Real flow numbers of Blanuša snarks

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Integral flows on graphs

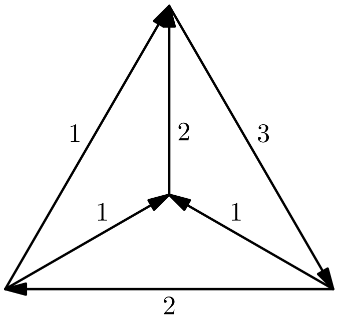
A nowhere-zero (NZ) k-flow [Tutte 1949]:

- Orientation
- Function $\varphi : E(G) \rightarrow \mathbb{N}$
  - $0 < \varphi(e) < k$
  - $\sum_{e\in\{v\}^+} \varphi(e) = \sum_{e\in\{v\}^-} \varphi(e)$

The flow number of a graph $\Phi_{\mathbb{Z}}(G)$:

- The smallest k such that G has a NZ k-flow.

Example of a nowhere-zero 3-flow



Possible values of $\Phi_{\mathbb{Z}}$

Only bridgeless graphs may have NZ flows.

There are graphs with flow numbers 2, 3, 4 and 5.

Conjecture (Tutte's 5-flow conjecture)

Every bridgeless graph has a NZ 5-flow.

Possible values of $\Phi_{\mathbb{Z}}$

Theorem (Seymour)

Every bridgeless graph has a NZ 6-flow.

Real flows on graphs

A real NZ r-flow:

- Orientation
- Function $\varphi : E(G) \to \mathbb{R}$
  - $1 \leq \varphi(e) \leq r - 1$
  - $\sum_{e \in \{v\}^+} \varphi(e) = \sum_{e \in \{v\}^-} \varphi(e)$

Real flow number

- $\Phi_{\mathbb{R}}(G) = \inf\{r \mid G \text{ has a NZ } r\text{-flow}\}$

Real flows on graphs

Why $1 \leq \varphi(e) \leq r - 1$?

- $1 \leq \varphi(e)$
  The flow $\varphi(e)/a$ is also a NZ-flow.
- $\varphi(e) \leq r - 1$
  The maximal possible flow value is the same as for the integer case.

1993 – Godyn, Tarsi a Zhang – dual concept to the fractional colorings.

Basic properties of real NZ flows

- The infimum from the definition is a minimum.
- The real flow number is rational.
- If $\Phi_{\mathbb{R}}(G) = p/q$ then it is sufficient to use values with the denominator q. to create a real NZ p/q-flow.

The real flow number and the flow number

Theorem (Goddyn, Tarsi, Zhang)

$\Phi_{\mathbb{Z}}(G) = \lceil \Phi_{\mathbb{R}}(G) \rceil$.

Theorem (Goddyn, Tarsi, Zhang)

$\Phi_{\mathbb{R}}(G) = \Phi_{\mathbb{Q}}(G)$.

Snarks

Snark is a non-trivial cubic graph without 3-edge-colouring.

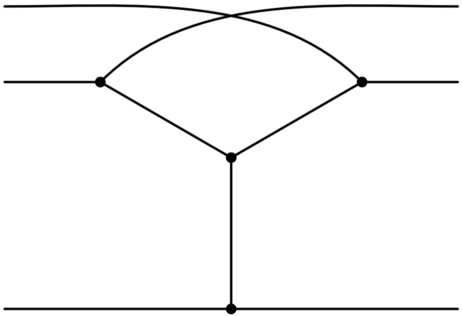
3-edge-colouring = nowhere-zero 4-flow.
It may be useful to study circular edge colourings and real nowhere-zero flows simultaneously.

Snarks

Objects of our attention:

- Isaacs snarks
- Blanuša snarks
- Goldberg snarks

Isaacs snarks



Isaacs snarks

Upper bound - E. Steffen.

Theorem

The real flow number of the Isaacs snark I_{2k+1} is
$4 < \Phi_{\mathbb{R}}(I_{2k+1}) \leq 4 + 1/k$.

Lower bound - joint work with M. Škoviera.

Real flows and orientations of a graph

Theorem (Goddyn, Tarsi, Zhang)

Let G be a bridgeless graph. Then G has a real nowhere-zero $(p/q+1)$-flow if and only if there exists an orientation O of G such that for each set S of vertices of G we have

$$q/p \leq |S^+|/|S^-| \leq p/q.$$

The bound

- There must exist an orientation and a subset of vertices such that
$$\frac{|S^+|}{|S^-|} = \frac{p}{q}$$
- Moreover, let us assume that both $G(S)$ and $G(V(G) - S)$ are connected.
- Therefore, the following holds for the boundary of S:

$$|\delta_G S| = |S^+| + |S^-| \geq p + q.$$

The bound

Theorem

Let G be a graph such that $\Phi_{\mathbb{R}}(G) = p/q + 1$ where p and q are two relatively prime positive integers. Then there exists a subset $S \subseteq V(G)$ such that both subgraphs of G induced by S and $V(G) - S$ are connected and

$$\delta_G(S) \geq p + q.$$

The bound for snarks

Since snarks are 3-regular, the following holds

- A snark with real flow number at most $4 + 1/k$ has at least $8k - 2$ vertices.
- A snark with at most $8k + 4$ vertices has its real flow number at least $4 + 1/k$.

Isaacs snarks

Since Isaacs snark I_k has $8k + 4$ vertices:

Theorem

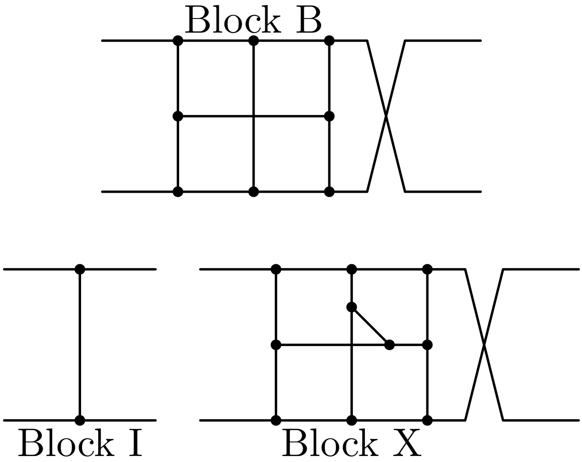
The real flow number of the Isaacs snark I_{2k+1} is $\Phi_{\mathbb{R}}(I_{2k+1}) = 4 + 1/k$.

Circular chromatic index

Theorem (Ghebleh, Kráľ, Norine, Thomas)

- $\chi_c(I_3) = 7/2$
- $\chi_c(I_5) = 17/5$
- $\chi_c(I_{2k+1}) = 10/3$ *for* $k \geq 3$.

Blanuša snarks



Balanced valuations - simplified

We take:

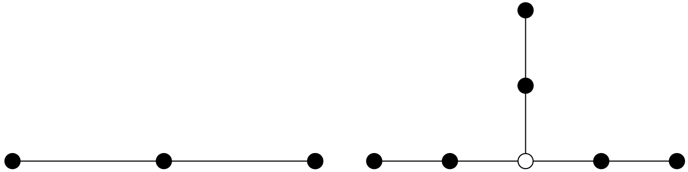
– a fixed nowhere-zero $(3+\varepsilon)$-flow φ, $\varepsilon < 1/2$,

– a positive orientation O of G.

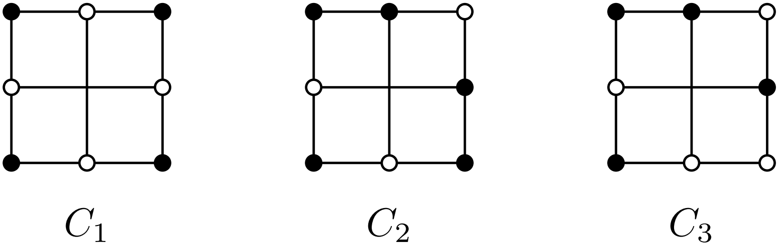
- Two incoming edges – white vertex.
- Two outgoing edges – black vertex.

Number of black vertices = number of white vertices.

Forbidden substructures

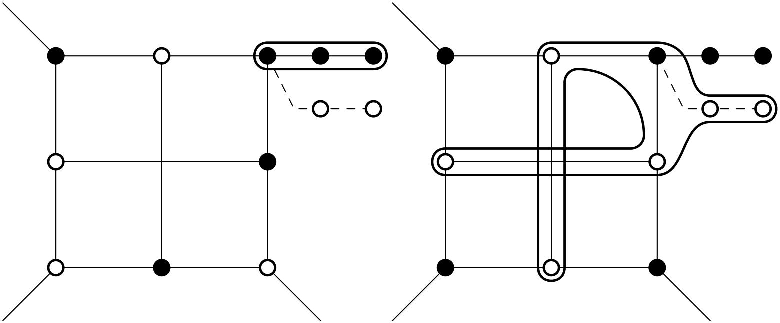


Possible colourings of the basic block



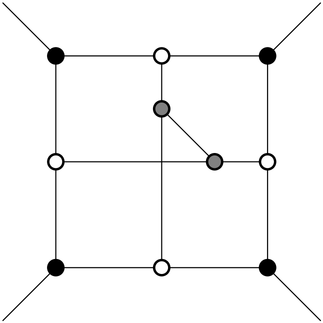
The colouring C_1

The colouring C_1 can not combine with the others:

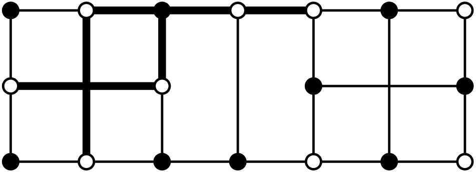


The colouring C_1 is unusable

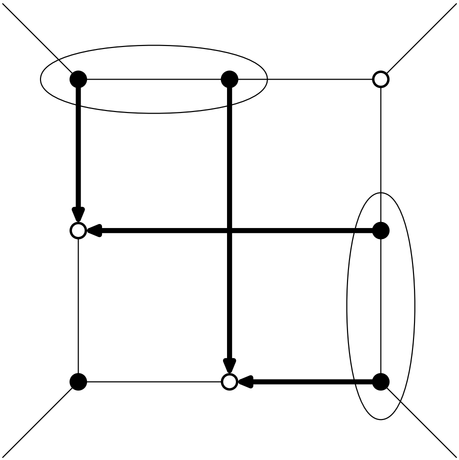
The middle blocks have to be coloured as follows:



The colouring C_1 is unusable



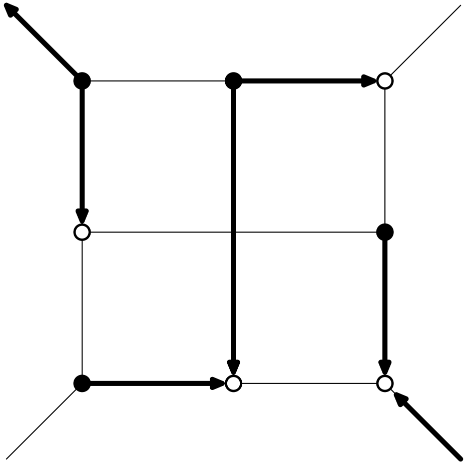
The colouring C_2 is unusable



Tight edges

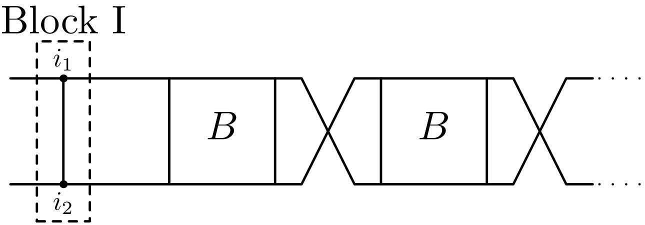
We create a modular flow φ' in $\mathbb{R}/(4+\varepsilon)\mathbb{Z}$ from the flow φ. We take the orientation so that all values are in $\langle 1, 2+\varepsilon/2\rangle$.
Tight edge – an edge with flow value $\langle 1, 1+\varepsilon\rangle$ in φ'.
Semi-tight edge – an edge with flow value $\langle 1, 1+2\varepsilon\rangle$ in φ'.

Structure of tight edges in block B.



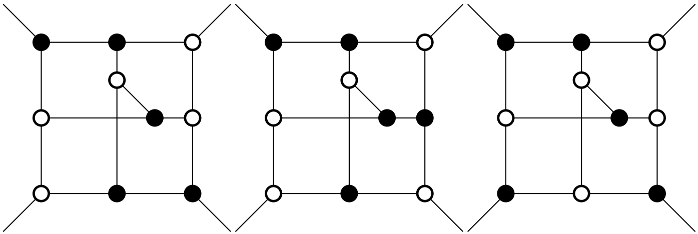
Blanuša snarks of type I

The total flow difference on neighbouring edges of Block I is at most 2ε.



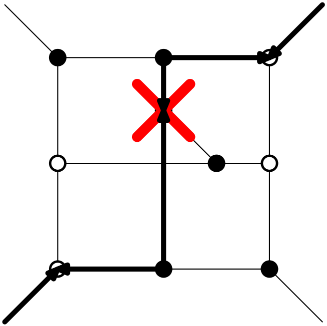
Blanuša snarks of type II

Possible colourings of block X.



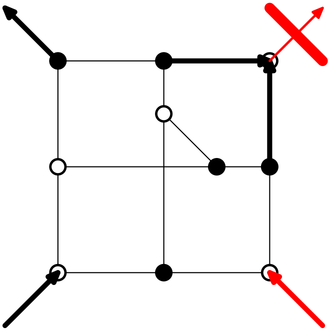
Colouring 1

Two tight edges have incompatible orientation.



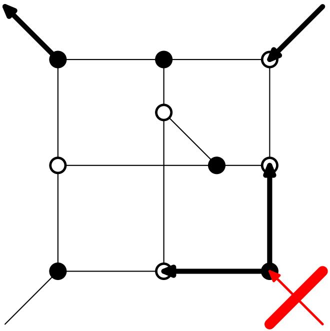
Colouring 2

One of the edges on right side has to be tight – the bottom right one.



The value of the upper right edge is at most $1+2\varepsilon$ - impossible.

Colouring 3



The value of the bottom right edge is at most $1 + 2\varepsilon$ - impossible.

Real flow number of the Blanuša snarks

It is easy to construct 4.5-flows on the Blanuša snarks that are different from the Petersen graph.

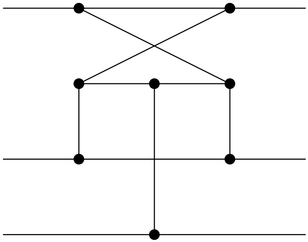
- $\Phi_{\mathbb{R}}(B_1^i) = 5$.
- $\Phi_{\mathbb{R}}(B_j^i) = 4 + 1/2$, $j \geq 2$.

Circular chromatic index

Theorem (Mazák, Ghebleh)

- $\chi_c(B_n^1) = 3 + 2/n$
- $\chi_c(B_n^2) = 3 + 1/\lfloor 1 + 3n/2 \rfloor.$

Goldberg snarks



Real flow number

Goldberg snark G_{2k+1} has its real flow number

$$4 + 1/(2k+1) \leq \Phi_{\mathbb{R}}(G_{2k+1}) \leq 4 + 1/k.$$

Circular chromatic index

Theorem (Ghebleh)

- $\chi_c(G_3) = 3 + 1/3$
- $\chi_c(G_{2k+1}) = 3 + 1/4$ *for* $k > 1$.

Thank you for your attention.