On the acyclic edge-colouring of graphs

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November 20, 2008

Acyclic edge-colouring

- proper k-edge-colouring
- acyclic k-edge-colouring
  - proper k-edge-colouring
  - no 2-coloured cycle

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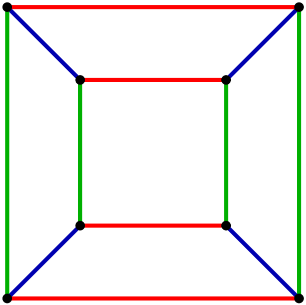

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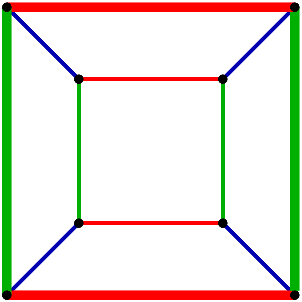

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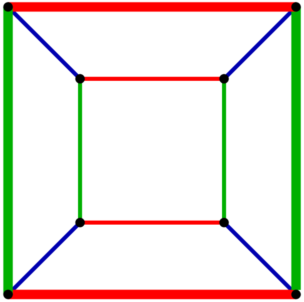

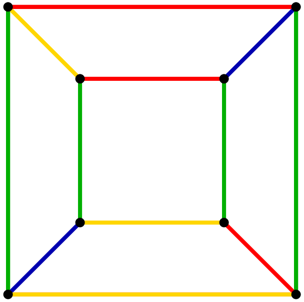

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- ACYCLIC CHROMATIC INDEX:
  - Alon, McDiarmid, Reed 1991: $a'(G) \leq 60\Delta(G)$
  - Molloy, Reed 1998: $a'(G) \leq 16\Delta(G)$
  - Conjecture Alon, Sudakov, Zaks 2001: $\Delta(G) \leq a'(G) \leq \Delta(G) + 2$

Acyclic edge-colouring – results

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- Alon, Sudakov, Zaks, 2001: true for almost all Δ-regular graphs
- Muthu, Narayanan, Subramanian, 2005: $a'(G) \leq 4.52\Delta$ for any G of girth at least 220
- Muthu, Narayanan, Subramanian, 2006: true for grid-like graphs
- Muthu, Narayanan, Subramanian, 2007: true for outer-planar graphs
- Nešetřil, Wormald, 2005: $a'(G) = \Delta + 1$ for almost every Δ-regular graph G

SUBCUBIC GRAPHS

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Theorem (E.M., Mazák, 2008)

G subcubic connected,

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PROOF.

- by induction on the number of vertices of G
- the proof splits into several cases depending on δ and the girth

$\longrightarrow$ Sketch of the induction step for $\delta = 3$ and even girth $g \geq 6$

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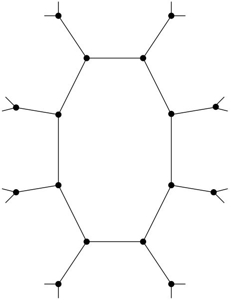

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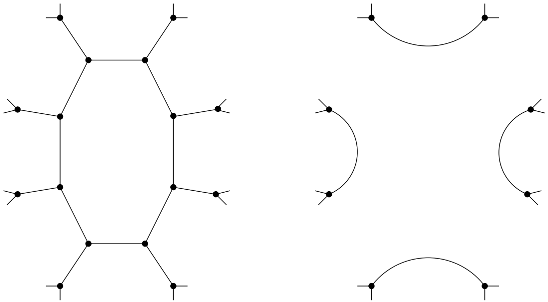

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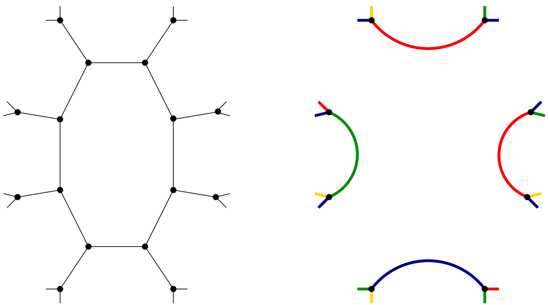

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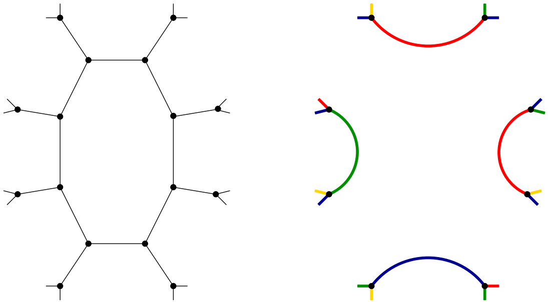

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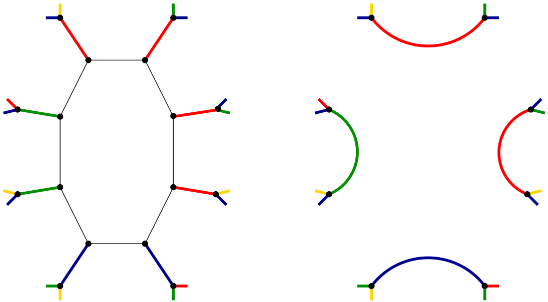

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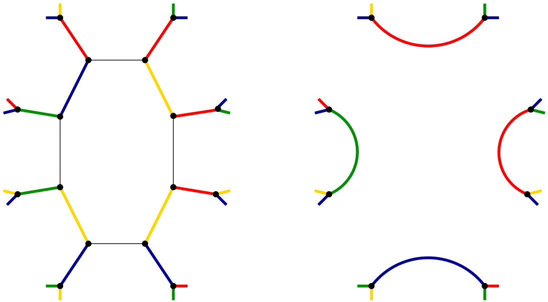

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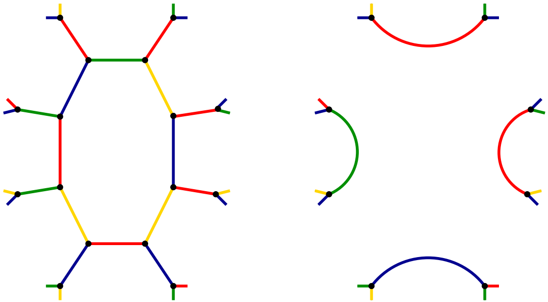

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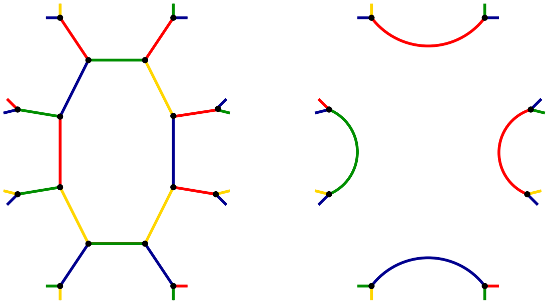

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- suppose $a'(G) = 3$
  - $\longrightarrow$ every colour is present at every vertex
  - $\longrightarrow$ edges of any colour form a perfect matching of G
  - $\longrightarrow$ any two colours induce a 2-factor of G

Thank you!