Evaluating game trees in the priced information model

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The function evaluation problem

$$f = (x \textit{ and } y) \textit{ or } (x \textit{ and } z)$$

- x, y, z: binary variables
- for some inputs it is possible to evaluate f without reading all variables

Example:

- $(x, y, z) = (0, 1, 1)$

  It is enough to know the value of x

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The function evaluation problem

Input:

- a function f over the variables $x_1, \ldots, x_n$
- each variable has a positive cost of reading its value

Goal:

- Devise an algorithm for evaluating f which incurs as little cost as possible reading its variables

Applications

Applications of the function evaluation problem:

- Reliability testing / diagnosis

  Telecommunications: testing connectivity of networks

  Manufacturing: testing machines before shipping

- Databases

  Query optimization

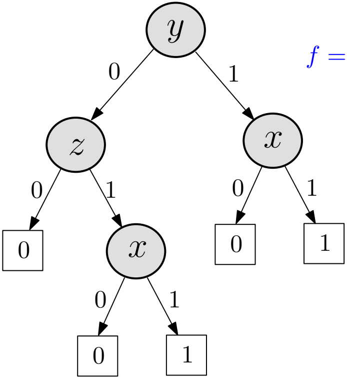
- Artificial Intelligence

  Finding optimal derivation strategies in knowledge-based systems

- Decision-making strategies (AND-OR trees)
- Computer-aided game playing *for two-player zero-sum games with perfect information, e.g.* **chess** (game trees)

Algorithms for evaluating f

- Dynamically select the next variable based on the values of the variables read so far
- Stop when the value of f is determined



$f = (x \; and \; y) \; or \; (x \; and \; z)$

Evaluation measure

Evasive Functions

- For any possible algorithm, all the variables must be read in the worst case.
- $f = (x \textit{ and } y) \textit{ or } (x \textit{ and } z)$
- Some important functions are evasive (e.g. game trees, AND/OR trees and threshold trees).

- Worst case analysis cannot distinguish among the performance of different algorithms.
- Instead, we use **competitive analysis** (Charikar et al. 2002)

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- Instead, we use **competitive analysis** (Charikar et al. 2002)

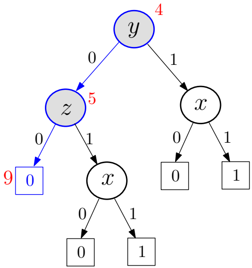
Cost of evaluation

$f = (x \textit{ and } y) \textit{ or } (x \textit{ and } z)$
$cost(x) = 3$, $cost(y) = 4$, $cost(z) = 5$

Asignment (x, y, z)	Value of f	Cheapest Proof	Cost
(0,0,0)	0	{x}	3
(1,1,0)	1	{x,y}	3+4=7

The competitive ratio

(x,y,z)	$f(x,y,z)$	Cost of Cheapest Proof	Algorithm Cost	Ratio
(0,0,0)	0	3	9	3



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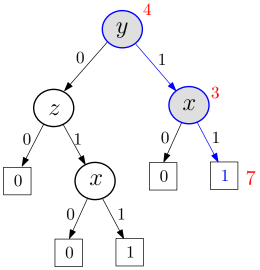

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(1,0,0)	0	9	9	1
(0,1,0)	0	3	7	7/3
(0,0,1)	0	3	12	4
(1,1,0)	1	7	7	1
(1,0,1)	1	8	12	3/2
(0,1,1)	0	3	7	7/3
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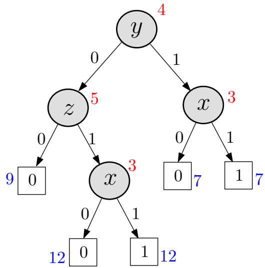

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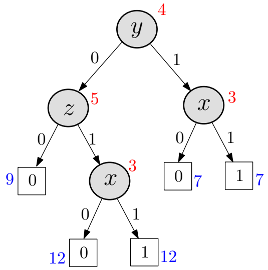

Measures of algorithm's performance

Competitive ratio of algorithm A for (f, c):

$$\max_{\text{all assignments } \sigma} \frac{\text{cost of } A \text{ to evaluate } f \text{ on } \sigma}{\text{cost of cheapest proof of } f \text{ on } \sigma}$$

In this talk:

Extremal *competitive ratio of A for f*:

$$\max_{\substack{\text{all assignments } \sigma,\\ \text{all cost vectors } c}} \frac{\text{cost of } A \text{ to evaluate } f \text{ on } (\sigma, c)}{\text{cost of cheapest proof of } f \text{ on } (\sigma, c)}$$

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Measure of function's complexity

Extremal competitive ratio of f :

$$\min_{\substack{\textit{all deterministic algorithms } A \\ \textit{that evaluate } f}} \left\{ \textit{extremal competitive ratio of } A \textit{ for } f \right\}$$

The function evaluation problem

Given:
a function f over the variables $x_1, x_2, \ldots, x_n$

Combinatorial Goal:

- Determine the extremal competitive ratio of f

Algorithmic Goal:

- Devise an algorithm for evaluating f that:
  1. achieves the optimal (or close to optimal) extremal competitive ratio
  2. is efficient *(runs in time polynomial in the size of f)*

The algorithm knows the reading costs.

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Related work - other models/measures

- **Non-uniform costs & competitive analysis**

  Charikar et al. [STOC 2000, JCSS 2002]

- **Unknown costs**

  Cicalese and Laber [SODA 2006]

- **Restricted costs (selection and sorting)**

  Gupta and Kumar [FOCS 2001], Kannan and Khanna [SODA 2003]

- **Randomized algorithms**

  Snir [TCS 1985], Saks and Wigderson [FOCS 1986], Laber [STACS 2004]

- **Stochastic models**

  Random input, uniform probabilities

  Tarsi [JACM 1983], Boros and Ünlüyurt [AMAI 1999]
  Charikar et al. [STOC 2000, JCSS 2002], Greiner et al. [AI 2005]

  Random input, arbitrary probabilities

  Kaplan et al. [STOC 2005]

  Random costs

  Angelov et al. [LATIN 2008]

How to evaluate general functions?

Good algorithms are expected to test. . .

- cheap variables - well defined
- important variables ???

Cicalese and Laber created
a linear program that captures the impact of the variables

The minimal proofs

f - a function over $V = \{x_1, \ldots, x_n\}$

R - range of f

Definition

Let $r \in R$. A **minimal proof for** $f(x) = r$ is a minimal set of variables $P \subseteq V$ such that *there is an assignment σ of values to the variables in P such that f_σ is constantly equal to r*.

Example: $f(x_1, x_2, x_3) = (x_1 \text{ and } x_2) \text{ or } (x_1 \text{ and } x_3)$, $R = \{0, 1\}$

minimal proofs for $f(x) = 1$: $\{\{x_1, x_2\}, \{x_1, x_3\}\}$

minimal proofs for $f(x) = 0$: $\{\{x_1\}, \{x_2, x_3\}\}$

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The Linear Program by Cicalese-Laber

$$\textbf{LP}(f) \left\{ \begin{array}{llll} \textit{Minimize } \sum_{x \in V} s(x) & & & \\ \textit{s.t.} & & & \\ \sum_{x \in P} s(x) & \geq & 1 & \text{for every minimal proof } P \text{ of } f \\ \quad s(x) & \geq & 0 & \text{for every } x \in V \end{array} \right.$$

Intuitively, $s(x)$ measures the **impact** of variable x.

The feasible solutions to the **LP**(f) are precisely the fractional hitting sets of the set of minimal proofs of f.

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LP(f): example

$f(x_1, x_2, x_3) = (x_1 \textit{ and } x_2) \textit{ or } (x_1 \textit{ and } x_3)$

minimal proofs for $f = 1$: $\{\{x_1, x_2\}, \{x_1, x_3\}\}$

minimal proofs for $f = 0$: $\{\{x_1\}, \{x_2, x_3\}\}$

$$\textbf{LP}(f) \begin{cases} \textit{Minimize } s_1 + s_2 + s_3 \\ \textit{s.t.} \\ \quad s_1 + s_2 \geq 1 \\ \quad s_1 + s_3 \geq 1 \\ \quad s_1 \geq 1 \\ \quad s_2 + s_3 \geq 1 \\ s_1, s_2, s_3 \geq 0 \end{cases}$$

Optimal solution: $s = (1, 1/2, 1/2)$

The Linear Programming Approach

LPA**(f: function)**

While the value of f is unknown

Select a feasible solution $s()$ of **LP**(f)

Read the variable u which minimizes $c(x)/s(x)$
(cost/impact)

$c(x) = c(x) - s(x)c(u)/s(u)$

$f \leftarrow$ restriction of f after reading u

End While

The LPA bounds the extremal competitive ratio

The selection of solution s determines both the **computational efficiency** and the **performance** (extremal competitive ratio) of the algorithm.

Key Lemma (Cicalese-Laber 2006)

Let $\mathbf{K}$ *be a positive number. If*

$$\mathit{ObjectiveFunctionValue}(s) \leq \mathbf{K}\,,$$

for every selected solution s

then

$$\mathit{ExtremalCompetitiveRatio}(f) \leq \mathbf{K}\,.$$

The fractional cover number

The **fractional cover number** of a function f is defined as

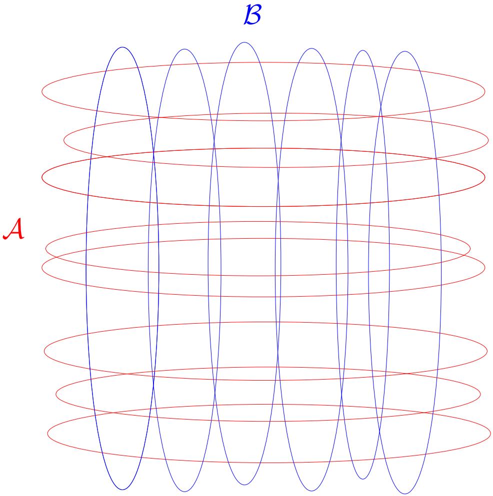
$$\Delta(f) = \max_{f' \text{ restriction of } f} \left\{ \textit{opt-value}(\mathbf{LP}(\mathbf{f'})) \right\}$$

Key Lemma (Cicalese-Laber 2006)

For every function f,

$$\textit{ExtremalCompetitiveRatio}(f) \leq \Delta(f)\,.$$

Cross-intersecting families



- cross-intersecting: $A \in \mathcal{A}, B \in \mathcal{B} \Rightarrow A \cap B \neq \emptyset$

Minimal proofs and cross-intersecting families

$f : S_1 \times \cdots \times S_n \to S$, a function over $V = \{x_1, \dots, x_n\}$

R - range of f

For $r \in R$, let $\mathcal{P}(r)$ denote the set of **minimal proofs for** $f(x) = r$.

Then:

- each $\mathcal{P}(r)$ is a non-empty Sperner family
- for every $r \neq r'$, the families $\mathcal{P}(r)$ and $\mathcal{P}(r')$ are cross-intersecting

The Linear Program by Cicalese-Laber

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$$\mathcal{P} = \cup_{r\in R}\mathcal{P}(r)$$

union of pairwise cross-intersecting Sperner families

For every function $f : S_1 \times \cdots \times S_n \to S$, the $\textbf{LP}(f)$ seeks a *minimal fractional hitting set of a union of pairwise cross-intersecting Sperner families*.

Cross-intersecting lemma

Cross-Intersecting Lemma (Cicalese-Laber 2008)

Let $\mathcal{A}$ and $\mathcal{B}$ be two non-empty **cross-intersecting** *families of subsets of V.*
Then, there is a fractional hitting set s of $\mathcal{A} \cup \mathcal{B}$ such that

$$\|s\|_1 = \sum_{x \in V} s(x) \leq \max\{|P| : P \in \mathcal{A} \cup \mathcal{B}\}\,.$$

- geometric proof
- generalizes to any number of pairwise cross-intersecting families

Applications of the cross-intersecting lemma

1. $f : S_1 \times \cdots \times S_n \to S$, nonconstant:
   $ExtremalCompetitiveRatio(f) \leq PROOF(f)$ [Cicalese-Laber 2008]
   ($PROOF(f) =$ size of the largest minimal proof of f)

2. Monotone Boolean functions:
   $ExtremalCompetitiveRatio(f) = PROOF(f)$ [Cicalese-Laber 2008]

3. Game trees: $ExtremalCompetitiveRatio(f) \leq TBA$

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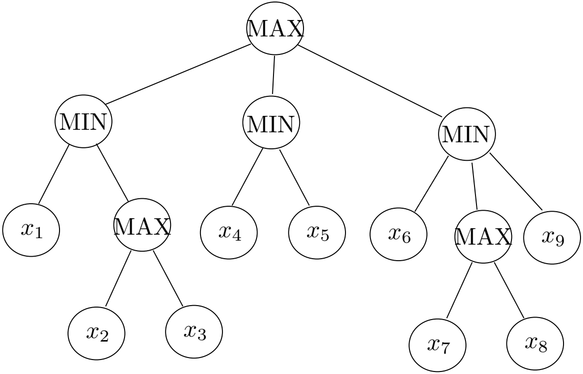
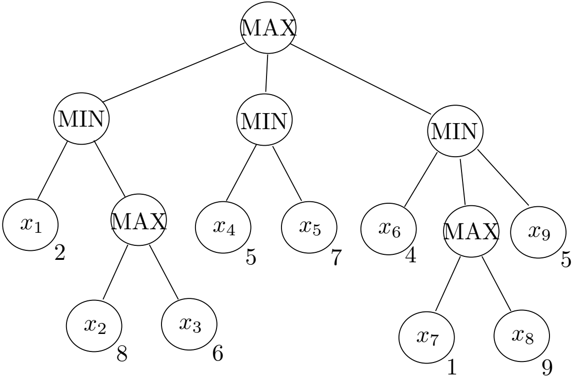
Game trees



game tree

Game trees



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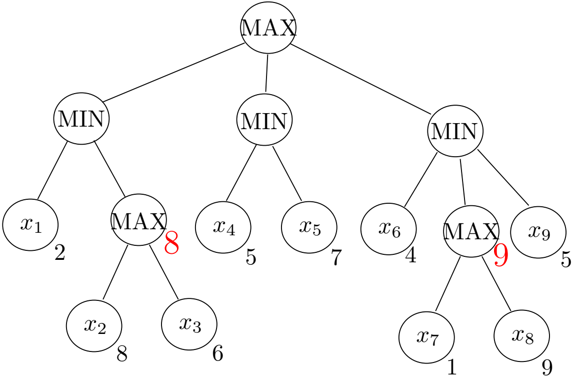

game tree

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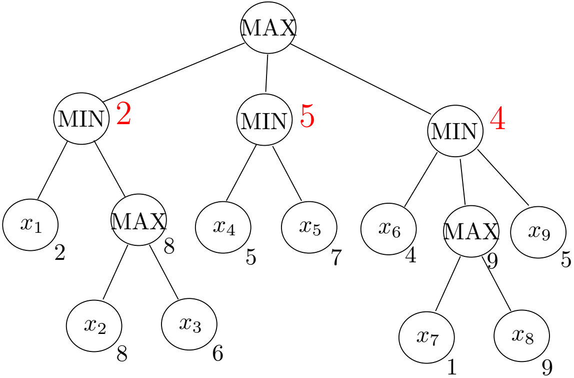
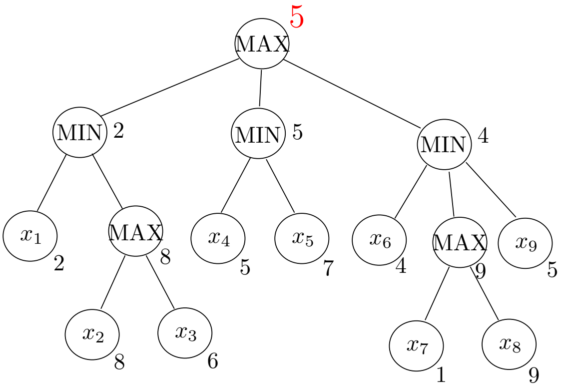


game tree

Game trees



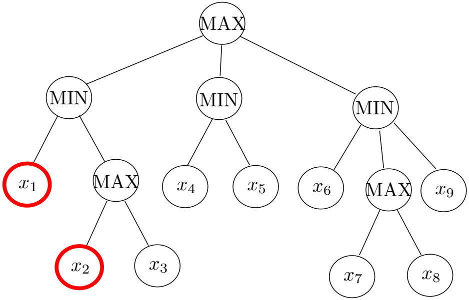
game tree

Minterms of game trees

Minterm: minimal set $A \subseteq V$ of variables such that

$$\textit{value of } f \geq \textit{value of } A := \textit{min value of variables in } A$$

Minterms can prove a lower bound for the value of f.



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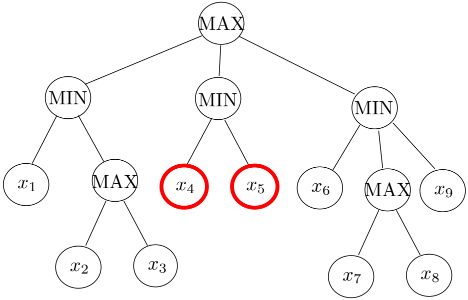

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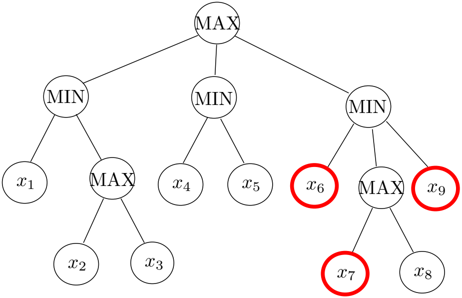
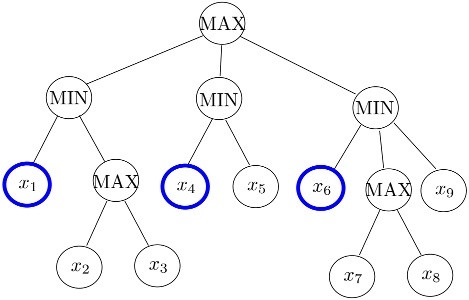


Maxterms of game trees

Maxterm: minimal set $B \subseteq V$ of variables such that

value of f $\leq$ *value of B* $:=$ *max value of variables in B*

Maxterms can prove an upper bound for the value of f.



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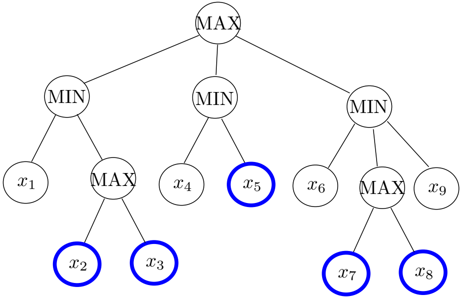


Lower bound for the extremal competitive ratio

- $k(f) = \max\{|A| : A \text{ minterm of } f\}$
- $l(f) = \max\{|B| : B \text{ maxterm of } f\}$

Theorem (Cicalese-Laber 2005)

Let f be a game tree with no minterms or maxterms of size 1. Then,

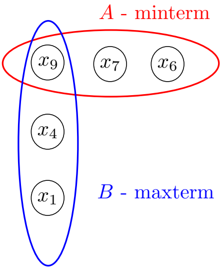
$$ExtremalCompetitiveRatio(f) \geq \max\{k(f), l(f)\} .$$

Minimal proofs of game trees

To prove that the value of *f* is **b**, we need:

- a minterm of value **b** [*proves* $f \geq$ **b**]
- a maxterm of value **b** [*proves* $f \leq$ **b**]

Every minimal proof = union of a minterm and a maxterm



A first upper bound

It follows that

$PROOF(f)$ = size of the largest minimal proof of f
$= k(f) + l(f) - 1.$

Theorem (Cicalese-Laber 2008)

$f : S_1 \times \cdots \times S_n \to S$, nonconstant:
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For a game tree f,
$ExtremalCompetitiveRatio(f) \leq k(f) + l(f) - 1$.

Lower bound: $\max\{k(f), l(f)\}$

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Can we close the gap?

Yes:

Claim

For every restriction f' of f, there is a fractional hitting set s of the set of minimal proofs of f' such that

$$\|s\|_1 \leq \max\{k(f), l(f)\}\,.$$

$$\Delta(f) = \max_{f' \text{ restriction of } f} \left\{ \textit{opt-value}(\mathbf{LP}(\mathbf{f}')) \right\} \leq \max\{k(f), l(f)\}$$

By the **Key Lemma**,

$$\textit{ExtremalCompetitiveRatio}(f) \leq \max\{k(f), l(f)\}$$

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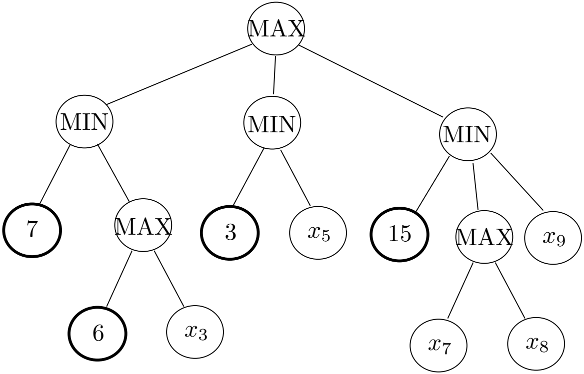
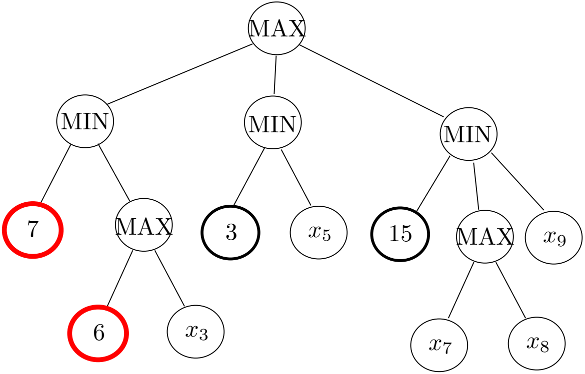
Restrictions of game trees



restriction of f

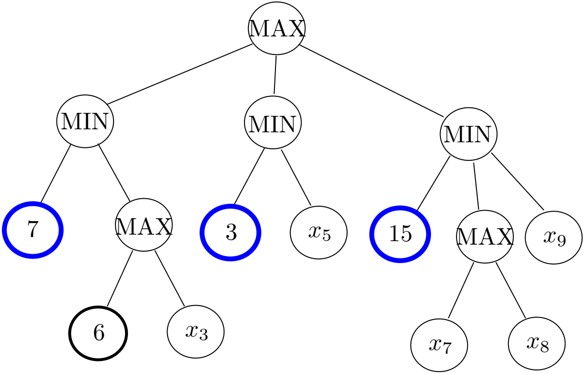
$$f'(x_3, x_5, x_7, x_8, x_9)$$

Restrictions of game trees



$$\tilde{f}(x_3, x_5, x_7, x_8, x_9) \geq 6$$

Restrictions of game trees



$$\tilde{f}(x_3, x_5, x_7, x_8, x_9) \leq 15$$

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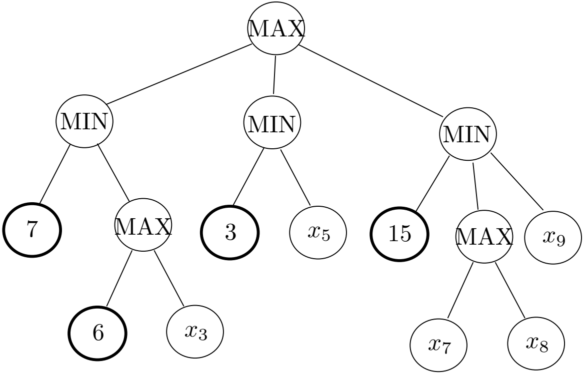
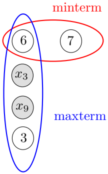


$$\tilde{f}(x_3, x_5, x_7, x_8, x_9) \in [6, 15] = [LB, UB]$$

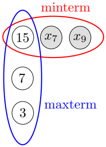
Minimal proofs of a restriction of a game tree

maxterm proofs

$b = LB$

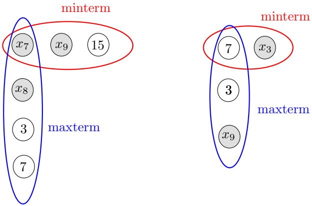


minterm proofs



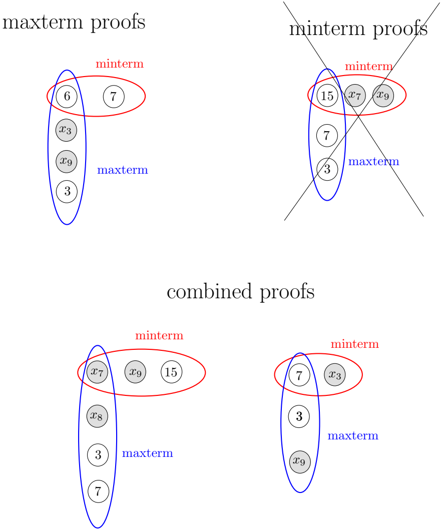
$b = UB$

combined proofs



$$LB \leq b \leq UB$$

Case 1: No maxterm has been fully evaluated yet



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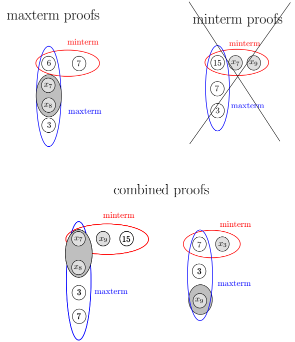
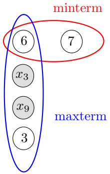


Let s be (the characteristic vector of) **a minimal hitting set of the shaded sets**.

$$\|s\|_1 \leq k(f) \leq \max\{k(f), l(f)\}$$

Case 2: There is a fully evaluated maxterm

maxterm proofs



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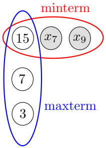

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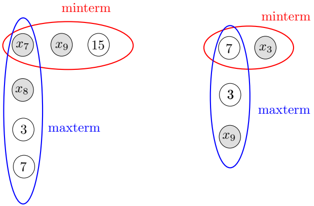
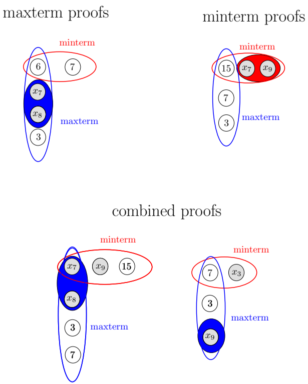


Case 2: There is a fully evaluated maxterm



R: the family of the minterm proofs

B: the family of the maxterm parts of the non-minterm proofs

Case 2: There is a fully evaluated maxterm

- **R** and **B** are non-empty sets
- **R** and **B** are cross-intersecting
- every minimal proof contains a member of $\mathbf{R} \cup \mathbf{B}$

By the **Cross-intersecting lemma**, there exists a feasible solution s to the $\mathbf{LP}(\mathbf{f}')$ such that

$$\|s\|_1 \leq \max\{|P| : P \in \mathbf{R} \cup \mathbf{B}\} \leq \max\{k(f), l(f)\}\,.$$

Putting together the two cases, it follows that

$$\Delta(f) \leq \max\{k(f), l(f)\}$$

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The extremal competitive ratio for game trees

Theorem

Let f be a game tree with no minterms or maxterms of size 1. Then,

$$\textit{ExtremalCompetitiveRatio}(f) = \max\{k(f), l(f)\}\,.$$

Theorem

Let f be a game tree. Let p (q) denote the number of minterms (maxterms) of f of size 1. Then

$$\textit{ExtremalCompetitiveRatio}(f) = \begin{cases} \max\{k(f), l(f)\}, & \text{if } p = q = 0 \text{ or } p = q = 1; \\ \max\{k(f), l(f) - p\}, & \text{if } p \geq 1 \text{ and } q = 0; \\ \max\{k(f) - q, l(f)\}, & \text{if } p = 0 \text{ and } q \geq 1. \end{cases}$$

There is a polynomial-time algorithm for evaluating game trees with optimal extremal competitiveness.

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Value dependent costs

Suppose that **the cost of reading a variable can depend on the variable's value:**

$$c(x) = \begin{cases} 50, & \text{if } x = 0; \\ 1000, & \text{if } x = 1. \end{cases}$$

Theorem

Let f be a monotone Boolean function or a game tree. Then,

$$\mathit{ExtremalCompetitiveRatio}(f, r) = r \cdot \mathit{ECR}(f) - r + 1\,,$$

where

$$r = \max_{x \in V} \frac{c_{max}(x)}{c_{min}(x)}\,.$$

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LPA has a very broad applicability

LPA does not depend on the structure of f

It can be used to derive upper bounds on the extremal competitive ratios of **very different functions**:

- f = minimum of a list:
  $ExtremalCompetitiveRatio(f) \leq n - 1$ [Cicalese-Laber 2005]
- f = the sorting function: $ExtremalCompetitiveRatio(f) \leq n - 1$
  [Cicalese-Laber 2008]
- $f : S_1 \times \cdots \times S_n \to S$, nonconstant:
  $ExtremalCompetitiveRatio(f) \leq PROOF(f)$ [Cicalese-Laber 2008]
- f = monotone Boolean function:
  $ExtremalCompetitiveRatio(f) = PROOF(f)$ [Cicalese-Laber 2008]
- f = game tree: $ExtremalCompetitiveRatio(f) \leq max\{k(f), l(f)\}$

Summary

We have seen:

- the Linear Programming Approach for the development of competitive algorithms for the function evaluation problem,
- the related combinatorial notion of the **fractional cover number**,
- the **extremal competitive ratio** for **game trees**,
- the more general model of value dependent costs.

Some Open Questions

- Is the extremal competitive ratio always integer?
- Find the extremal comp. ratio of general Boolean functions.
- Is there a polynomial algorithm with optimal extremal comp. ratio for evaluating monotone Boolean functions (given by an oracle/by the list of minterms)?

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The end

THANK YOU