On the structure of 1-planar graphs

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20.11.2008

Planar graphs

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The family $\mathcal{P}$ of planar graphs is well explored; there are hundreds of results concerning various colourings, hamiltonicity/longest cycles, local properties etc.

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- the crossing number
- the thickness of a graph
- embeddings into plane which have constant number of crossings per edge (k-planarity)

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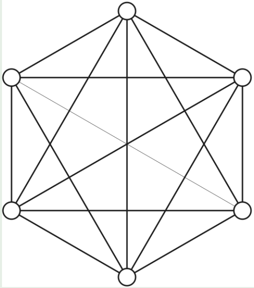
The graph K_6 and its drawings:



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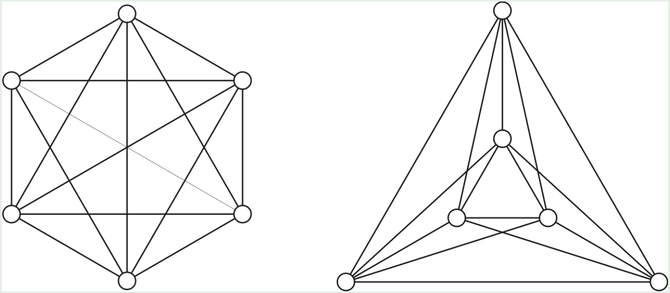

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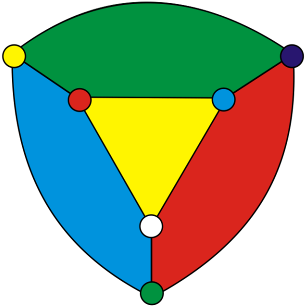

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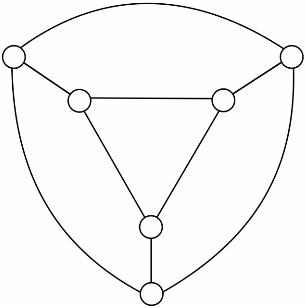

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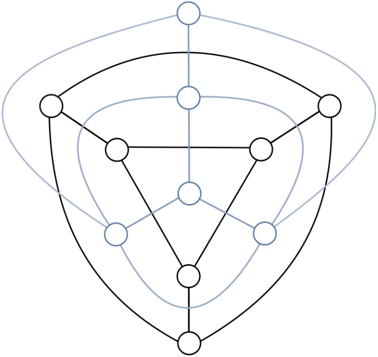

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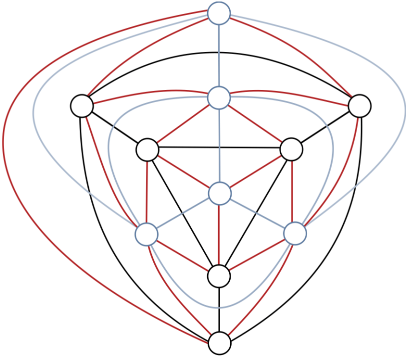

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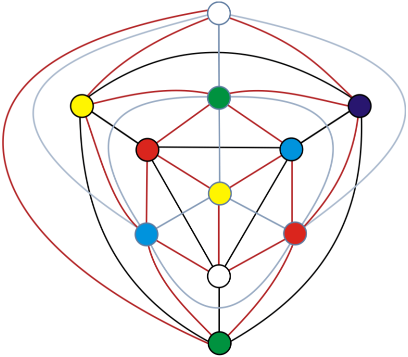

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Comparing to the family of planar graphs, the family $\overline{\mathcal{P}}$ of 1-planar graphs is still little explored (there are max. 30 papers).

Theorem (Kuratowski 1930)

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Theorem (Hopcroft, Tarjan 1974)

There is linear algorithm (in terms of order of a graph) for planarity testing.

Observation

The family of 1-planar graphs is not closed under edge contractions; thus, one cannot consider a characterization of 1-planar graphs in terms of forbidden minors.

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Theorem (Mohar, Korzhik 2008)

The problem of recognition of 1-planar graphs is NP-complete.

Definition

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- notation:

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Lemma

$M(1, \mathcal{P}) = 0$,
$M(2, \mathcal{P}) = 1$,
$M(n, \mathcal{P}) = 3n - 6$ *for all* $n \geq 3$.
For all $n \in \mathbb{N}$, $m(n, \mathcal{P}) = M(n, \mathcal{P})$.

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Corollary

For each graph $G \in \mathcal{P}$, $\delta(G) \leq 5$; *the bound 5 is best possible.*

Theorem (Pach, Tóth 1997; Gärtner, Thiele, Ziegler)

$M(n, \overline{\mathcal{P}}) = 4n - 8$ *for all* $n \geq 12$.

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For each graph $G \in \overline{\mathcal{P}}$, $\delta(G) \leq 7$; *the bound 7 is best possible.*

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Lemma (D.Hudák, T.M.)

$M(n, \overline{\mathcal{P}}) = \binom{n}{2}$ *for* $n \leq 6$,
$M(7, \overline{\mathcal{P}}) = 19$,
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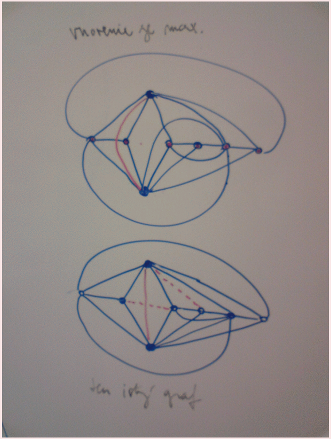
Lemma (D. Hudák,T.M.)

$m(7, \overline{\mathcal{P}}) \leq 18$,
$m(n, \overline{\mathcal{P}}) \leq 4n - 9$ *for* $n = 3k, k \geq 3$.

Observation

Maximal 1-planar drawings need not yield maximal 1-planar graphs:



- notation:

$g(G)\dots$ the girth of G (that is, the length of the shortest cycle in G)

$g(\mathcal{H})\dots \sup\limits_{G\in\mathcal{H}} g(G)$

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Theorem

$g(\mathcal{P}_1) = g(\mathcal{P}_2) = +\infty$,
$g(\mathcal{P}_3) = 5$,
$g(\mathcal{P}_4) = g(\mathcal{P}_5) = 3$.

Theorem (I.Fabrici, T.M. 2007)

$g(\overline{\mathcal{P}}_3) \geq 7$,
$g(\overline{\mathcal{P}}_5) = 4$,
$g(\overline{\mathcal{P}}_6) = g(\mathcal{P}_7) = 3$.

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Recently, R. Soták constructed a 1-planar graph of minimum degree 4 and girth 5.

We conjecture $g(\overline{\mathcal{P}}_3) = 7$ and $g(\overline{\mathcal{P}}_4) = 5$.

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Theorem (Borodin 1979)

Every planar graph is acyclically 5-colourable.

Observation (Vizing)

For each $\Delta \leq 5$ there exists a planar graph G with $\Delta(G) = \Delta$ and with edge chromatic number equal to $\Delta + 1$ (that is, G is class two graph).

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Theorem (Sanders, Zhao 2001)

Each planar graph G with $\Delta(G) \geq 7$ has edge chromatic number equal to $\Delta(G)$ (is of class one).

Theorem (Borodin 1984)

Each 1-planar graph is 6-colourable.

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Theorem (Fabrici, Madaras 2007)

Each 1-planar graph of girth at least 5 is 5-colourable.

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Other colourings (and other questions concerning standard vertex colouring) for 1-planar graphs were not studied yet.

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Formally, for given family $\mathcal{G}$ of plane graphs and a plane graph H, we test the validity of the following statement:

Each graph $G \in \mathcal{G}$ that contains H as a subgraph, contains also a subgraph $K \cong H$ such that each vertex of K has (in G) degree at most $\varphi(H, \mathcal{G}) < +\infty$.

(the number $\varphi(H, \mathcal{G})$ does not depend on G; for certain $\mathcal{G}, H$ need not exist)

Theorem (A. Kotzig 1955)

Each 3-connected graph $G \in \mathcal{P}$ contains an edge such that the sum of degrees of its endvertices is at most 13; moreover, if $G \in \mathcal{P}_4$, then G contains an edge such that the sum of degrees of its endvertices is at most 11. The bounds 13 and 11 are best possible.

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Theorem (I. Fabrici, S. Jendroľ 1997)

Each 3-connected graph $G \in \mathcal{P}$ which contains a k-vertex path, contains also a k-vertex path such that each vertex of this path has degree (in G) at most $5k$. The bound $5k$ is best possible.

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Theorem (Borodin 1989)

Each graph $G \in \mathcal{P}_5$ contains a 3-cycle such that sum of degrees of its vertices is at most 17. The bound 17 is best possible.

Theorem (I. Fabrici, T.M. 2007)

Each 3-connected graph $G \in \overline{\mathcal{P}}$ contains an edge such that degrees of its endvertices are at most 20. The bound 20 is best possible.

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Theorem (I. Fabrici, T.M. 2007)

Each graph $G \in \overline{\mathcal{P}}_6$ contains

- *a 3-cycle with all vertices of degree at most 10; the bound 10 is sharp,*
- *a 3-star with all vertices of degree at most 15,*
- *a 4-star with all vertices of degree at most 23.*

Theorem (I. Fabrici, T.M. 2007)

Each graph $G \in \overline{\mathcal{P}}_7$ contains

- *a 5-star with all vertices of degree at most 11,*
- *a 6-star with all vertices of degree at most 15.*

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Theorem (D. Hudák, T.M.)

Each graph $G \in \overline{\mathcal{P}}_7$ contains

- *a $(7, 7)$-edge,*
- *a graph K_4 with all vertices of degree at most 13,*
- *a graph $K^*_{2,3}$ ($K_{2,3}$ with extra edge in smaller bipartition) with all vertices of degree at most 13,*
- *a 5-cycle with all vertices of degree at most 9.*

Theorem (D. Hudák, T.M. 2008)

Each graph $G \in \overline{\mathcal{P}}_5$ of girth 4 contains

- *a $(5, \leq 6)$-edge,*
- *a 4-cycle with all vertices of degree at most 9,*
- *a 4-star with all vertices of degree at most 11.*

Theorem (D. Hudák, T.M. 2008)

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- *a 4-cycle with all vertices of degree at most 9,*
- *a 4-star with all vertices of degree at most 11.*

Here, the assumption on girth 4 is essential – if $G \in \overline{\mathcal{P}}_5$, then there is no finite bound for degrees of vertices of $C_4 \subseteq G$ or $K_{1,4} \subseteq G$ which is independent on G. In other words, for any m there exists a graph $G_m \in \overline{\mathcal{P}}_5$ such that each 4-cycle $C_4 \subseteq G_m$ contains a vertex of degree at least m (similarly for 4-star).

Thanks for your attention :-)