Classification and enumeration of discrete group actions on Riemann surfaces of small genera

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Background

Motivation:
study of algebraic curves, map theory, group theory, chemistry, crystallography, physics...

General proposition:
Every finite group acts as an automorphism group of a surface (Greenberg).

Problem:
Given class $\mathbf{R}$ of Riemann surfaces with genus $g \geq 2$, describe the class $\mathbf{G}$ of finite groups s. t. $\mathrm{G} \in \mathbf{G}$ acts as a group of orientation-preserving automorphisms of a surface $\mathcal{S} \in \mathbf{R}$.

In the words of Greenberg:
Study all orientation-preserving self-homeomorphisms of surfaces from $\mathbf{R}$.

History:
Riemann, Hurwitz, Klein, Schwarts, Wiman...,
...Singermann, Jones, Conder, Mednykh, Nedela...

Glossary

Riemann-Hurwitz equation($\mathcal{S}_g \to \mathcal{S}_\gamma$) relates genera of a surface and its cover with an orbifold and automorphism group

$$2g-2 = |\mathrm{G}|\left(2\gamma - 2 + \sum_{i=1}^{r}\left(1-\frac{1}{m_i}\right)\right); \forall i : m_i \geq 2 \in \mathbb{Z};$$

Universal cover $\widetilde{M}$ is a tesselation of elliptic ($g=0$), Euclidean ($g=1$) or hyperbolic ($g \geq 2$) plane;

Fuchsian group F is a discrete group with the presentation

$$\langle x_1, \dots, x_r, a_1, b_1, \dots, a_\gamma, b_\gamma \mid x^{m_1} = \dots = x_r^{m_r} = \prod_{i=1}^{\gamma}[a_i, b_i] \prod_{j=1}^{r} x_j = 1 \rangle;$$

Quotient $\bar{M}$ is an one-vertex map on the surface $\mathcal{S}_\gamma$

Quotient surface $\mathcal{S}_\gamma$ is an orbifold with the signature

$$[\gamma; m_1, m_2, \dots, m_r],$$

where orbifold is as surface of genus γ with r points (branch-points) chosen, endowed with indexes $m_1, \dots m_r$.

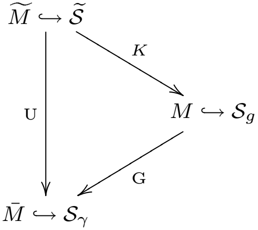
Combinatorial approach

1. An automorphism (orientation-preserving) of a vertex-transitive map M extends to a self-homeomorphism φ of the surface $\mathcal{S}$
2. Every finite group of automorphisms of a surface S is a group of automorphisms of a (Cayley) vertex-transitive map on S
3. It is sufficient to study the class of vertex-transitive maps on $\mathbf{R}$ instead of surfaces over $\mathbf{R}$ – "dimension reduction" of the problem;
4. Lots of results and techniques of map theory, group theory are known. Techniques are more convenient to use (especially for me:)), the software can help (GAP, Magma. . .).

Core idea

Theorem

Let M be a vertex-transitive map of genus g and let $\mathrm{G} \leq \mathrm{Aut}^{+}(M)$ be a vertex-transitive subgroup. Let $\bar{M} = M/\mathrm{G}$ be its one-vertex quotient on an orbifold $\mathcal{O}(\gamma; m_1, m_2 \ldots, m_r)$. Then there exist a torsion-free normal subgroup $K \lhd \mathrm{F} \cong \mathrm{Aut}^{+}(\widetilde{M}) \cong \pi_1(\mathcal{O})$ of genus g such that $\mathrm{G} \cong \mathrm{F}/K$ and $M \cong \widetilde{M}/K$. In particular, the index $[\mathrm{F} : K]$ is given by Riemann-Hurwitz equation relating M and $\bar{M}$.



Algorithm: Solving Riemann-Hurwitz equation (numerically)

Rieman-Hurwitz equation again

$$2g-2=|\mathrm{G}|\left(2\gamma-2+\sum_{i=1}^{r}\left(1-\frac{1}{m_i}\right)\right);\forall i: m_i\geq 2\in\mathbb{Z};$$

At first we solve it numerically.
We have to observe and meet the following criteria:

1. $\gamma\leq g$,
2. $r\leq 2g+2$,
3. $\forall i:\ |\mathrm{G}|\equiv 0 \mod m_i$,
4. $|\mathrm{G}|\leq 84(g-1)$.

Now we can formulate an algorithm which determines all possible numeric solutions of RHE for given genus. The solutions obtained by (brute-force) computation are tuples

$$[g,\gamma,|\mathrm{G}|,\{m_1,\ldots,m_r\}].$$

Note that

$$(\gamma;\{m_1,\ldots,m_r\})$$

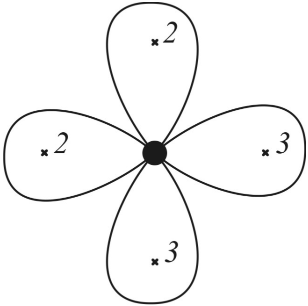
is known as orbifold signature of the orbifold $\mathcal{O}$ which the quotient $\bar{M}$ is embedded in.

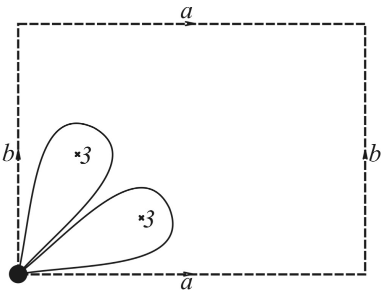
Algorithm: Setting the presentation of $\pi_1(\mathcal{O})$

Canonical quotient

1. quotient map $\bar{M}$ is a buquet of r loops,
2. every loop is the boundary of a face containing exactly one branch-point with respective branch-index m_i,
3. outer face of the map is an r-gon containing no branch-point.

Examples

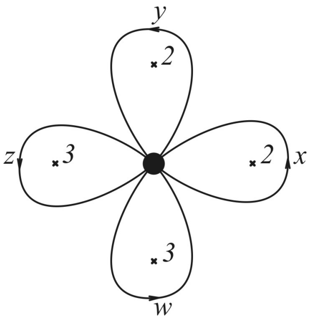




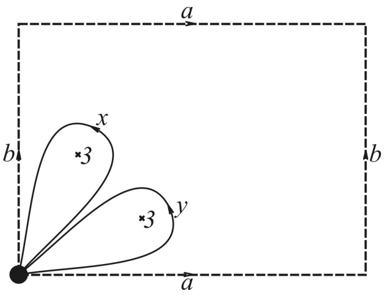
Algorithm: Setting the presentation of $\pi_1(\mathcal{O})$ (continued)

We adapt the classical concept of voltage-assignments by Gross and Tucker to use it for maps on orbifolds.

Presentations



$\langle x, y, z, w \mid x^2 = y^2 = z^3 = w^3 = xyzw = 1 \rangle$



$\langle x, y, a, b \mid x^3 = y^3 = [a, b]xy = 1 \rangle$

What about non-canonical quotients?

Algorithm: Setting quotients (up to isomorphism)

Observations:

1. $|\mathrm{F} : K| = |\mathrm{G}|$,
2. $K \trianglelefteq \mathrm{F}$,
3. $\mathrm{F} \longrightarrow \mathrm{G}$ is order preserving, i.e.
   - no generator of F is sent to identity,
   - no relation of F is collapsed.

The problem reduces to classification of order-preserving, torsion-free normal subgroups (subgroups of genus g) of F, where F ranges through all admissible signatures.

Low-index subgroups procedure is the tool of first choice. We adapted one implementation by P. Dobcsányi.

Finally we use GAP to check whether $\mathrm{F} \rightarrow \mathrm{G}$ is order-preserving and reveal the structure description of G;

We want more: How every kernel does look like?

Example: Genus 2 quotients

G	# K's	Orbifold	G	# K's	Orbifold
C_2	1	$\mathcal{O}(0; 2^6)$	C_{10}	1	$\mathcal{O}(0; 2, 5, 10)$
C_2	1	$\mathcal{O}(1; 2^2)$	$C_2 \times C_6$	1	$\mathcal{O}(0; 2, 6^2)$
C_3	3	$\mathcal{O}(0; 3^4)$	$C_3 \rtimes C_4$	1	$\mathcal{O}(0; 3, 4^2)$
C_4	1	$\mathcal{O}(0; 2^2, 4^2)$	D_{12}	3	$\mathcal{O}(0; 2^3, 3)$
$C_2 \times C_2$	10	$\mathcal{O}(0; 2^5)$	$C_8 \rtimes C_2$	1	$\mathcal{O}(0; 2, 4, 8)$
C_5	3	$\mathcal{O}(0; 5^3)$	$C_2 \ltimes (C_2^2 \times C_3)$	1	$\mathcal{O}(0; 2, 4, 6)$
C_6	1	$\mathcal{O}(0; 3, 6^2)$	$\mathrm{SL}_2(3)$	1	$\mathcal{O}(0; 3, 3, 4)$
C_6	2	$\mathcal{O}(0; 2^2, 3^2)$	$\mathrm{GL}_2(3)$	1	$\mathcal{O}(0; 2, 3, 8)$
D_6	1	$\mathcal{O}(0; 2^2, 3^2)$			
C_8	1	$\mathcal{O}(0; 2, 8^2)$			
Q_8	1	$\mathcal{O}(0; 4^3)$			
D_8	3	$\mathcal{O}(0; 2^3, 4)$			

Classification up to isomorphisms of groups

Former results:

- 1991 Broughton – genera $g = 2$, $g = 3$;
- 1997 Bogopolski – genus $g = 4$;
- 1990 Kuribayashi and Kimura – genus $g = 5$;
- 2008 J.K. and R.N. – genera $g = 2 \ldots 15$.

Census

g	# coverings	bound for $\lvert G \rvert$	
2	21	48	(Klein)
3	49	168*	(Klein)
4	63	120	(Gordan)
5	92	192	(Wiman)
6	87	150	(Wiman, $\lvert G \rvert < 420$)
7	147	504*	
8	108	336	
9	260	320	
10	225	432	

Generalized Archimedean solids (maps)

A map M is Archimedean (of genus g) if the following holds:

1. Underlying surface $\mathcal{S}$ of M is orientable of genus g (≥ 0),
2. $\mathrm{Aut}^{+}(M)$ acts transitively on vertices of M,
3. Underlying graph is simple,
4. Face-width $r(M) \geq 3$.

From (2) and (3) – Mader $\Longrightarrow$ M is 3-connected
From (3) and (4) – Jendroľ and Voss $\Longrightarrow$ M is polyhedral

Classification "by hand"

$g = 0$ 5 Platonic solids, 13 other maps, ∞-many prisms,

$g = 1$ ∞-many maps of 10 local types (Grünbaum),

$g \geq 2$ finitely many Archimedean solids; [K. and N. up to genus 5].

Concluding remarks and suggestions

1. one-vertex quotients on orientable surfaces with empty boundary – classification of vertex-transitive maps on orientable surfaces up to genus g (K. & N., $g = 2$: 13, $g = 3$: 123, $g = 4$: 136, $g = 5$: 397 polyhedral ones);
2. analyse of kernels of order-preserving epimorphisms – study of outer automorphisms of Fuchsian groups (special cases studied by G. Jones, A. Breda. . .);
3. every non-orientable map have orientable double cover, so we can classify non-orientable compact surfaces with empty boundary (partial results archieved);
4. general problem of coverings of spaces. The numerical conditions reads as follows

$$2g + k - 2 = |\mathrm{G}| \left(2\gamma + k' - 2 + \sum_{i=1}^{r} \left(1 - \frac{1}{m_i}\right) + \frac{1}{2} \sum_{i=1}^{k'} \sum_{j=1}^{h_i} \left(1 - \frac{1}{n_{ij}}\right)\right).$$

5. we can focus to special quotients to help the enumerations of special classes coverings (cyclic case – N. & Mednykh, K. – actions up to $g = 30$); what about Abelian case?
6. theory of maps on orbifolds (enumeration and classification problems solved on "solid ground")