Regular embeddings of graphs into surfaces

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Banská Bystrica

Wien, November 2008

November 20, 2008

Maps and graph embeddings

Maps Map is a 2-cell decomposition of a surface

Category OrMaps Maps on orientable surfaces together with orientation-preserving homomorphisms

Category Maps Maps on surfaces (possibly with non-empty boundary) together with homomorphisms

Graph embedding: $i : K \hookrightarrow S$, S compact connected surface, $i(K)$ defines a map on S,

Symmetries of graph embeddings

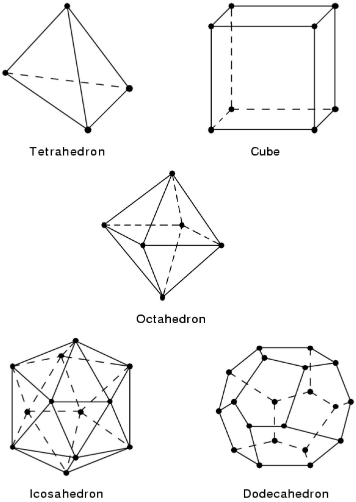
Automorphisms: $\psi \in Aut(K)$ is an automorphism of embedding $i : K \hookrightarrow S$ if $i(\psi(x)) = i(x)$ for every dart (arc) of K,

Orientation preserving automorphism: If S is orientable then $Aut(i)$ contains a subgroup $Aut^+(i)$ of index at most two of orientation preserving automorphisms,

A bound on $|Aut(i)|$ $|Aut(i)| \leq 2|D|$ and $|Aut^+(i)| \leq |D|$, if we have $=$ the embedding is called **regular** or **orientably regular** respectively,

Problem 1 Given K under what conditions K admits a regular (orientably regular) embeddings?

Problem 2 Given K describe all regular (orientably regular) embeddings of K.



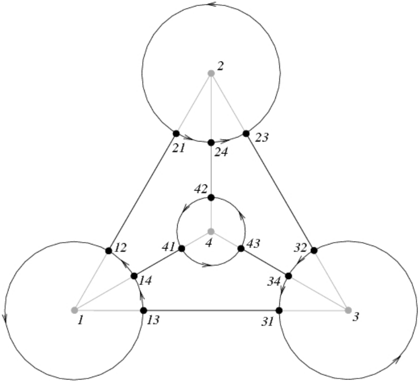
Five Plato's Solids

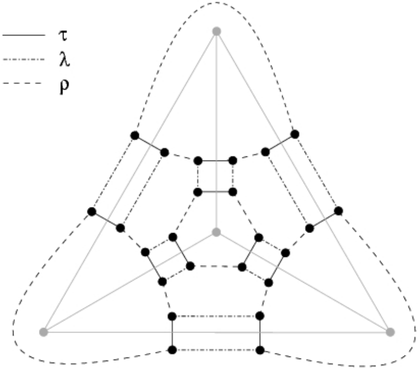
Combinatorial Maps

- ORMAPS: $(D; R, L)$, $R, L \in Sym(D)$, $L^2 = 1$, $\langle R, L \rangle$ is transitive on D,
- MAPS: $(F; r, \ell, t)$, $r, \ell, t \in Sym(F)$, $r^2 = \ell^2 = t^2 = (\ell t)^2 = 1$, $\langle r, \ell, t \rangle$ is transitive on F,
- ORDINARY MAPS: L, r, t, ℓ are fixed-point-free, but the category is not closed under taking quotients!

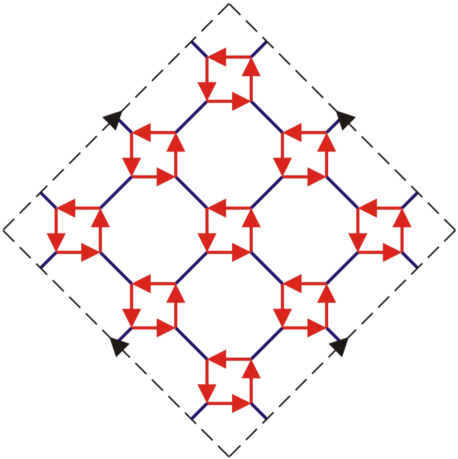
!!! Map = a particular action diagram of a permutation group = a particular 3-edge coloured cubic graph !!!

Combinatorial maps





Feynmann diagrams



Automorphisms of graph embeddings

orientation preserving automorphisms if $i(K)$ is described in terms of $(D; R, L)$ then $Aut^+(i)$ consists of permutations in $Sym(D)$ commuting with R and L

semiregularity $Aut^+(i)$ is the centralizer of $\langle R, L\rangle$ in the symmetric group $Sym(D)$, in particular the action is **semiregular** on the set of darts (arcs) of K

regular embedding An embedding $i : K \hookrightarrow S$ is (orientably) regular if the action of $Aut^+(i)$ is regular,

the following conditions are equivalent

i is regular on $D(K)$,
$\langle R, L\rangle$ is regular on $D(K)$,
$|Aut^+(i)| = |\langle R, L\rangle|$.

Automorphisms-general case

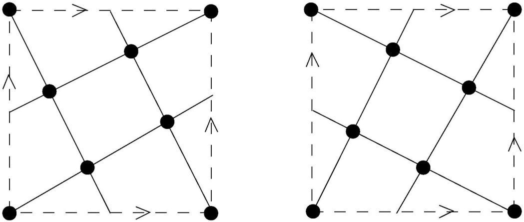
Observation For some trivial maps $Aut(i)$ does not embedd into $Aut(K)$,

Semiregularity The action of $Aut(i)$ is defined on flags, and it is semiregular. How to interpret flags in K?

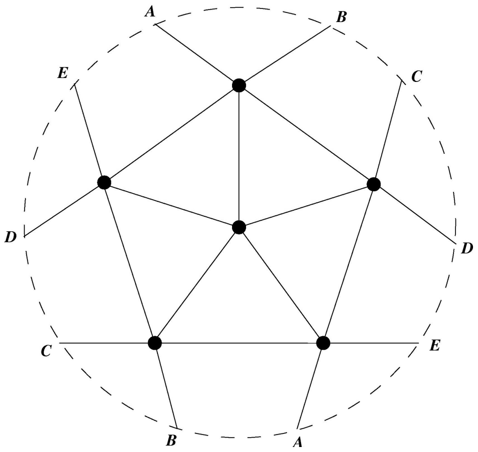
Valency condition If $val(v) \geq 3$ for every vertex v then flags can be defined in terms of K, then $Aut(i) \leq Aut(K)$.

regularity $i : K \hookrightarrow S$ is regular if $Aut(i)$ acts regularly on the flags,

Orientably regular but not regular embedding



Regular embedding of K_6 into PP



Sabidussi-like characterization

Theorem

A connected graph K admits a regular embedding into a closed surface if and only if there exists a one-regular subgroup $G \leq Aut(K)$ with cyclic vertex-stabilizers.

Theorem

(1999,Gardiner,N.,Širáň,Škoviera) A connected graph K admits a regular embedding into a closed surface if and only if there exists a subgroup $G \leq Aut(K)$ satisfying

- *G is arc-transitive,*
- *G_v is dihedral of order $2 \cdot val(K)$,*
- *G_e is dihedral of order 4*

3-valent regular embeddings of graphs

s	Types	Bipartite?	Smallest example	Uniq. minimal?	or.reg.	regul
1	1	Sometimes	F026	No	Yes	No
2	$1, 2^1$	Sometimes	F004 (K_4)	No	Yes	Yes
2	2^1	Sometimes	F084	No	No	Yes
2	2^2	Sometimes	F448C	No	No	No
3	$1, 2^1, 2^2, 3$	Always	F006 ($K_{3,3}$)	No	Yes	Yes
3	$2^1, 2^2, 3$	Always	F020B (GP(10,3))	No	No	Yes
3	$2^1, 3$	Never	F010 (Petersen)	No	No	Yes
3	$2^2, 3$	Never	F028 (Coxeter)	No	No	No
3	3	Sometimes	F110	No	No	No
4	$1, 4^1$	Always	F014 (Heawood)	Yes	Yes	No
4	4^1	Sometimes	F102 (S(17))	No	No	No
4	4^2	Sometimes	3^{10}-fold cover of F468?	No	No	No
5	$1, 4^1, 4^2, 5$	Always	Biggs-Conway graph	Yes	Yes	No
5	$4^1, 4^2, 5$	Always	F030 (Tutte's 8-cage)	No	No	No
5	$4^1, 5$	Never	S_{10} graph?	No	No	No
5	$4^2, 5$	Never	F234B (Wong's graph)	No	No	No
5	5	Sometimes	$M_{24} \wr C_2$ graph?	No	No	No

M.Conder+ N.: The 17 families of finite symmetric cubic graphs

Constructions: Cayley maps

Let G be a group and ρ be a cyclic permutation of a set of generators X closed under taking inverses $X = X^{-1}$.

A Cayley map $CM(G, \rho)$ is a map whose set of darts is $D = G \times X$, $L(g, x) = (gx, x^{-1})$ and $R(g, x) = (g, \rho x)$.

Observation: **Every Cayley map is a vertex-transitive map - G acts as a group of map automorphisms.**

Constructions: Regular Cayley Maps

Biggs 1972: If ρ extends to a group automorphism then $CM(G, \rho)$ is regular,

Širáň, Škoviera 1992,1994: Systematic investigation of regular CM, in particular, balanced and antibalanced case.

Jajcay, Širáň 1999 Structure of $Aut(M)$ of a regular CM, in particular, $Aut(M) \cong G \cdot \langle \rho \rangle$ is a product of the colour group G and cyclic group,

Balanced case $Aut(M)$ is a split extension (semidirect product) with G normal,

Survey Richter, R. Bruce; Šir aň, J.; Jajcay, R.; Tucker, T.; Watkins, M.: Cayley maps. J. Combin. Theory Ser. B 95 (2005), no. 2, 189–245.

Example: Standard embeddings of cubes

- Q_n can be defined as a Cayley graph $Cay(\mathbb{Z}_2^n, \mathcal{B})$,
- where $\mathcal{B} = \{e_1, e_2, \ldots, e_n)$ is the standard base,
- Clearly, there is a linear transformation T taking
- $e_i \mapsto e_{i+1}$, for $i = 1, \ldots, n-1$ and $e_n \mapsto e_1$
- It follows that $CM(\mathbb{Z}_2^n, \rho)$, where $\rho = T|_{\mathcal{B}}$, is a regular embedding of Q_n.

Regular Cayley maps

Skew-morphism A permutation ψ of elements of a group G is called a skew-morphism if $\psi(1_G) = 1_G$ and $\psi(a \cdot b) = \psi(a) \cdot (\psi(b))^{\pi(a)}$, where $\pi : G \to Z$ is a power function.

J+S characterization $CM(G, \rho)$ is orientably regular iff ρ extends to a skew-morphism of G.

bal.reg.emb. of circulants $G = Z_n$, ρ is given by a multiplication by a coprime to n

emb. of circulants Problem: Determine all skew-morphisms of an additive cyclic group containing an orbit closed under taking inverse elements,

Example: Regular complete maps

Problem: Determine orientably regular embeddings of K_n.

Biggs construction: $F = (F, +)$ additive group of a finite field, $F^* = (F, \cdot)$ multiplicative group of $F = GF(p^e)$, if t is a primitive element of F and $\rho(x) = xt$ for $x \neq 0$ then $CM(F, \rho)$ is orientably regular,

James+Jones An automorphism group of an orietable regular complete map is **sharply doubly transitive**

Zassenhaus SDT groups are affine groups $F : \langle t \rangle$,

isomorphism problem complete regular maps of the same order depends only on the choice of t, $M_t \cong M_u$ iff t and u are conjugate elements,

Regular complete maps

Theorem

(James and Jones 1985) Orientably regular embedding of K_n exists iff $n = p^e$. In particular, there are $\Phi(p^e - 1)/e$ such maps.

Theorem

(James 1983, Wilson 1989) Regular embedding of K_n exist only for $n = 2, 3, 4, 6$. The embeddings are unique up to Petrie duality.

Constructions: Lifting techniques

Observation: Cayley map can be viewed as a regular cover $CM(G,\rho) \to B(\rho)$ over a one-vertex map with $CT(p) = G$, in particular balanced Cayley maps are regular covers over the two one-vertex regular maps and all the automorphisms (the cyclic group) lifts along p,

Regular covers: There is a well-known generalization of CM construction discovered as byproduct of the proof MAP COLOR THEOREM (Ringel and others) and formalized by Alpert, Gross and Tucker in terms of **ordinary voltage assignments**, base graph can be any connected graph as well as the voltage group G can be arbitrary,

Lifting graph and map automorphisms

Gvozdjak,Siran 1993 Lifting condition: An automorphism ψ of X lifts along a regular covering $p : X^\xi \to X$, $\xi : X \to G$ ordinary v.a. iff

lifting condition: $\xi(W) = 1_G$ implies $\xi(\psi W) = 1_G$

for every closed walk W of X.

Malnic,N.,Skoviera, 2000,2002 further development of the theory of lifting graph and map automorphisms,

in particular: coverings between regular maps are regular

observation if a map does not come from a simple group it has a regular quotient

Lifting problem: Case of Abelian $CT(p)$

homology assignments v.a. can be viewed as a morphisms from the fundamental grupoid into the voltage group G,

if G is abelian $\pi(X)$ can be replaced by its abelianization,

in this case the lifting problem can be solved solved using methods of linear algebra,

Surowski ... homology and cohomology on maps,

Siran, Malnic, Potocnik, Du, Kwak,... two particular cases G is cyclic, G is elementary abelian,

Cyclic and elementary abelian CT(p)

Proposition. Let X be a connected graph with a spanning tree T. Then the mapping $f \to M_T(f)$ is a homomorphism from $Aut(X)$ to the unitary group $U(\beta(X), \mathbb{Z})$.

Theorem

(Siran) T-reduced v.a. in a cyclic group satisfying the lifting conditions are eigenvectors of $M_T(f)$

Theorem

(Malnic) Let $\vec{\xi}$ be a T-reduced v.a. from $\mathbb{Z}_p^\beta$ on a connected graph with cycle rank β. Let f be a graph automorphism. Then $\vec{\xi}$ satisfies the lifting condition iff the column space $C([\vec{\xi}])$ is an $M_T(f)$-invariant subspace of $\mathbb{Z}_p^\beta$

Example: Lifts over the hexagonal embedding of $K_{3,3}$

dim	Invariant subspace	$[\xi_{C_1}]^t$	$[\xi_{C_2}]^t$	$[\xi_{C_3}]^t$	$[\xi_{C_4}]^t$	condition
1	$\mathbb{V}_1(\lambda)$	$[1]$	$[-\lambda^2]$	$[-\lambda]$	$[0]$	$p \equiv 1\ (3)$ $\lambda^3 \equiv -1(p)$
1	$\mathbb{V}_2(\lambda)$	$[1]$	$[\lambda]$	$[\lambda^2]$	$[0]$	$p \equiv 1\ (3)$ $\lambda^3 \equiv -1(p)$
2	$\mathbb{U}_1$	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ -1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} -2 \\ 0 \end{bmatrix}$	none
2	$\mathbb{U}_2$	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$	$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$	none
3	$\mathbb{U}_1 \oplus \mathbb{V}_1(\lambda)$	$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ -1 \\ -\lambda^2 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 1 \\ -\lambda \end{bmatrix}$	$\begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}$	$p \equiv 1\ (3)$ $\lambda^3 \equiv -1(p)$
3	$\mathbb{U}_1 \oplus \mathbb{V}_2(\lambda)$	$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ -1 \\ \lambda \end{bmatrix}$	$\begin{bmatrix} 1 \\ 1 \\ \lambda^2 \end{bmatrix}$	$\begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}$	$p \equiv 1\ (3)$ $\lambda^3 \equiv -1(p)$
4	$\mathbb{V}$	$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$	none

Table: G-invariant voltage spaces, Case $p > 3$. (Here $a \equiv b(n)$ means $a = b(\mod n)$)

Operations of maps preserving both the graph and regularity

Petrie dual: $(F; \lambda, \rho, \tau) \mapsto (F; \lambda\tau, \rho, \tau)$,

regularity of PD the monodromy group is the same and $Aut(M)$ is the centralizer,

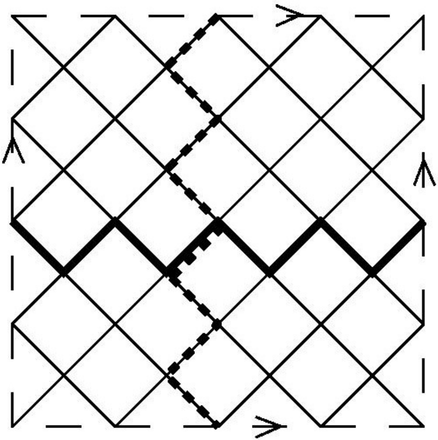
observation do not preserve regularity of chiral maps,

congruent maps: $(D, R, L) \mapsto (D, R^e, L)$, where e is coprime to $ord(R)$, introduced by S. Wilson,

both graph and group is preserved, but the maps may be different, complete maps, icosahedron,

N.Škoviera 1996,97 A systematic investigation of exponents, Exponent group $|Ex(M)|$,

Petrie dual operator



Regular embeddings of the same graph with the same group

Theorem N+Š: Two orientably regular embeddings of the same graph with the same group are congruent. The number of isoclasses is $\Phi(val(K))/|Ex(M)|$,

Hužvar investigated the same problem for the class of MAPS, the respective operators are exponents and Petrie duality,

General strategy

- constructions
- identification of automorphism groups
- given graph and group determining isoclasses using exponents
- Group theoretical formulation: find generating pairs $\langle \rho, \lambda \rangle = G \in Aut(K)$ of subgroups, $\rho \in G_v$, $\lambda \in G_e$, $\lambda^2 = 1$ and determine their equivalence classes in $Aut(K)$,
- Equivalence $= (\rho_2, \lambda_2) = (\rho_1^g, \lambda_2^g)$ for some $g \in Aut(K)$.

Regular embeddings of some popular graphs

complete graphs + James and Jones: exist only if $n = p^e$, SDS groups, $\phi(p^e - 1)/e$ of maps,

complete graphs James, Wilson, none for $n = 5$ and $n \geq 7$.

complete bipartite graphs + long story, N.,Škoviera, Zlatoš, Jones, Du, Kwak, Kwon,... completed recently by Jones, groups are extensions of Z_2 by $C_n \cdot C_n$,

relation to the theory of dessign denfants and algebraic curves,

complete bipartite graphs reflexible done by Kwak and Kwon,

Cubes Q_n

N., Škoviera 1997 a construction by modification of the standard embedding,

Du,Kwak, Nedela 2004 odd n case, important is understanding of permutation group of degree n generated by a regular element of order n and G_v of order 2^k.

Kwon 2002 a construction of many new maps for even n,

Kwon,N. 2004 nonorientable case, no maps for $n \geq 3$,

Catalano,N. 2007 new maps not in Kwon family

2008 two independent groups (N., Catalano, Kwon), (Conder, Wilson,...) The Kwon and CN family cover all the maps.

open problems and work in progress

- Complete multipartite graphs $K_n[m]$, known only for $m = p$, or $n = 2(3)$.
- Circulants, skew-morphisms of cyclic groups,
- lifting problem with abelian $CT(p)$ other cases than elementary abelian and cyclic.