Matroids, minors and complexes

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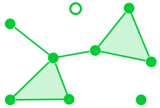
Complexes

Definition

A complex $\mathcal{C}$ on E is a collection of subsets of E (faces) such that

$$\text{if } \sigma \subset \tau \in \mathcal{C}, \text{ then } \sigma \in \mathcal{C}.$$



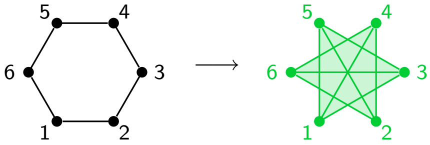
Complexes include:

- independence complexes of graphs,
- matroids.

Independence complexes

Definition

The independence complex of a graph G is the complex on $V(G)$ whose faces are the independent sets of G.



Matroids

Definition

A complex $\mathcal{M}$ on E is a matroid if for every $\sigma, \tau \in \mathcal{M}$ such that $|\sigma| < |\tau|$, there is $x \in \tau \setminus \sigma$ such that $\sigma + x$ is a face.

- independent set of $\mathcal{M}$ = face

Definition

The cycle matroid $\mathcal{M}(G)$ of a graph G: the matroid on $E(G)$ whose independent sets are the acyclic sets of edges.

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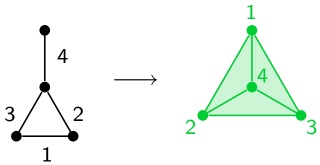

Minors of graphs

Definition

A minor of a graph G: any graph that can be obtained from G by a series of vertex or edge deletions and edge contractions.

Two out of many theorems and problems concerning minors:

- Kuratowski Theorem: a graph is planar if and only if it contains no minor in $\{K_5, K_{3,3}\}$
- Tutte's 4-flow conjecture: every graph with no Petersen minor has a nowhere-zero 4-flow

Matroid minors

Definition

Let $\mathcal{M}$ be a matroid on E and $X \subset E$. The deletion and contraction of e are the matroids

$$\mathcal{M} \setminus e = \{I \subset E - \{e\} : I \in \mathcal{M}\},$$
$$\mathcal{M} / e = \{I \subset E - \{e\} : I + e \in \mathcal{M}\}.$$

A minor of $\mathcal{M}$: any matroid obtained by a sequence of contractions and deletions.

Some forbidden minor characterizations of Tutte:

- binary matroids: no $U_{2,4}$ minor
- regular matroids: no $U_{2,4}$, F_7 or F_7^* minor
- graphic matroids: no $U_{2,4}$, F_7, F_7^*, $M^*(K_{3,3})$ or $M^*(K_5)$ minor

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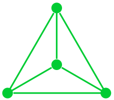


Complex minors

We may define deletion, contraction and minors for complexes in exactly the same way as for matroids:

$$\mathcal{C} \setminus e = \{ I \subset E - \{e\} : I \in \mathcal{C} \} ,$$
$$\mathcal{C} \,/\, e = \{ I \subset E - \{e\} : I + e \in \mathcal{C} \} .$$

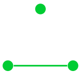
- these correspond to 'induced subcomplexes' and 'links'
- caution: need to contract one element at a time, since $(\mathcal{C}/e)/f$ need not equal $\mathcal{C}/\{e, f\}$ (with the natural definition)

A characterization of matroids and independence complexes

T_i = the complex on 3 points with i faces of size 2.

Theorem (TK 2006)

A complex is a matroid if and only if it contains no minor isomorphic to T_1.



Theorem (TK 2006)

A complex is an independence complex if and only if it contains no minor isomorphic to T_3.

How about complexes with no T_2 minor?

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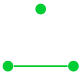

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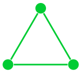

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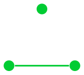

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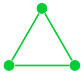

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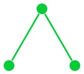

Decreasing functions

Definition

A function $f : 2^E \to \mathbb{N}$ is decreasing if for each pair of subsets $X \subset Y \subset E$,

$$f(X) \geq \min\{f(Y), |X|\}.$$

Observation

Any decreasing f determines a complex $\mathcal{C}(f)$ on E whose faces are all the $\sigma \subseteq E$ such that $f(\sigma) \geq |\sigma|$.

Complexes with no T_2-minor

Definition

$f : 2^E \to \mathbb{N}$ is admissible if it is decreasing and for each $X, Y \subset E$,

$$f(X \cup Y) \geq \min \{f(X), f(Y), |X \cap Y| + 1\}.$$

Theorem

A complex $\mathcal{C}$ on E has no T_2-minor if and only if there is an admissible $f : 2^E \to \mathbb{N}$ such that $\mathcal{C} = \mathcal{C}(f)$.

The matroid intersection theorem

- rank $\mathrm{r}_{\mathcal{M}}(X)$ of a set $X \subset E$ in a matroid $\mathcal{M}$ on E = the size of maximal independent sets contained in X

Theorem (Edmonds 1965)

Let $\mathcal{M}$ and $\mathcal{N}$ be matroids on E. They have a common independent set of size n if and only if

$$r_{\mathcal{M}}(X) + r_{\mathcal{N}}(E - X) \geq n$$

for all $X \subset E$.

Applications include:

- Tutte's and Nash-Williams' characterization of graphs with k disjoint spanning trees,
- Hall's theorem on perfect matchings in bipartite graphs.

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A complex intersection theorem?

- a recent result of Aharoni and Berger allows one to replace $\mathcal{N}$ with an arbitrary complex and $r_{\mathcal{N}}(X)$ with the 'topological connectivity' of the induced subcomplex on X
- the connectivity equals rank in the matroidal case
- the straightforward generalization to two complexes does not work (the RHS may be twice the actual dimension of the intersection)
- however, it can be shown to work if the two complexes do not contain a T_2 minor

Matroid intersection and minors

Observation

Complexes that can be obtained as the intersections of two matroids on the same ground set form a minor-closed class.

Forbidden minors for being a matroid intersection include, e.g.:

- 'odd cycles',
- some complexes on 4 points:

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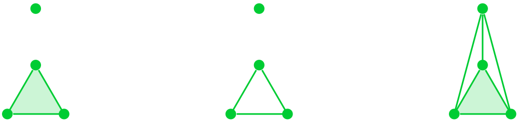

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Some questions

Some questions:

- is there a forbidden minor characterization of matroid intersections?
- more specifically, which 1-dimensional complexes are matroid intersections?
- does the matroid intersection theorem have interesting analogues in other minor-closed classes?
- do some properties of binary matroids extend to complexes without $U_{2,4}$ minors?
- what can be said about T_0-free matroids?
- matroid duality can be extended in a straightforward way to all complexes; which minor-closed classes are closed under duality?

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Thank you for your attention.