Circular Chromatic Index of Blanuša Snarks

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Outline

Definition

A *circular r-edge-colouring* is a mapping $c : E(G) \to [0, r)$ such that for any two adjacent edges e and f we have

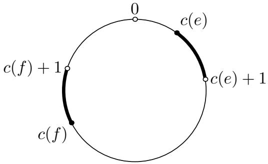
$$1 \leq |c(e) - c(f)| \leq r - 1.$$

Definition

The *circular chromatic index* is

$$\chi'_c(G) = \inf \{r \mid G \text{ has circular } r\text{-edge-colouring}\}.$$

Why circular?



Circular colouring: colour corresponds to a unit-length arc on a circle with length r.

Definition (Vince)

A *(p, q)-edge-colouring* is a mapping $c : E(G) \to \{0, 1, \ldots p-1\}$ such that for any two adjacent edges e and f we have

$$q \leq |c(e) - c(f)| \leq p - q.$$

Properties of circular edge-colourings

- The infimum is always attained and is rational.
- Sufficient to consider $p \leq |E(G)|$ and $q \leq |V(G)|/2$.
- $\chi'(G) = \lceil \chi'_c(G) \rceil$
- NP-complete for cubic graphs.

Snarks and their properties

Definition

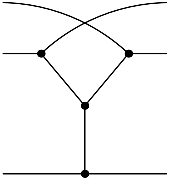
Snarks – bridgeless cubic graphs which are not 3-edge-colourable.

- Vizing: 3 or 4 colours
- $\chi'_c \in (3, 11/3]$ for a snark

  $\chi'_c > 7/2$ for the Petersen graph only?

Known Values

- Flower snarks
  - $\chi'_c(F_3) = 3.5$
  - $\chi'_c(F_5) = 3.4$
  - $\chi'_c(F_{2k+1}) = 3.\overline{3}$ for $k \geq 3$
- Goldberg snarks
  - $\chi'_c(G_3) = 3.\overline{3}$
  - $\chi'_c(G_{2k+1}) = 3.25$ for $k \geq 2$



Girth Conjecture (Jaeger and Swart)

There are no snarks with girth $g > 6$.

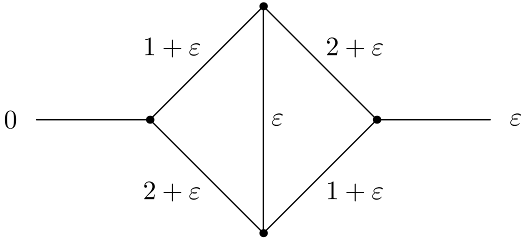
Theorem (Kochol 1996)

There exist snarks with arbitrarily high girth.

Theorem (Kaiser, Kráľ, Škrekovski 2003)

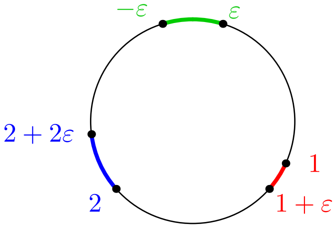
For every $\varepsilon > 0$ there exist g such that every snark G with girth at least g has $\chi'_c(G) \leq 3 + \varepsilon$.

Classes with index converging to 3 can be constructed by inserting diamonds into edges of any snark.

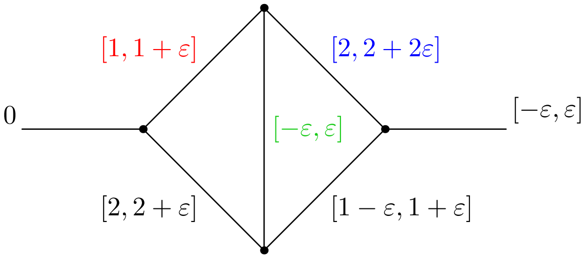


No equivalent of Parity Lemma works.

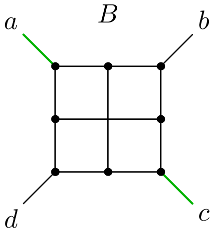
Lower bound for a diamond



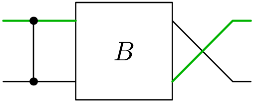
Technique of circular intervals.



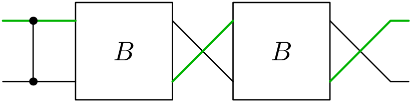
Building blocks of Blanuša snarks



Edges a, c receive the same colour in a 3-edge-colouring.

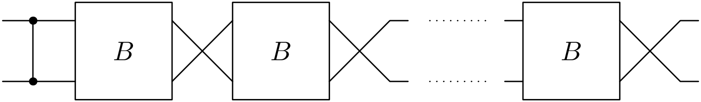


Petersen graph



Blanuša snark

Generalized type 1 Blanuša snarks

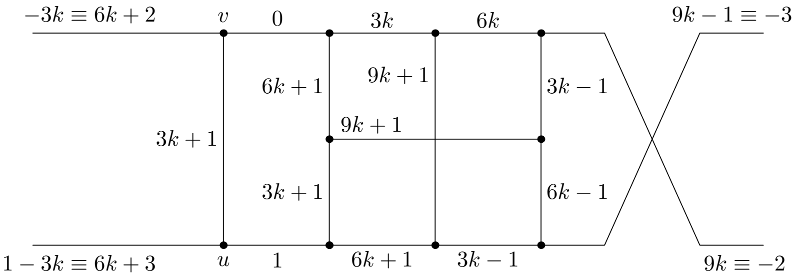


Graph B^1_m with m copies of block B.

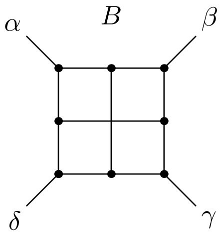
$$\chi'_c(B^1_m) = 3 + \frac{2}{3m}$$

Upper bound

$(9m + 2,\ 3m)$-edge-colouring of B_m^1



Lower bound



$$|\alpha-\gamma| \le 2\varepsilon$$
$$|\alpha-\gamma| + |\beta-\delta| \le 3\varepsilon$$

For m blocks:

$$2 \le \text{total change of colour} \le 3m\varepsilon.$$

$$\chi'_c(B^1_m) = 3+\varepsilon \ge 3 + \frac{2}{3m}$$

Indices of generalized Blanuša snarks are $\left\{3 + \frac{1}{n}; n \geq 2\right\}$.

Conjecture

Circular chromatic index of a snark belongs to $\left\{3 + \frac{2}{k}; k \geq 3\right\}$.

Conjecture (Zhu 2006)

There is no infinite increasing sequence of indices of snarks.

Theorem (Lukoťka, M.)

For any rational number $r \in [3, 3 + 1/3]$ there exist a snark with circular chromatic index r.

Lower bound for snarks of given order

Theorem (Mačaj, M.)

Let G be a snark on $2k$ vertices with girth at least 5. Then

$$\chi'_c(G) \geq 3 + \frac{2.5}{k}.$$

For the generalized type 1 Blanuša snark of order $2k$ we have

$$\chi'_c = 3 + \frac{2.\overline{6}}{k}.$$

Thank you for your attention.