Distinguishing finite and infinite graphs

- with special emphasis on Cartesian products -

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Contents

1. The distinguishing number

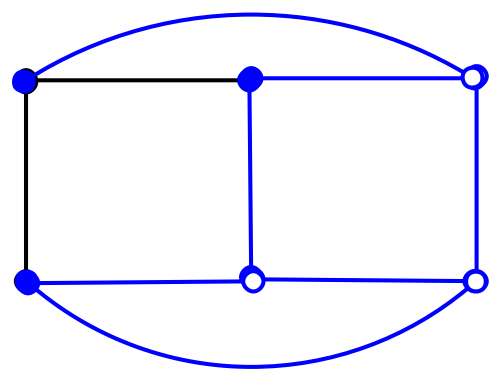
$D(G)$ of a graph G is the least natural number d such that G has a labeling with $D(G)$ labels that is not preserved by any nontrivial automorphism:

$D(P_n) = 2$ for $n > 1$

$D(P_\infty) = 1$

$D(T_2) = 2$

$D(K_3 \square K_2) = 2$



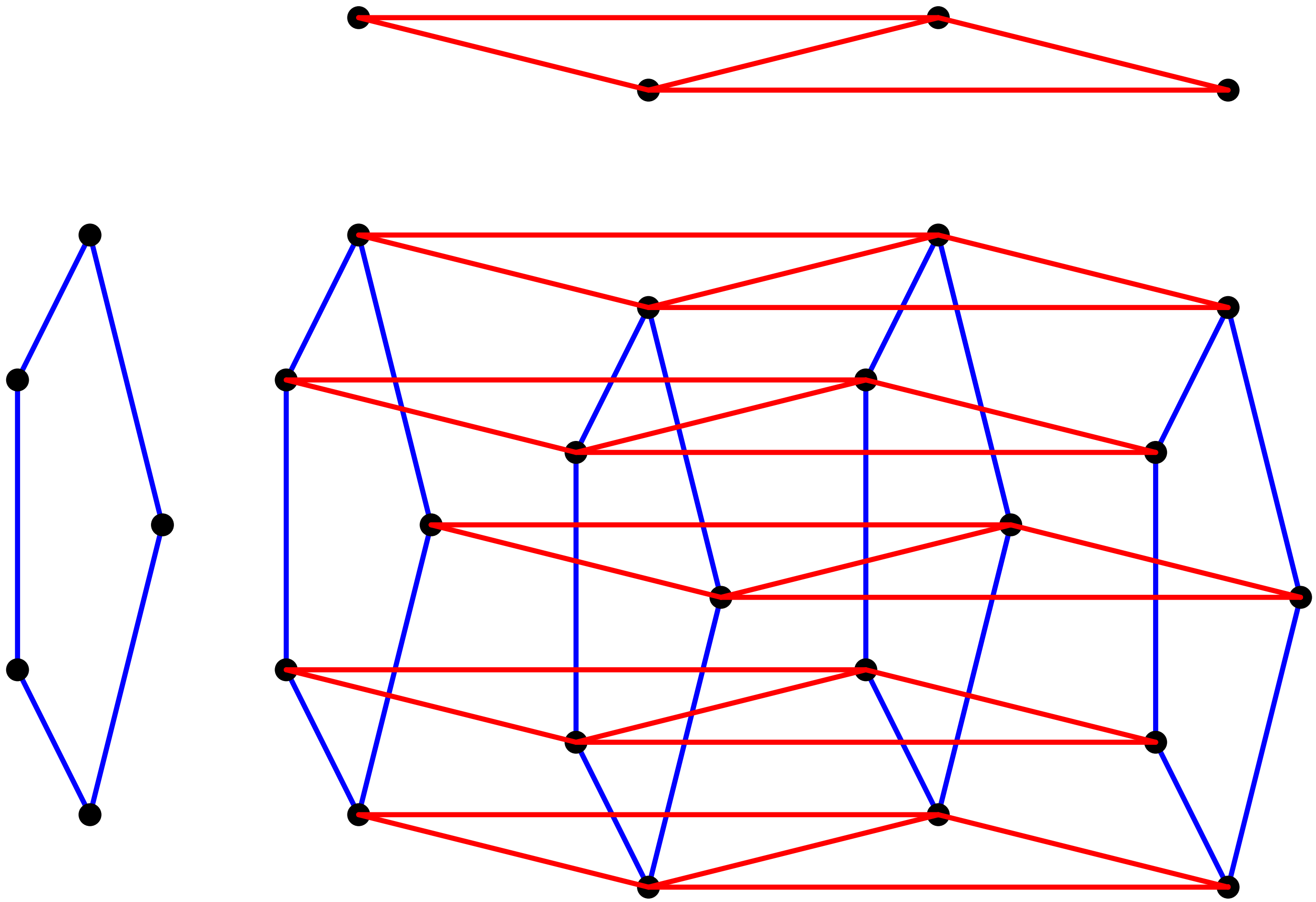
Proposition *The automorphisms of a Cartesian product of prime graphs are induced by automorphisms of the factors and permutations of isomorphic factors.*

The Cartesian product $G \square H$

$$V(G \square H) = V(G) \times V(H),$$

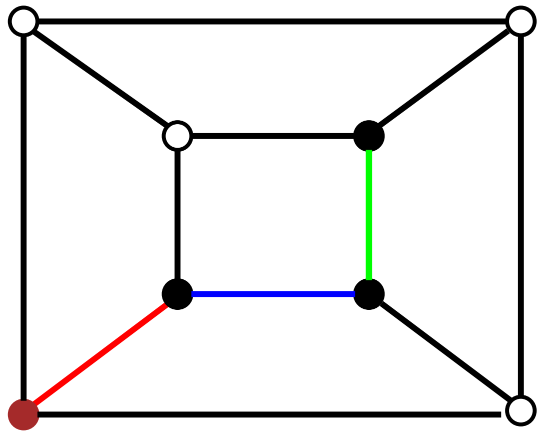
$E(G \square H)$ that is the set of all pairs $[(u, v), (x, y)]$

where either $u = x$ and $[v, y] \in E(H)$ or $[u, x] \in E(G)$ and $v = y$

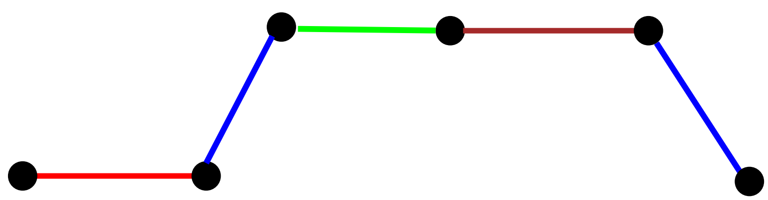


Bogstad and Cowen, 2004, determined $D(Q_k)$:

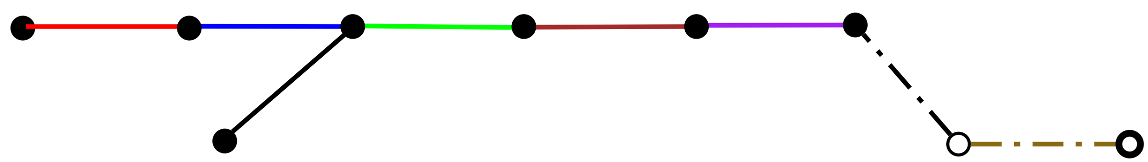
$D(Q_2) = D(Q_3) = 3$; the figure shows $D(Q_3) \leq 3$



$D(Q_k) = 2$ for $k \geq 4$



We needed $k+2$ black vertices for the distinguishing coloring. One can do with k for $k \geq 7$:



Thus, one needs fewer than k black vertices to distinguish Q_k. How many suffice?

Let B be a smallest set of distinguishing black vertices. Any automorphism that stabilizes it is the identity. Clearly $|B| \leq k$.

What is we look for a smallest set S such that every automorphism α that fixes every element in S is the identity. It is plausible that S can be as small as $\log k$ for Q_k.

But, since S is so small, we $d_G(u,v) \neq d_G(x,y)$ unless $\{u,v\} = \{x,y\}$. But then every automorphism that stabilizes S also fixes every vertex, and thus is the identity.

Theorem **(Debra Boutin)** *Let B be a smallest set of black vertices that distinguishes Q_k. If $k \geq 5$, then*

$$\lceil \log_2 k \rceil + 1 \leq |B| \leq 2\lceil \log_2 k \rceil - 1$$

.

Suppose $\alpha B = \beta B$.

Then $\beta^{-1}\alpha B = B$. Hence $\beta^{-1}\alpha = id$ and $\alpha = \beta$.

This means, if we wish to check whether α and β are the same, we have to check whether $\alpha B = \beta B$, where B has size

$$< 2 \log_2 k.$$

Boutin's proof uses a tedious construction. She also can probably prove that

$$\lceil \log_3(2k+1) \rceil + 1 \leq |B| \leq 2\lceil \log_3(2k+1) \rceil - 1$$

for K_3^k.

Can one prove a general theorem for K_n^k by probabilistic methods?

2. Distinguishing the infinite hypercube

- mainly with Werner Klöckl -

The vertices of the infinite hypercube $Q_{\aleph_0}$ are the infinite 01-sequences; any two of them being adjacent if they differ in exactly one place.

$Q_{\aleph_0}$ is a component of the Cartesian product of $\aleph_0$ copies of K_2, the so-called weak Cartesian product.

Theorem $D(Q_{\aleph_0}) = 2$.

Proof. Let P be a one-sided infinite path that contains exactly one edge of every set of parallel edges of $Q_{\aleph_0}$. Color its vertices black and all others white. This is a distinguishing coloring. $\square$

Corollary *Let G be the weak Cartesian product of $\aleph_0$ complete graphs K_2 or K_3. Then $D(G) = 2$.*

Proof. A triangle is fixed if two of its vertices are fixed.

Choose the edges of P such that it contains exactly one edge of every set of parallel edges for every factor K_2 and one edge of every set of parallel triangles (K_3-fibers) for every factor K_3. □

This construction also works for the Cartesian product of finitely many K_2-s and K_3-s if there is at least factor is a K_2 and one a K_3.

Then P is a finite path.

We choose its first edge from a triangle and the last such that it is not in a triangle.

Theorem *Let G be the weak Cartesian product of countably many finite or countable prime (e.g. complete) graphs. Then $D(G) = 2$.*

Remark: To any two natural numbers k, n one can always find k finite complete graphs K_i, $1 \leq i \leq k$ such that

$$D(\prod_{1 \leq i \leq k}^{\square} K_i) > n.$$

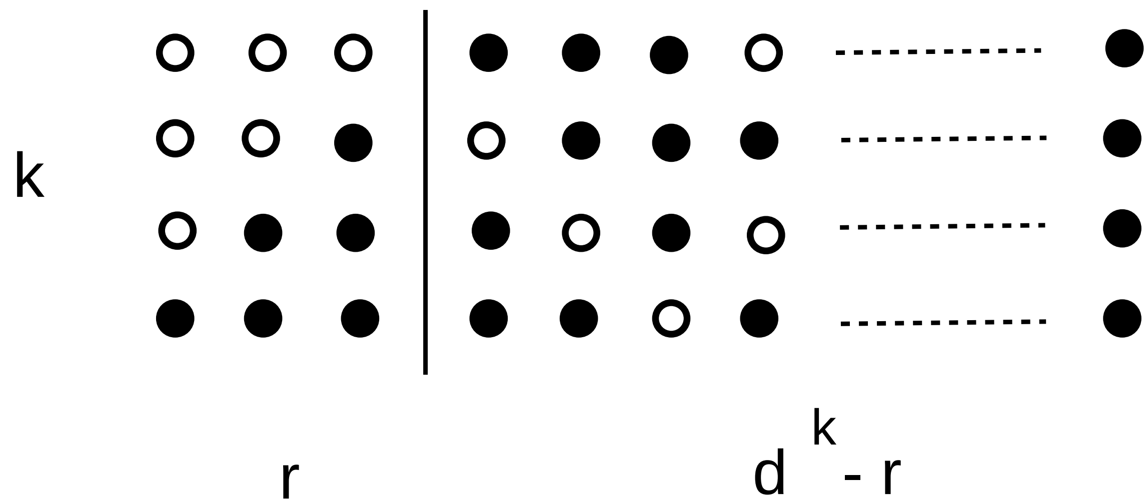
Up to now all graphs were countable. Now a result for an uncountable graph.

Theorem *For any infinite cardinal $\mathfrak{n}$ the distinguishing number of $Q_{\mathfrak{n}}$ is 2.*

Proof by transfinite induction.

3. Distinguishing products of two complete graphs

- with Janja Jerebic and Sandi Klavžar -



$\mathbb{N}_d^k$ - set of vectors of length k with integer entries between 1 and d

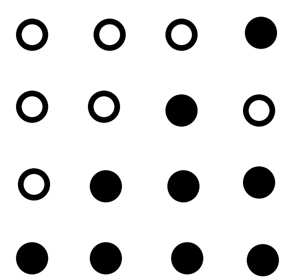
(Here $k = 3$ and $d = 2$; $D(K_4 \,\square\, K_{2^4-4+1}) = 2$.)

Let $\pi \in S_k$ and $\mathbf{v} = (v_1, \ldots, v_k) \in \mathbb{N}_d^k$. Set $\pi\mathbf{v} = (v_{\pi^{-1}(1)}, \ldots, v_{\pi^{-1}(k)})$

We say $X = \{\mathbf{v}^1, \ldots, \mathbf{v}^r\}$ is column-invariant if $\exists \pi \in S_k$ such that

$$\{\mathbf{v}^1, \ldots, \mathbf{v}^r\} = \{\pi\mathbf{v}^1, \ldots, \pi\mathbf{v}^r\}$$

For example, the following vectors are column-invariant:



Lemma **(Switching Lemma)** *Let* $k, d \geq 2$ *and* $1 \leq r < d^k$. *Then*

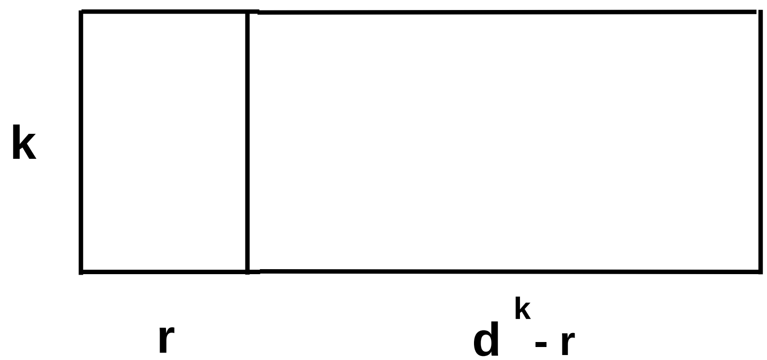
every set of r vectors from $\mathbb{N}_d^k$ is column invariant

if and only if

every set of $d^k - r$ vectors from $\mathbb{N}_d^k$ is column invariant.

Theorem **(Basic)** *Let* $2 \leq d, k < n$ *and* $(d-1)^k < n \leq d^k - k + 1$. *Then*

$$D(K_k \,\square\, K_n) = d$$



Theorem **(Bounding)** *Let $k, d \geq 2$ and $1 \leq r \leq k-2$. Then the following implications hold*

$$D(K_r \square K_k) \geq d+1 \;\Rightarrow\; D(K_k \square K_{d^k - r}) = d+1$$
$$D(K_r \square K_k) \leq d \;\Rightarrow\; D(K_k \square K_{d^k - r}) = d.$$

Thus $\qquad d \leq D(K_k \square K_n) \leq d+1 \qquad$ for $\qquad d = \lceil n^{1/k} \rceil.$

4. A recursion for the distinguishing number of $K_k \square K_n$

- mainly with Janja Jerebic and Sandi Klavžar -

Distinguishing(k, n)

INPUT: Integers k, n with $1 \leq k \leq n$

OUTPUT: $D(K_k \square K_n)$

1. $d = \lfloor n^{1/k} \rfloor + 1$
2. **if** $(d-1)^k \leq n \leq d^k - k + 1$
3. **then** $D(K_k \square K_n) = d$
4. **else** determine $D(K_k \square K_n)$ from $D(K_{d^k - n} \square K_k)$ by an application of the Bounding Theorem

Analysis of the recursion

Step 3 returns the distinguishing number

Step 4, is executed only if $d^k - k + 1 < n$. Since $d \geq 2$

$$\begin{aligned} 2^k - k + 1 &< n, \\ 2^k &< 2n, \\ k - 1 &< \log_2 n. \end{aligned}$$

Hence $d^k - n < k - 1 < \log_2 n$.

We must thus consider $K_{k_1} \square K_k$, where $k_1 = d^k - n < \log_2 n$.

If Distinguishing(k_1, k) also enters the recursive step, then with a call of Distinguishing(k_2, k_1), where $k_2 < \log_2 k$.

Since $k_i \geq 1$ the number of recursive steps is bounded by the *iterated logarithm* $\log_2^* n$.

$\log_2^* 2 = 1$, $\log_2^* 4 = 2$, $\log_2^* 16 = 3$, $\log_2^* 65536 = 4$, $\log_2^*(2^{65536}) = 5$.

Theorem **(Finite $K_k \square K_n$)** *The distinguishing number $D(K_k \square K_n)$ of the product of two complete graphs K_k and K_n, where $1 \leq k \leq n$, can be determined in $O(\log^* n)$ time.*

Here any finite number d is the distinguishing number of some product of complete graphs. In the infinite case we have:

Theorem **(Infinite $K_{\mathfrak{n}} \square K_{\mathfrak{m}}$)** *For infinite cardinals $\mathfrak{n}$ we have:*

$$D(K_{\mathfrak{n}} \square K_{2^{\mathfrak{n}}}) = 2.$$

If $2^{\mathfrak{n}} < \mathfrak{m}$, then $D(K_{\mathfrak{n}} \square K_{\mathfrak{m}}) > \mathfrak{n}$.

If the generalized continuum hypothesis does not hold, then there are cardinals

$$\mathfrak{n} \text{ and } \mathfrak{m}$$

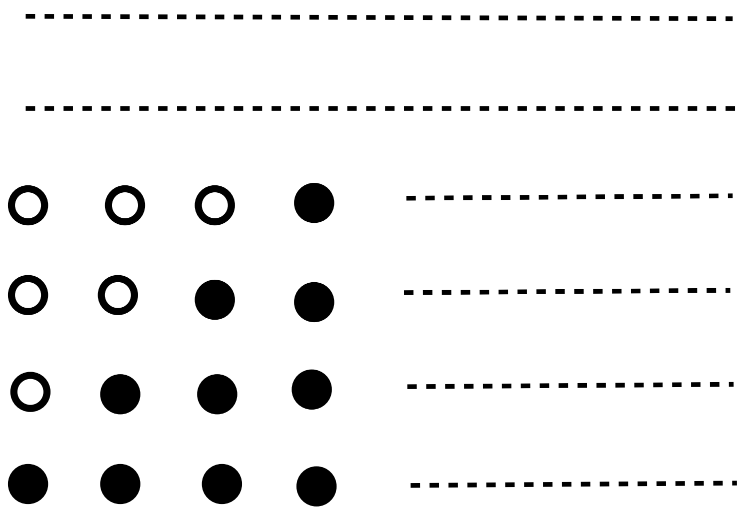
such that

$$\mathfrak{n} < \mathfrak{m} < 2^{\mathfrak{n}}.$$

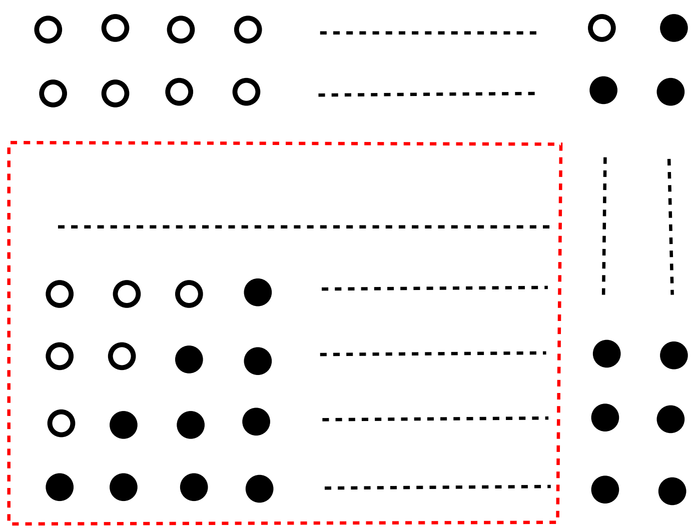
We do not know whether $D(K_{\mathfrak{n}} \square K_{\mathfrak{m}}) = 2$ in this case.

We only prove $D(K_{\aleph_0} \square K_{\aleph_0}) = 2$.

To see this one simply labels as in the figure.



To show $D(K_{\mathfrak{n}} \square K_{\mathfrak{n}}) = 2$ for arbitrary $\mathfrak{n}$ one well-orders the vertices of the factors and proceeds by transfinite induction.



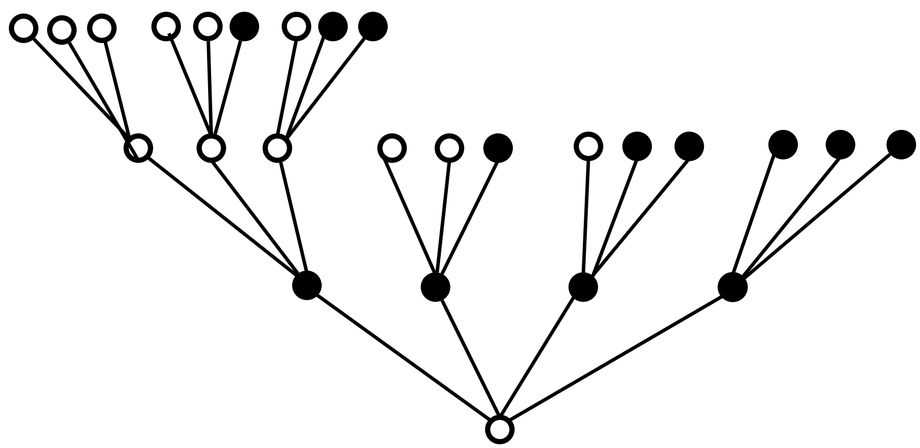
For $D(K_{\mathfrak{n}} \square K_{\mathfrak{m}})$ the Switching Lemma is needed.

5. Infinite trees and tree-like graphs

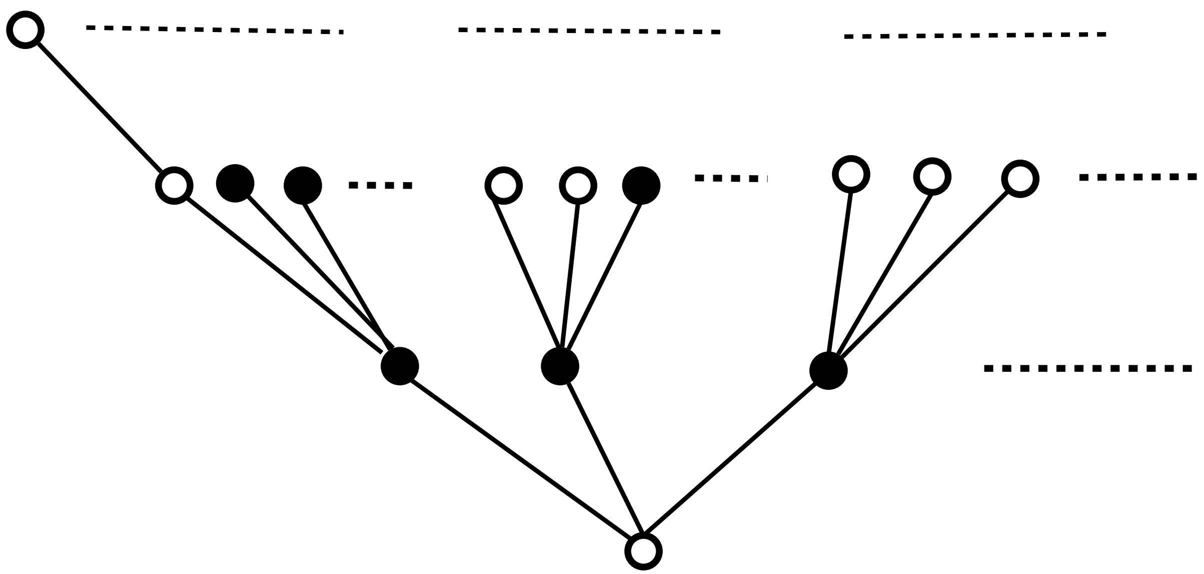
- with Sandi Klavžar and Vladimir Trofimov -

Theorem *The distinguishing number of the homogeneous tree* $T_{\mathfrak{n}}$ *of finite or infinite degree* $\mathfrak{n}$ *is 2.*

Proof for $\mathfrak{n} = 4$



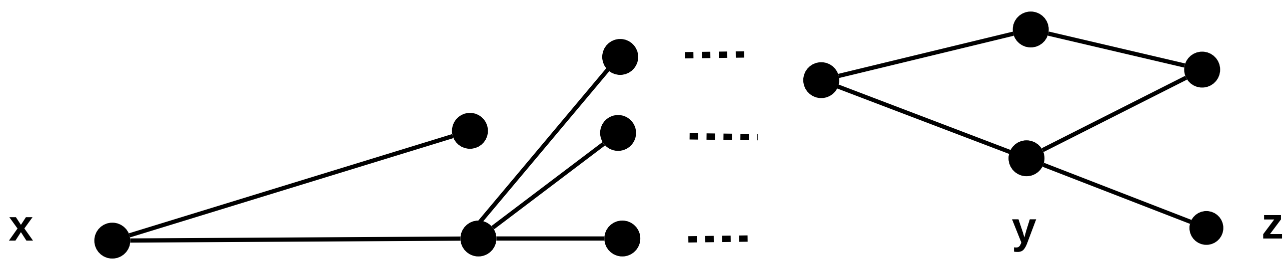
Proof for $\mathfrak{n} = \aleph_0$



Theorem *Let* Γ *be a connected graph with* $d(v) \leq 2^{\aleph_0} \quad \forall v \in V(\Gamma)$*.*
Suppose there is a vertex x *in* Γ *with the following property:*

$$\forall y \in V(\Gamma)\ \exists z \in V(\Gamma) \text{ such that } \{y\} = \Gamma(z) \cap B_x(d(x,z) - 1).$$

Then $D(\Gamma) \leq 2$*.*



6. General countable graphs

- with Sandi Klavžar and Vladimir Trofimov -

Theorem *Let G be a connected, infinite graph with largest degree $\Delta(G) < \infty$. Then $D(G) \leq \Delta(G)$.*

In the finite case the bound is $\Delta(G) + 1$.

Theorem *The distinguishing number of the random graph is 2.*

Property of the random graph R: For any finite disjoint subsets X and Y of $V(R)$, there are infinitely many vertices z of R such that

$$zx \in E(R) \text{ for all } x \in X \text{ and}$$
$$zy \notin E(R) \text{ for all } y \in Y.$$

7. Appendix - Exact formulas and examples for finite graphs

- with Janja Jerebic and Sandi Klavžar -

Theorem (Basis for explicit results) *Let $k, d \geq 2$, $1 \leq r \leq k-2$. Then $D(K_k \,\square\, K_{d^k-r}) = d+1$ if and only if every set consisting of r vectors from $\mathbb{N}_d^k$ is column-invariant.*

Proof. If every set of r vectors from $\mathbb{N}_d^k$ is column-invariant,
then $D(K_k \,\square\, K_r) \geq d+1$,
and thus $D(K_k \,\square\, K_{d^k-r}) = d+1$
by (i) of the Bounding Theorem.

If there is a set of r vectors from $\mathbb{N}_d^k$ that is not column-invariant,
then $D(K_k \,\square\, K_r) \leq d$,
and thus $D(K_k \,\square\, K_{d^k-r}) \neq d+1$
by (ii) of the Bounding Theorem. $\square$

Proposition 4.2 *Let* $d \geq 2$, $3 \leq k \leq d$. *Then*

$$D(K_k \,\Box\, K_{d^k-1}) = d.$$

Proposition 4.3 *Let* $k, d \geq 2$ *and* $0 \leq r < \log_d k$. *Then*

$$D(K_k \,\Box\, K_{d^k-r}) = d+1.$$

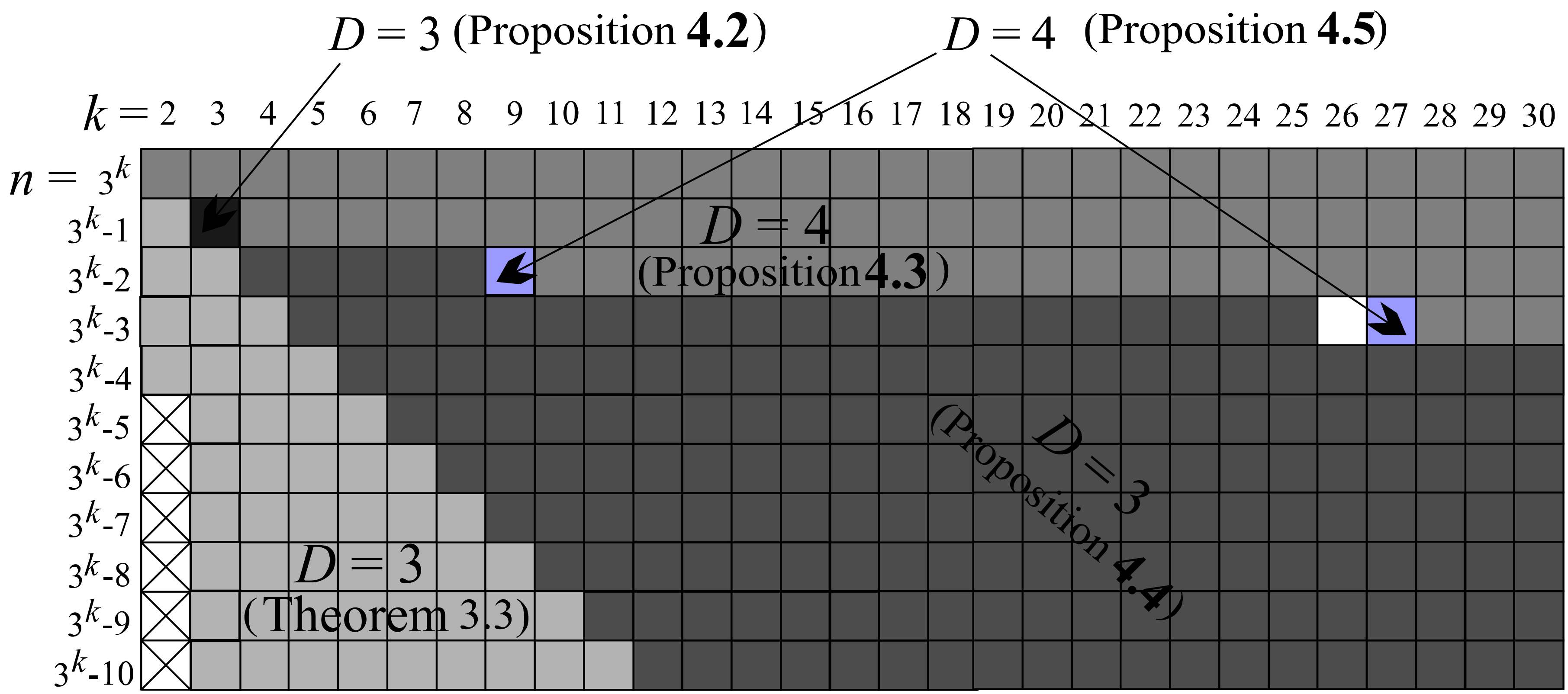
Proposition 4.4 *Let* $d, r \geq 2$ *and* $r+2 \leq k \leq d^r - r + 1$. *Then*

$$D(K_k \,\Box\, K_{d^k-r}) = d.$$

Proposition 4.5 *Let* $d, r \geq 2$ *and* $d^r - \log_d r < k \leq d^r$. *Then*

$$D(K_k \,\Box\, K_{d^k-r}) = d+1.$$

The rare case when the recursion applies (white field) for $2^k < n \leq 3^k$



(Theorem 3.3 is the Basic Theorem)

Summary for $D(K_k \square K_n)$

$D(K_k \square K_n)$ satisfies the inequality

$$d \le D(K_k \square K_n) \le d + 1,$$

where $d = \lceil n^{1/k} \rceil$ and $1 \le k \le n$.

It can be determined explicitly unless

$$d^r - r + 2 \le k \le d^r - \log_d r.$$

Then it can be computed by at most $\log_2^* n$ calls of a recursion.

Can one replace the recursion by an explicit formula?