) = 4km(g - 1)$$

Two extreme cases:

- $g - 1$ divides $|G|$;

Regular maps on a given surface – two extreme cases

Let G be the automorphism groups of a regular orientable map of type $\{m, k\}$ of genus g. Euler's formula gives:

$$|G|(km - 2k - 2m) = 4km(g - 1)$$

Two extreme cases:

- $g - 1$ divides $|G|$;
- $g - 1$ and $|G|$ are relatively prime.

Regular maps on a given surface – two extreme cases

Let G be the automorphism groups of a regular orientable map of type $\{m, k\}$ of genus g. Euler's formula gives:

$$|G|(km - 2k - 2m) = 4km(g - 1)$$

Two extreme cases:

- $g - 1$ divides $|G|$;
- $g - 1$ and $|G|$ are relatively prime.

Similarly, if G is the automorphism groups of a reflexible regular map of type $\{m, k\}$ of Euler characteristic χ

Regular maps on a given surface – two extreme cases

Let G be the automorphism groups of a regular orientable map of type $\{m, k\}$ of genus g. Euler's formula gives:

$$|G|(km - 2k - 2m) = 4km(g - 1)$$

Two extreme cases:

- $g - 1$ divides $|G|$;
- $g - 1$ and $|G|$ are relatively prime.

Similarly, if G is the automorphism groups of a reflexible regular map of type $\{m, k\}$ of Euler characteristic χ (recall: $\chi = 2 - 2g$ or $\chi = 2 - h$):

Regular maps on a given surface – two extreme cases

Let G be the automorphism groups of a regular orientable map of type $\{m, k\}$ of genus g. Euler's formula gives:

$$|G|(km - 2k - 2m) = 4km(g - 1)$$

Two extreme cases:

- $g - 1$ divides $|G|$;
- $g - 1$ and $|G|$ are relatively prime.

Similarly, if G is the automorphism groups of a reflexible regular map of type $\{m, k\}$ of Euler characteristic χ (recall: $\chi = 2 - 2g$ or $\chi = 2 - h$):

$$|G|(km - 2k - 2m) = 4km(-\chi)$$

Regular maps on a given surface – two extreme cases

Let G be the automorphism groups of a regular orientable map of type $\{m, k\}$ of genus g. Euler's formula gives:

$$|G|(km - 2k - 2m) = 4km(g - 1)$$

Two extreme cases:

- $g - 1$ divides $|G|$;
- $g - 1$ and $|G|$ are relatively prime.

Similarly, if G is the automorphism groups of a reflexible regular map of type $\{m, k\}$ of Euler characteristic χ (recall: $\chi = 2 - 2g$ or $\chi = 2 - h$):

$$|G|(km - 2k - 2m) = 4km(-\chi)$$

and the two extreme cases are

Regular maps on a given surface – two extreme cases

Let G be the automorphism groups of a regular orientable map of type $\{m, k\}$ of genus g. Euler's formula gives:

$$|G|(km - 2k - 2m) = 4km(g - 1)$$

Two extreme cases:

- $g - 1$ divides $|G|$;
- $g - 1$ and $|G|$ are relatively prime.

Similarly, if G is the automorphism groups of a reflexible regular map of type $\{m, k\}$ of Euler characteristic χ (recall: $\chi = 2 - 2g$ or $\chi = 2 - h$):

$$|G|(km - 2k - 2m) = 4km(-\chi)$$

and the two extreme cases are

- χ divides $|G|$;

Regular maps on a given surface – two extreme cases

Let G be the automorphism groups of a regular orientable map of type $\{m, k\}$ of genus g. Euler's formula gives:

$$|G|(km - 2k - 2m) = 4km(g - 1)$$

Two extreme cases:

- $g - 1$ divides $|G|$;
- $g - 1$ and $|G|$ are relatively prime.

Similarly, if G is the automorphism groups of a reflexible regular map of type $\{m, k\}$ of Euler characteristic χ (recall: $\chi = 2 - 2g$ or $\chi = 2 - h$):

$$|G|(km - 2k - 2m) = 4km(-\chi)$$

and the two extreme cases are

- χ divides $|G|$;
- χ and $|G|$ are relatively prime.

Summary of results of Conder-Š-Tucker

The case when $g-1$ divides $|\mathrm{Aut}^o(M)|$

Summary of results of Conder-Š-Tucker

The case when $g-1$ divides $|\mathrm{Aut}^o(M)|$

We have classified all regular orientable maps M *of genus* $g>1$ *such that* $g-1$ *is a prime dividing* $|\mathrm{Aut}^o(M)|$.

Summary of results of Conder-Š-Tucker

The case when $g-1$ divides $|\mathrm{Aut}^o(M)|$

We have classified all regular orientable maps M *of genus* $g>1$ *such that* $g-1$ *is a prime dividing* $|\mathrm{Aut}^o(M)|$.

- Three infinite families of (chiral) maps (Belolipetsky-Jones)

Summary of results of Conder-Š-Tucker

The case when $g-1$ divides $|\mathrm{Aut}^o(M)|$

We have classified all regular orientable maps M of genus $g>1$ such that $g-1$ is a prime dividing $|\mathrm{Aut}^o(M)|$.

- Three infinite families of (chiral) maps (Belolipetsky-Jones)

The case when $g-1$ and $|\mathrm{Aut}^o(M)|$ are relatively prime

Summary of results of Conder-Š-Tucker

The case when $g-1$ divides $|\mathrm{Aut}^o(M)|$

We have classified all regular orientable maps M *of genus* $g>1$ *such that* $g-1$ *is a prime dividing* $|\mathrm{Aut}^o(M)|$*.*

- Three infinite families of (chiral) maps (Belolipetsky-Jones)

The case when $g-1$ and $|\mathrm{Aut}^o(M)|$ are relatively prime

We have taken a major step forward by producing a classification of all regular orientable maps M *of genus* g *for which* $g-1$ *and* $|\mathrm{Aut}^o(M)|$ *are relatively prime.*

Summary of results of Conder-Š-Tucker

The case when $g-1$ divides $|\mathrm{Aut}^o(M)|$

We have classified all regular orientable maps M of genus $g>1$ such that $g-1$ is a prime dividing $|\mathrm{Aut}^o(M)|$.

- Three infinite families of (chiral) maps (Belolipetsky-Jones)

The case when $g-1$ and $|\mathrm{Aut}^o(M)|$ are relatively prime

We have taken a major step forward by producing a classification of all regular orientable maps M of genus g for which $g-1$ and $|\mathrm{Aut}^o(M)|$ are relatively prime.

- Seven infinite families of maps

Summary of results of Conder-Š-Tucker

The case when $g - 1$ divides $|\mathrm{Aut}^o(M)|$

We have classified all regular orientable maps M of genus $g > 1$ such that $g - 1$ is a prime dividing $|\mathrm{Aut}^o(M)|$.

- Three infinite families of (chiral) maps (Belolipetsky-Jones)

The case when $g - 1$ and $|\mathrm{Aut}^o(M)|$ are relatively prime

We have taken a major step forward by producing a classification of all regular orientable maps M of genus g for which $g - 1$ and $|\mathrm{Aut}^o(M)|$ are relatively prime.

- Seven infinite families of maps

As a consequence of these, we have obtained *a complete classification of all regular orientable maps of genus $p + 1$ where p is prime.*

Specific corollaries include:

Specific corollaries include:

(1) If p is a prime such that $p-1$ is not divisible by 3, 5 or 8, then every regular orientable map of genus $g = p + 1$ is reflexible;

Specific corollaries include:

(1) If p is a prime such that $p-1$ is not divisible by 3, 5 or 8, then every regular orientable map of genus $g = p+1$ is reflexible;

(2) If M is a regular but chiral map of genus $g = p+1$, where p is prime, and $p-1$ is not divisible by 5 or 8, then either M or its dual has multiple edges;

Specific corollaries include:

(1) If p is a prime such that $p-1$ is not divisible by 3, 5 or 8, then every regular orientable map of genus $g = p+1$ is reflexible;

(2) If M is a regular but chiral map of genus $g = p+1$, where p is prime, and $p-1$ is not divisible by 5 or 8, then either M or its dual has multiple edges;

(3) If M is a reflexible regular map of orientable genus $g = p+1$, where p is prime and $p > 13$, then either M or its dual has multiple edges, and if $p \equiv 1 \mod 6$, then both M and its dual have multiple edges.

Specific corollaries include:

(1) If p is a prime such that $p-1$ is not divisible by 3, 5 or 8, then every regular orientable map of genus $g = p+1$ is reflexible;

(2) If M is a regular but chiral map of genus $g = p+1$, where p is prime, and $p-1$ is not divisible by 5 or 8, then either M or its dual has multiple edges;

(3) If M is a reflexible regular map of orientable genus $g = p+1$, where p is prime and $p > 13$, then either M or its dual has multiple edges, and if $p \equiv 1 \bmod 6$, then both M and its dual have multiple edges.

Infinitely many gaps in the spectrum of chiral regular maps

Specific corollaries include:

(1) If p is a prime such that $p-1$ is not divisible by 3, 5 or 8, then every regular orientable map of genus $g = p+1$ is reflexible;

(2) If M is a regular but chiral map of genus $g = p+1$, where p is prime, and $p-1$ is not divisible by 5 or 8, then either M or its dual has multiple edges;

(3) If M is a reflexible regular map of orientable genus $g = p+1$, where p is prime and $p > 13$, then either M or its dual has multiple edges, and if $p \equiv 1 \bmod 6$, then both M and its dual have multiple edges.

Infinitely many gaps in the spectrum of chiral regular maps and

Specific corollaries include:

(1) If p is a prime such that $p-1$ is not divisible by 3, 5 or 8, then every regular orientable map of genus $g = p+1$ is reflexible;

(2) If M is a regular but chiral map of genus $g = p+1$, where p is prime, and $p-1$ is not divisible by 5 or 8, then either M or its dual has multiple edges;

(3) If M is a reflexible regular map of orientable genus $g = p+1$, where p is prime and $p > 13$, then either M or its dual has multiple edges, and if $p \equiv 1 \bmod 6$, then both M and its dual have multiple edges.

Infinitely many gaps in the spectrum of chiral regular maps and in the spectrum of reflexible regular orientable maps with *simple* underlying graphs.

Specific corollaries include:

(1) If p is a prime such that $p-1$ is not divisible by 3, 5 or 8, then every regular orientable map of genus $g = p+1$ is reflexible;

(2) If M is a regular but chiral map of genus $g = p+1$, where p is prime, and $p-1$ is not divisible by 5 or 8, then either M or its dual has multiple edges;

(3) If M is a reflexible regular map of orientable genus $g = p+1$, where p is prime and $p > 13$, then either M or its dual has multiple edges, and if $p \equiv 1 \bmod 6$, then both M and its dual have multiple edges.

Infinitely many gaps in the spectrum of chiral regular maps and in the spectrum of reflexible regular orientable maps with *simple* underlying graphs.

Another consequence: A new proof of the classification result of Breda, Nedela, Š for regular maps on surfaces of genus $p+2$ for odd primes p.

Specific corollaries include:

(1) If p is a prime such that $p-1$ is not divisible by 3, 5 or 8, then every regular orientable map of genus $g = p + 1$ is reflexible;

(2) If M is a regular but chiral map of genus $g = p + 1$, where p is prime, and $p - 1$ is not divisible by 5 or 8, then either M or its dual has multiple edges;

(3) If M is a reflexible regular map of orientable genus $g = p + 1$, where p is prime and $p > 13$, then either M or its dual has multiple edges, and if $p \equiv 1 \bmod 6$, then both M and its dual have multiple edges.

Infinitely many gaps in the spectrum of chiral regular maps and in the spectrum of reflexible regular orientable maps with *simple* underlying graphs.

Another consequence: A new proof of the classification result of Breda, Nedela, Š for regular maps on surfaces of genus $p + 2$ for odd primes p.

Also: Characterization of regular maps of Euler characteristic $-2p$.

Regular maps of Euler characteristic $-3p$

Regular maps of Euler characteristic $-3p$

A $(k, m, 2)$-group is any group with presentation

Regular maps of Euler characteristic $-3p$

A $(k, m, 2)$-group is any group with presentation
$\langle (x, y, z),\ (yz)^k = (zx)^m = \ldots = 1 \rangle$

Regular maps of Euler characteristic $-3p$

A $(k, m, 2)$-group is any group with presentation

$\langle (x, y, z),\ (yz)^k = (zx)^m = \ldots = 1 \rangle$

where, as usual, exponents are true orders.

Regular maps of Euler characteristic $-3p$

A $(k, m, 2)$-group is any group with presentation

$\langle (x, y, z),\ (yz)^k = (zx)^m = \ldots = 1 \rangle$

where, as usual, exponents are true orders.

The Euler characteristic of a finite $(k, m, 2)$-group G is

Regular maps of Euler characteristic $-3p$

A $(k, m, 2)$-group is any group with presentation
$\langle (x, y, z),\ (yz)^k = (zx)^m = \ldots = 1 \rangle$
where, as usual, exponents are true orders.

The Euler characteristic of a finite $(k, m, 2)$-group G is
$|G|(1/2k + 1/2m - 1/4)$,

Regular maps of Euler characteristic $-3p$

A $(k, m, 2)$-group is any group with presentation
$\langle (x, y, z),\ (yz)^k = (zx)^m = \ldots = 1\rangle$
where, as usual, exponents are true orders.

The Euler characteristic of a finite $(k, m, 2)$-group G is
$|G|(1/2k + 1/2m - 1/4)$, which is χ of the corresponding regular map.

Regular maps of Euler characteristic $-3p$

A $(k, m, 2)$-group is any group with presentation
$\langle (x, y, z),\ (yz)^k = (zx)^m = \ldots = 1 \rangle$
where, as usual, exponents are true orders.

The Euler characteristic of a finite $(k, m, 2)$-group G is
$|G|(1/2k + 1/2m - 1/4)$, which is χ of the corresponding regular map.

It turns out that all $(k, m, 2)$-groups with $\chi = -3p$ have 'reasonably' bounded order for p large enough:

Regular maps of Euler characteristic $-3p$

A $(k, m, 2)$-group is any group with presentation

$\langle (x, y, z),\ (yz)^k = (zx)^m = \ldots = 1 \rangle$

where, as usual, exponents are true orders.

The Euler characteristic of a finite $(k, m, 2)$-group G is $|G|(1/2k + 1/2m - 1/4)$, which is χ of the corresponding regular map.

It turns out that all $(k, m, 2)$-groups with $\chi = -3p$ have 'reasonably' bounded order for p large enough:

Proposition. *Let G be a $(k, m, 2)$-group with $\chi = -3p$. If $p > 53$, then $km - 2k - 2m = tp$ and $|G| = 12km/t$ for some $t \in \{2, 4, 12\}$.*

Regular maps of Euler characteristic $-3p$

A $(k, m, 2)$-group is any group with presentation
$\langle (x, y, z),\ (yz)^k = (zx)^m = \ldots = 1 \rangle$
where, as usual, exponents are true orders.

The Euler characteristic of a finite $(k, m, 2)$-group G is
$|G|(1/2k + 1/2m - 1/4)$, which is χ of the corresponding regular map.

It turns out that all $(k, m, 2)$-groups with $\chi = -3p$ have ‘reasonably’ bounded order for p large enough:

Proposition. *Let G be a $(k, m, 2)$-group with $\chi = -3p$. If $p > 53$, then $km - 2k - 2m = tp$ and $|G| = 12km/t$ for some $t \in \{2, 4, 12\}$.*

The proof uses Gorenstein-Walter

Regular maps of Euler characteristic $-3p$

A $(k, m, 2)$-group is any group with presentation
$\langle (x, y, z),\ (yz)^k = (zx)^m = \ldots = 1 \rangle$
where, as usual, exponents are true orders.

The Euler characteristic of a finite $(k, m, 2)$-group G is
$|G|(1/2k + 1/2m - 1/4)$, which is χ of the corresponding regular map.

It turns out that all $(k, m, 2)$-groups with $\chi = -3p$ have ‘reasonably’ bounded order for p large enough:

Proposition. *Let G be a $(k, m, 2)$-group with $\chi = -3p$. If $p > 53$, then $km - 2k - 2m = tp$ and $|G| = 12km/t$ for some $t \in \{2, 4, 12\}$.*

The proof uses Gorenstein-Walter and arguments showing that regular maps with $\chi = -3$ do not have normal p-covers.

Regular maps of Euler characteristic $-3p$

A $(k,m,2)$-group is any group with presentation
$\langle (x,y,z),\ (yz)^k = (zx)^m = \ldots = 1 \rangle$
where, as usual, exponents are true orders.

The Euler characteristic of a finite $(k,m,2)$-group G is
$|G|(1/2k + 1/2m - 1/4)$, which is χ of the corresponding regular map.

It turns out that all $(k,m,2)$-groups with $\chi = -3p$ have 'reasonably' bounded order for p large enough:

Proposition. *Let G be a $(k,m,2)$-group with $\chi = -3p$. If $p > 53$, then $km - 2k - 2m = tp$ and $|G| = 12km/t$ for some $t \in \{2,4,12\}$.*

The proof uses Gorenstein-Walter and arguments showing that regular maps with $\chi = -3$ do not have normal p-covers. Easy: $t \leq 12$.

Regular maps of Euler characteristic $-3p$

A $(k, m, 2)$-group is any group with presentation
$\langle (x, y, z),\ (yz)^k = (zx)^m = \ldots = 1 \rangle$
where, as usual, exponents are true orders.

The Euler characteristic of a finite $(k, m, 2)$-group G is
$|G|(1/2k + 1/2m - 1/4)$, which is χ of the corresponding regular map.

It turns out that all $(k, m, 2)$-groups with $\chi = -3p$ have 'reasonably' bounded order for p large enough:

Proposition. *Let G be a $(k, m, 2)$-group with $\chi = -3p$. If $p > 53$, then $km - 2k - 2m = tp$ and $|G| = 12km/t$ for some $t \in \{2, 4, 12\}$.*

The proof uses Gorenstein-Walter and arguments showing that regular maps with $\chi = -3$ do not have normal p-covers. Easy: $t \leq 12$. Longer arguments are needed to exclude $t \in \{1, 3, 6, 8\}$.

The hard cases

The hard cases

If groups with $\chi = -3p$ were almost Sylow-cyclic,

The hard cases

If groups with $\chi = -3p$ were almost Sylow-cyclic, meaning cyclicity of all odd-order Sylow subgroups and dihedrality of the even-order ones,

The hard cases

If groups with $\chi = -3p$ were almost Sylow-cyclic, meaning cyclicity of all odd-order Sylow subgroups and dihedrality of the even-order ones, we could use the Conder-Potočnik-Š classification (with some extra work).

The hard cases

If groups with $\chi = -3p$ were almost Sylow-cyclic, meaning cyclicity of all odd-order Sylow subgroups and dihedrality of the even-order ones, we could use the Conder-Potočnik-Š classification (with some extra work). Unfortunately, in many of our cases the Sylow 3-subgroups are not cyclic.

The hard cases

If groups with $\chi = -3p$ were almost Sylow-cyclic, meaning cyclicity of all odd-order Sylow subgroups and dihedrality of the even-order ones, we could use the Conder-Potočnik-Š classification (with some extra work). Unfortunately, in many of our cases the Sylow 3-subgroups are not cyclic.

HC1: $t = 2$, that is, G is a $(k, m, 2)$-group of order $6km$.

The hard cases

If groups with $\chi = -3p$ were almost Sylow-cyclic, meaning cyclicity of all odd-order Sylow subgroups and dihedrality of the even-order ones, we could use the Conder-Potočnik-Š classification (with some extra work). Unfortunately, in many of our cases the Sylow 3-subgroups are not cyclic.

HC1: $t = 2$, that is, G is a $(k, m, 2)$-group of order $6km$.

From Gorenstein-Walter the case $G/O \cong PGL(2, 7)$ as a $(7, 8, 2)$-group survives...

The hard cases

If groups with $\chi = -3p$ were almost Sylow-cyclic, meaning cyclicity of all odd-order Sylow subgroups and dihedrality of the even-order ones, we could use the Conder-Potočnik-Š classification (with some extra work). Unfortunately, in many of our cases the Sylow 3-subgroups are not cyclic.

HC1: $t = 2$, that is, G is a $(k, m, 2)$-group of order $6km$.

From Gorenstein-Walter the case $G/O \cong PGL(2,7)$ as a $(7,8,2)$-group survives... but G fails to be almost Sylow-cyclic if 3 divides $|O|$.

HC2: $t = 4$, that is, G is a $(k, m, 2)$-group of order $3km$.

The hard cases

If groups with $\chi = -3p$ were almost Sylow-cyclic, meaning cyclicity of all odd-order Sylow subgroups and dihedrality of the even-order ones, we could use the Conder-Potočnik-Š classification (with some extra work). Unfortunately, in many of our cases the Sylow 3-subgroups are not cyclic.

HC1: $t = 2$, that is, G is a $(k, m, 2)$-group of order $6km$.

From Gorenstein-Walter the case $G/O \cong PGL(2,7)$ as a $(7,8,2)$-group survives... but G fails to be almost Sylow-cyclic if 3 divides $|O|$.

HC2: $t = 4$, that is, G is a $(k, m, 2)$-group of order $3km$.

In this case the Gorenstein-Walter survivor is $G/O \cong S_2$, a Sylow 2-subgroup of G.

The hard cases

If groups with $\chi = -3p$ were almost Sylow-cyclic, meaning cyclicity of all odd-order Sylow subgroups and dihedrality of the even-order ones, we could use the Conder-Potočnik-Š classification (with some extra work). Unfortunately, in many of our cases the Sylow 3-subgroups are not cyclic.

HC1: $t = 2$, that is, G is a $(k, m, 2)$-group of order $6km$.

From Gorenstein-Walter the case $G/O \cong PGL(2, 7)$ as a $(7, 8, 2)$-group survives... but G fails to be almost Sylow-cyclic if 3 divides $|O|$.

HC2: $t = 4$, that is, G is a $(k, m, 2)$-group of order $3km$.

In this case the Gorenstein-Walter survivor is $G/O \cong S_2$, a Sylow 2-subgroup of G. But here the Sylow 3-subgroups are not cyclic.

The hard cases

If groups with $\chi = -3p$ were almost Sylow-cyclic, meaning cyclicity of all odd-order Sylow subgroups and dihedrality of the even-order ones, we could use the Conder-Potočnik-Š classification (with some extra work). Unfortunately, in many of our cases the Sylow 3-subgroups are not cyclic.

HC1: $t = 2$, that is, G is a $(k, m, 2)$-group of order $6km$.

From Gorenstein-Walter the case $G/O \cong PGL(2, 7)$ as a $(7, 8, 2)$-group survives... but G fails to be almost Sylow-cyclic if 3 divides $|O|$.

HC2: $t = 4$, that is, G is a $(k, m, 2)$-group of order $3km$.

In this case the Gorenstein-Walter survivor is $G/O \cong S_2$, a Sylow 2-subgroup of G. But here the Sylow 3-subgroups are not cyclic.
Method: Reductions modulo a minimal odd-order normal subgroup.

The hard cases

If groups with $\chi = -3p$ were almost Sylow-cyclic, meaning cyclicity of all odd-order Sylow subgroups and dihedrality of the even-order ones, we could use the Conder-Potočnik-Š classification (with some extra work). Unfortunately, in many of our cases the Sylow 3-subgroups are not cyclic.

HC1: $t = 2$, that is, G is a $(k, m, 2)$-group of order $6km$.

From Gorenstein-Walter the case $G/O \cong PGL(2, 7)$ as a $(7, 8, 2)$-group survives... but G fails to be almost Sylow-cyclic if 3 divides $|O|$.

HC2: $t = 4$, that is, G is a $(k, m, 2)$-group of order $3km$.

In this case the Gorenstein-Walter survivor is $G/O \cong S_2$, a Sylow 2-subgroup of G. But here the Sylow 3-subgroups are not cyclic. Method: Reductions modulo a minimal odd-order normal subgroup.

$t = 12$, that is, $(k, m, 2)$-groups of order km:

The hard cases

If groups with $\chi = -3p$ were almost Sylow-cyclic, meaning cyclicity of all odd-order Sylow subgroups and dihedrality of the even-order ones, we could use the Conder-Potočnik-Š classification (with some extra work). Unfortunately, in many of our cases the Sylow 3-subgroups are not cyclic.

HC1: $t = 2$, that is, G is a $(k, m, 2)$-group of order $6km$.

From Gorenstein-Walter the case $G/O \cong PGL(2,7)$ as a $(7,8,2)$-group survives... but G fails to be almost Sylow-cyclic if 3 divides $|O|$.

HC2: $t = 4$, that is, G is a $(k, m, 2)$-group of order $3km$.

In this case the Gorenstein-Walter survivor is $G/O \cong S_2$, a Sylow 2-subgroup of G. But here the Sylow 3-subgroups are not cyclic.
Method: Reductions modulo a minimal odd-order normal subgroup.

$t = 12$, that is, $(k, m, 2)$-groups of order km: Easy!

The hard cases

If groups with $\chi = -3p$ were almost Sylow-cyclic, meaning cyclicity of all odd-order Sylow subgroups and dihedrality of the even-order ones, we could use the Conder-Potočnik-Š classification (with some extra work). Unfortunately, in many of our cases the Sylow 3-subgroups are not cyclic.

HC1: $t = 2$, that is, G is a $(k, m, 2)$-group of order $6km$.

From Gorenstein-Walter the case $G/O \cong PGL(2, 7)$ as a $(7, 8, 2)$-group survives... but G fails to be almost Sylow-cyclic if 3 divides $|O|$.

HC2: $t = 4$, that is, G is a $(k, m, 2)$-group of order $3km$.

In this case the Gorenstein-Walter survivor is $G/O \cong S_2$, a Sylow 2-subgroup of G. But here the Sylow 3-subgroups are not cyclic. Method: Reductions modulo a minimal odd-order normal subgroup.

$t = 12$, that is, $(k, m, 2)$-groups of order km: Easy! (Done before, BNŠ.)

The result

The result

Theorem. *Up to isomorphism and duality, any regular map with* $\chi = -3p$, $p > 53$, *belongs to one of the following families, listed by the groups* G*:*

The result

Theorem. *Up to isomorphism and duality, any regular map with* $\chi = -3p$, $p > 53$, *belongs to one of the following families, listed by the groups* G:

(a) *If* $p \equiv -8 \pmod{21}$ *and* $p \not\equiv -8 \pmod{49}$,

The result

Theorem. *Up to isomorphism and duality, any regular map with* $\chi = -3p$, $p > 53$, *belongs to one of the following families, listed by the groups* G*:*

(a) *If* $p \equiv -8 \pmod{21}$ *and* $p \not\equiv -8 \pmod{49}$, *then* G *is a* $((p+8)/3, 8, 2)$*-group isomorphic to one of the two extensions of* $Z_{(p+8)/21}$ *by* $PGL(2,7)$ *of order* $16(p+8)$*;*

The result

Theorem. *Up to isomorphism and duality, any regular map with* $\chi = -3p$, $p > 53$, *belongs to one of the following families, listed by the groups* G:

(a) *If* $p \equiv -8 \pmod{21}$ *and* $p \not\equiv -8 \pmod{49}$, *then* G *is a* $((p+8)/3, 8, 2)$*-group isomorphic to one of the two extensions of* $Z_{(p+8)/21}$ *by* $PGL(2,7)$ *of order* $16(p+8)$; *letting* $n = (p+8)/21$ *we have* $\langle (x,y,z),\ r^{7n} = s^8 = (rs)^2 = [x, r^7] = xr^2s^2r^{7i+1} = 1 \rangle$, $7i \equiv -3 \pmod{n}$

The result

Theorem. *Up to isomorphism and duality, any regular map with* $\chi = -3p$, $p > 53$, *belongs to one of the following families, listed by the groups* G:

(a) *If* $p \equiv -8 \pmod{21}$ *and* $p \not\equiv -8 \pmod{49}$, *then* G *is a* $((p+8)/3, 8, 2)$*-group isomorphic to one of the two extensions of* $Z_{(p+8)/21}$ *by* $PGL(2,7)$ *of order* $16(p+8)$; *letting* $n = (p+8)/21$ *we have*
$\langle (x,y,z),\ r^{7n} = s^8 = (rs)^2 = [x, r^7] = xr^2s^2r^{7i+1} = 1\rangle$, $7i \equiv -3 \pmod{n}$
$\langle (x,y,z),\ r^{7n} = s^8 = (rs)^2 = [x, r^7] = xsr^3s^3r^{7i+1} = 1\rangle$, $7i \equiv 2 \pmod{n}$.

The result

Theorem. *Up to isomorphism and duality, any regular map with* $\chi = -3p$, $p > 53$, *belongs to one of the following families, listed by the groups* G:

(a) *If* $p \equiv -8 \pmod{21}$ *and* $p \not\equiv -8 \pmod{49}$, *then* G *is a* $((p+8)/3, 8, 2)$*-group isomorphic to one of the two extensions of* $Z_{(p+8)/21}$ *by* $PGL(2,7)$ *of order* $16(p+8)$; *letting* $n = (p+8)/21$ *we have*

$\langle (x,y,z),\ r^{7n} = s^8 = (rs)^2 = [x, r^7] = xr^2s^2r^{7i+1} = 1 \rangle$, $7i \equiv -3 \pmod{n}$

$\langle (x,y,z),\ r^{7n} = s^8 = (rs)^2 = [x, r^7] = xsr^3s^3r^{7i+1} = 1 \rangle$, $7i \equiv 2 \pmod{n}$.

(b) If $p \equiv 1 \pmod{4}$, *then* G *is either one of the* $(2j, 2l, 2)$*-groups* $G_{j,l}$

The result

Theorem. *Up to isomorphism and duality, any regular map with* $\chi = -3p$, $p > 53$, *belongs to one of the following families, listed by the groups* G:

(a) *If* $p \equiv -8 \pmod{21}$ *and* $p \not\equiv -8 \pmod{49}$, *then* G *is a* $((p+8)/3, 8, 2)$*-group isomorphic to one of the two extensions of* $Z_{(p+8)/21}$ *by* $PGL(2,7)$ *of order* $16(p+8)$; *letting* $n = (p+8)/21$ *we have*

$\langle (x,y,z),\ r^{7n} = s^8 = (rs)^2 = [x, r^7] = xr^2s^2r^{7i+1} = 1\rangle$, $7i \equiv -3 \pmod{n}$

$\langle (x,y,z),\ r^{7n} = s^8 = (rs)^2 = [x, r^7] = xsr^3s^3r^{7i+1} = 1\rangle$, $7i \equiv 2 \pmod{n}$.

(b) If $p \equiv 1 \pmod{4}$, *then* G *is either one of the* $(2j, 2l, 2)$*-groups* $G_{j,l}$

$\langle (x,y,z),\ r^{2j} = s^{2l} = (rs)^2 = (rs^{-1})^2 = 1\rangle \cong D_j \times D_l$ *of order* $4jl$,

The result

Theorem. *Up to isomorphism and duality, any regular map with* $\chi = -3p$, $p > 53$, *belongs to one of the following families, listed by the groups* G:

(a) *If* $p \equiv -8 \pmod{21}$ *and* $p \not\equiv -8 \pmod{49}$, *then* G *is a* $((p+8)/3, 8, 2)$*-group isomorphic to one of the two extensions of* $Z_{(p+8)/21}$ *by* $PGL(2,7)$ *of order* $16(p+8)$; *letting* $n = (p+8)/21$ *we have*
$\langle (x,y,z),\ r^{7n} = s^8 = (rs)^2 = [x, r^7] = xr^2s^2r^{7i+1} = 1\rangle$, $7i \equiv -3 \pmod{n}$
$\langle (x,y,z),\ r^{7n} = s^8 = (rs)^2 = [x, r^7] = xsr^3s^3r^{7i+1} = 1\rangle$, $7i \equiv 2 \pmod{n}$.

(b) If $p \equiv 1 \pmod 4$, *then* G *is either one of the* $(2j, 2l, 2)$*-groups* $G_{j,l}$
$\langle (x,y,z),\ r^{2j} = s^{2l} = (rs)^2 = (rs^{-1})^2 = 1\rangle \cong D_j \times D_l$ *of order* $4jl$, *where* $j \geq l \geq 3$, *both* j, l *are odd*, $(j,l) \leq 3$, $(j-1)(l-1) = 3p+1$, *and* $j \equiv l \not\equiv 1 \pmod 3$,

The result

Theorem. *Up to isomorphism and duality, any regular map with* $\chi = -3p$, $p > 53$, *belongs to one of the following families, listed by the groups* G:

(a) *If* $p \equiv -8 \pmod{21}$ *and* $p \not\equiv -8 \pmod{49}$, *then* G *is a* $((p+8)/3, 8, 2)$*-group isomorphic to one of the two extensions of* $Z_{(p+8)/21}$ *by* $PGL(2,7)$ *of order* $16(p+8)$; *letting* $n = (p+8)/21$ *we have*

$\langle (x,y,z),\ r^{7n} = s^8 = (rs)^2 = [x, r^7] = xr^2s^2r^{7i+1} = 1\rangle$, $7i \equiv -3 \pmod{n}$

$\langle (x,y,z),\ r^{7n} = s^8 = (rs)^2 = [x, r^7] = xsr^3s^3r^{7i+1} = 1\rangle$, $7i \equiv 2 \pmod{n}$.

(b) *If* $p \equiv 1 \pmod{4}$, *then* G *is either one of the* $(2j, 2l, 2)$*-groups* $G_{j,l}$ $\langle (x,y,z),\ r^{2j} = s^{2l} = (rs)^2 = (rs^{-1})^2 = 1\rangle \cong D_j \times D_l$ *of order* $4jl$, *where* $j \geq l \geq 3$, *both* j, l *are odd*, $(j,l) \leq 3$, $(j-1)(l-1) = 3p+1$, *and* $j \equiv l \not\equiv 1 \pmod{3}$, *or one of the* $(6, 2l, 2)$*-groups* G_l *with presentation* $\langle (x,y,z),\ r^6 = s^{2l} = (rs)^2 = r^2s^2r^2s^{-2} = 1\rangle \cong (D_3 \times D_l).Z_3$

The result

Theorem. *Up to isomorphism and duality, any regular map with* $\chi = -3p$, $p > 53$, *belongs to one of the following families, listed by the groups* G:

(a) *If* $p \equiv -8 \pmod{21}$ *and* $p \not\equiv -8 \pmod{49}$, *then* G *is a* $((p+8)/3, 8, 2)$*-group isomorphic to one of the two extensions of* $Z_{(p+8)/21}$ *by* $PGL(2,7)$ *of order* $16(p+8)$; *letting* $n = (p+8)/21$ *we have*
$\langle (x,y,z),\ r^{7n} = s^8 = (rs)^2 = [x, r^7] = xr^2s^2r^{7i+1} = 1\rangle$, $7i \equiv -3 \pmod{n}$
$\langle (x,y,z),\ r^{7n} = s^8 = (rs)^2 = [x, r^7] = xsr^3s^3r^{7i+1} = 1\rangle$, $7i \equiv 2 \pmod{n}$.

(b) If $p \equiv 1 \pmod{4}$, *then* G *is either one of the* $(2j, 2l, 2)$*-groups* $G_{j,l}$
$\langle (x,y,z),\ r^{2j} = s^{2l} = (rs)^2 = (rs^{-1})^2 = 1\rangle \cong D_j \times D_l$ *of order* $4jl$, *where* $j \geq l \geq 3$, *both* j, l *are odd,* $(j,l) \leq 3$, $(j-1)(l-1) = 3p+1$, *and* $j \equiv l \not\equiv 1 \pmod{3}$, *or one of the* $(6, 2l, 2)$*-groups* G_l *with presentation*
$\langle (x,y,z),\ r^6 = s^{2l} = (rs)^2 = r^2s^2r^2s^{-2} = 1\rangle \cong (D_3 \times D_l).Z_3$
of order $36l$, *where* $l \equiv 2 \pmod{4}$ *and* $2l - 3 = p$.

Methods

Combinatorial group theory in conjunction with classical theorems:

Methods

Combinatorial group theory in conjunction with classical theorems:

Schur-Zassenhaus: *If N is a normal subgroup of the finite group G, such that the order $|N|$ and the index $|G : N|$ are coprime, then G contains a subgroup of order $|G : N|$, and any two such subgroups are conjugate in G.*

Methods

Combinatorial group theory in conjunction with classical theorems:

Schur-Zassenhaus: *If N is a normal subgroup of the finite group G, such that the order $|N|$ and the index $|G:N|$ are coprime, then G contains a subgroup of order $|G:N|$, and any two such subgroups are conjugate in G.*

Schur: *If$Z(G)$ has finite index m in G, then the commutator subgroup G' is finite and the order of every element of G' divides m.*

Methods

Combinatorial group theory in conjunction with classical theorems:

Schur-Zassenhaus: *If N is a normal subgroup of the finite group G, such that the order $|N|$ and the index $|G:N|$ are coprime, then G contains a subgroup of order $|G:N|$, and any two such subgroups are conjugate in G.*

Schur: *If$Z(G)$ has finite index m in G, then the commutator subgroup G' is finite and the order of every element of G' divides m.*

Ito: *If the group G is expressible as AB where A and B are abelian subgroups of G, then the commutator subgroup G' is abelian.*

Methods

Combinatorial group theory in conjunction with classical theorems:

Schur-Zassenhaus: *If N is a normal subgroup of the finite group G, such that the order $|N|$ and the index $|G:N|$ are coprime, then G contains a subgroup of order $|G:N|$, and any two such subgroups are conjugate in G.*

Schur: *If$Z(G)$ has finite index m in G, then the commutator subgroup G' is finite and the order of every element of G' divides m.*

Ito: *If the group G is expressible as AB where A and B are abelian subgroups of G, then the commutator subgroup G' is abelian.*

Suzuki and Wong: *Let G be a finite, non-solvable, almost Sylow-cyclic group. Then G has an index-2-subgroup $G_0 \cong H \times L$, where $H \cong Z_u \rtimes Z_v$, and $L \cong \mathrm{SL}(2,q)$ or $\mathrm{PSL}(2,q)$ for some prime $q > 3$, with u, v and $|L|$ being pairwise relatively prime.*

Methods

Combinatorial group theory in conjunction with classical theorems:

Schur-Zassenhaus: *If N is a normal subgroup of the finite group G, such that the order $|N|$ and the index $|G:N|$ are coprime, then G contains a subgroup of order $|G:N|$, and any two such subgroups are conjugate in G.*

Schur: *If$Z(G)$ has finite index m in G, then the commutator subgroup G' is finite and the order of every element of G' divides m.*

Ito: *If the group G is expressible as AB where A and B are abelian subgroups of G, then the commutator subgroup G' is abelian.*

Suzuki and Wong: *Let G be a finite, non-solvable, almost Sylow-cyclic group. Then G has an index-2-subgroup $G_0 \cong H \times L$, where $H \cong Z_u \rtimes Z_v$, and $L \cong \mathrm{SL}(2,q)$ or $\mathrm{PSL}(2,q)$ for some prime $q > 3$, with u, v and $|L|$ being pairwise relatively prime.*

... and, of course, the Gorenstein-Walter theorem ...

Directions of future research

Directions of future research

- Almost-Sylow-cyclicity of our groups failed for Sylow 3-subgroups

Directions of future research

- Almost-Sylow-cyclicity of our groups failed for Sylow 3-subgroups – but these were Abelian.

Directions of future research

- Almost-Sylow-cyclicity of our groups failed for Sylow 3-subgroups – but these were Abelian. Is this perhaps worth pursuing?

Directions of future research

- Almost-Sylow-cyclicity of our groups failed for Sylow 3-subgroups – but these were Abelian. Is this perhaps worth pursuing?
- Extension of the classification for regular maps on surfaces of Euler characteristic equal to small negative multiples of a prime?

Directions of future research

- Almost-Sylow-cyclicity of our groups failed for Sylow 3-subgroups – but these were Abelian. Is this perhaps worth pursuing?
- Extension of the classification for regular maps on surfaces of Euler characteristic equal to small negative multiples of a prime?
- How about $-\chi = pp'$ with prime $p > p' > 3$?

Directions of future research

- Almost-Sylow-cyclicity of our groups failed for Sylow 3-subgroups – but these were Abelian. Is this perhaps worth pursuing?
- Extension of the classification for regular maps on surfaces of Euler characteristic equal to small negative multiples of a prime?
- How about $-\chi = pp'$ with prime $p > p' > 3$? Advantage if 'gap' at characteristic $-p'$ ...

Directions of future research

- Almost-Sylow-cyclicity of our groups failed for Sylow 3-subgroups – but these were Abelian. Is this perhaps worth pursuing?
- Extension of the classification for regular maps on surfaces of Euler characteristic equal to small negative multiples of a prime?
- How about $-\chi = pp'$ with prime $p > p' > 3$? Advantage if 'gap' at characteristic $-p'$ ... but the number of G-W 'survivors' increases.

Directions of future research

- Almost-Sylow-cyclicity of our groups failed for Sylow 3-subgroups – but these were Abelian. Is this perhaps worth pursuing?
- Extension of the classification for regular maps on surfaces of Euler characteristic equal to small negative multiples of a prime?
- How about $-\chi = pp'$ with prime $p > p' > 3$? Advantage if 'gap' at characteristic $-p'$ ... but the number of G-W 'survivors' increases.
- Prime powers?

Directions of future research

- Almost-Sylow-cyclicity of our groups failed for Sylow 3-subgroups – but these were Abelian. Is this perhaps worth pursuing?
- Extension of the classification for regular maps on surfaces of Euler characteristic equal to small negative multiples of a prime?
- How about $-\chi = pp'$ with prime $p > p' > 3$? Advantage if 'gap' at characteristic $-p'$ ... but the number of G-W 'survivors' increases.
- Prime powers? Conder, Potočnik and Š:

Directions of future research

- Almost-Sylow-cyclicity of our groups failed for Sylow 3-subgroups – but these were Abelian. Is this perhaps worth pursuing?
- Extension of the classification for regular maps on surfaces of Euler characteristic equal to small negative multiples of a prime?
- How about $-\chi = pp'$ with prime $p > p' > 3$? Advantage if 'gap' at characteristic $-p'$ ... but the number of G-W 'survivors' increases.
- Prime powers? Conder, Potočnik and Š:

Up to isomorphism and duality, the complete list of automorphism groups of regular maps with $\chi = -p^2$, p *an odd prime, is:*

Directions of future research

- Almost-Sylow-cyclicity of our groups failed for Sylow 3-subgroups – but these were Abelian. Is this perhaps worth pursuing?
- Extension of the classification for regular maps on surfaces of Euler characteristic equal to small negative multiples of a prime?
- How about $-\chi = pp'$ with prime $p > p' > 3$? Advantage if 'gap' at characteristic $-p'$ ... but the number of G-W 'survivors' increases.
- Prime powers? Conder, Potočnik and Š:

Up to isomorphism and duality, the complete list of automorphism groups of regular maps with $\chi = -p^2$, p an odd prime, is:

- $p = 3,\ G \cong \langle (x, y, z),\ r^6 = s^6 = sr^2s^2y = 1\rangle,\ |G| = 108$

Directions of future research

- Almost-Sylow-cyclicity of our groups failed for Sylow 3-subgroups – but these were Abelian. Is this perhaps worth pursuing?
- Extension of the classification for regular maps on surfaces of Euler characteristic equal to small negative multiples of a prime?
- How about $-\chi = pp'$ with prime $p > p' > 3$? Advantage if 'gap' at characteristic $-p'$ ... but the number of G-W 'survivors' increases.
- Prime powers? Conder, Potočnik and Š:

Up to isomorphism and duality, the complete list of automorphism groups of regular maps with $\chi = -p^2$, p an odd prime, is:

- $p = 3,\ G \cong \langle (x, y, z),\ r^6 = s^6 = sr^2s^2y = 1\rangle,\ |G| = 108$
- $p = 3,\ G \cong \langle (x, y, z),\ r^6 = s^4 = (rs^{-1})^3x = 1\rangle,\ |G| = 216$

Directions of future research

- Almost-Sylow-cyclicity of our groups failed for Sylow 3-subgroups – but these were Abelian. Is this perhaps worth pursuing?
- Extension of the classification for regular maps on surfaces of Euler characteristic equal to small negative multiples of a prime?
- How about $-\chi = pp'$ with prime $p > p' > 3$? Advantage if 'gap' at characteristic $-p'$ ... but the number of G-W 'survivors' increases.
- Prime powers? Conder, Potočnik and Š:

Up to isomorphism and duality, the complete list of automorphism groups of regular maps with $\chi = -p^2$, p an odd prime, is:

- $p = 3,\ G \cong \langle (x, y, z),\ r^6 = s^6 = sr^2s^2y = 1\rangle,\ |G| = 108$
- $p = 3,\ G \cong \langle (x, y, z),\ r^6 = s^4 = (rs^{-1})^3x = 1\rangle,\ |G| = 216$
- $p = 7,\ G \cong \mathrm{PSL}(2, 13),\ |G| = 1092$, *with presentation*

$\langle (x, y, z),\ r^{13} = s^3 = rs^{-1}r^2s^{-1}r^2sr^{-1}sr^{-1}z = r^{-5}s^{-1}r^5sr^{-4}sy = 1\rangle$