Maximum weight of a connected graph of given order and size

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Fundamentals

G a graph (simple, finite, undirected)

weight of edge $e = xy \in E(G)$ $\quad w_G(e) := \deg_G(x) + \deg_G(y)$

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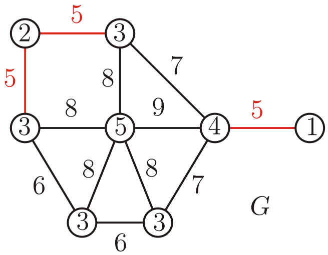

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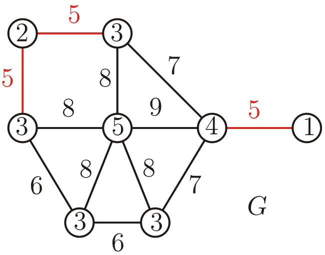

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$n \in \mathbb{Z}$, $n \geq 2$, $m \in \{1, \ldots, \binom{n}{2}\}$, $\mathcal{P}$ a graph property

$\mathcal{P}(n,m) := \{G \in \mathcal{P} : |V(G)| = n, |E(G)| = m\} \neq \emptyset$

$w(n, m, \mathcal{P}) := \max(w(G) : G \in \mathcal{P}(n,m))$

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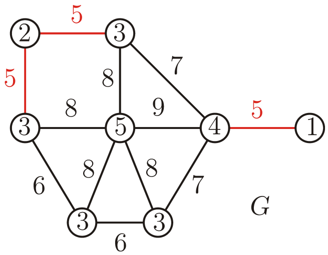

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$\mathcal{I}$ "to be a graph" $\rightarrow w(8, 11, \mathcal{I}) \geq 5$

Problem (Erdős 1990)

Given n and m determine $w(n, m, \mathcal{I})$.

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Theorem (Ivančo, Jendroľ 1991)

Let $a = \lceil \frac{1}{2}(1 + \sqrt{1 + 8m}) \rceil$ *and* $b = \frac{1}{2}(a^2 - a - 2m)$, *let* $h = \lceil \frac{1}{2}(2n - 1 - \sqrt{(2n-1)^2 - 8m}) \rceil$ *and let* $p, k \in \mathbb{Z}$ *be such that* $hk + p = m, h + k \leq n$ *and* $h(h-3) < 2p \leq h(h-1)$. *Let* $f(n, m) = h + k + \lfloor \frac{2p}{h} \rfloor$ *and let* $g(n, m)$ *be defined by*

$$g(n, m) = \begin{cases} 2a - 2 \textit{ if } b = 0; \\ 2a - 3 \textit{ if } b = 1; \\ 2a - 4 \textit{ if } 2 \leq b \leq \lfloor \frac{a}{2} \rfloor \textit{ or } b = 3; \\ 2a - 5 \textit{ if } \lfloor \frac{a}{2} \rfloor < b \leq \lceil \frac{a+2}{2} \rceil \textit{ or } a = 8 \textit{ and } b = 6; \\ 2a - 6 \textit{ in all other cases.} \end{cases}$$

Then $w(n, m, \mathcal{I}) \geq \max(f(n, m), g(n, m))$.

History (continued)

Ivančo, Jendrol' 1991: $w(n, m, \mathcal{I})$ for $m \in \{\binom{n}{2} - n + 2, \ldots, \binom{n}{2}\}$

Conjecture

$w(n, m, \mathcal{I}) = \max(f(n, m), g(n, m))$ *for all pairs* (n, m).

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Conjecture $\rightarrow$ Theorem (Jendroľ, Schiermeyer 2001)

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Conjecture → Theorem (Jendroľ, Schiermeyer 2001)

$\mathcal{B}$ "to be a bipartite graph"

Theorem (H, Jendroľ, Schiermeyer)

Let $n \in \mathbb{Z}$, $n \geq 2$, $m \in \{1, \dots, \lfloor \frac{n^2}{4} \rfloor\}$, $a^* = \lceil \frac{n - \sqrt{n^2 - 4m}}{2} \rceil$, $b^* = \lceil \frac{m}{a^*} \rceil$ *and* $s^* = a^* b^* - m$. *Then*

1. $a^* + b^* \leq w(n, m, \mathcal{B}) \leq a^* + b^* + 1$;
2. $w(n, m, \mathcal{B}) = a^* + b^*$ *for* $s^* = 0, 1$;
3. *if* $w(n, m, \mathcal{B}) = a^* + b^* + 1$, *there exists* $k \in \mathbb{Z}$ *with* $(a^* + k)(b^* - k - 1) = m$.

Sparse graphs

Fact

If $\mathcal{P}_1 \subseteq \mathcal{P}_2$, then $w(n, m, \mathcal{P}_1) \leq w(n, m, \mathcal{P}_2)$. Moreover, if $w(n, m, \mathcal{P}_1) \geq w$ and $w(n, m, \mathcal{P}_2) \leq w$, then $w(n, m, \mathcal{P}) = w$ for any $\mathcal{P}$ with $\mathcal{P}_1 \subseteq \mathcal{P} \subseteq \mathcal{P}_2$.

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G sparse ... $|E(G)| = m \le \binom{n}{2} - \binom{\lceil n/2 \rceil}{2} \le \frac{3n^2-4n+1}{8}$
optimum for $w(n,m,\mathcal{C})$: a subgraph of $G_{n,k} := D_k \oplus K_{n-k}$ (join)
$D_k = \overline{K_k}$ ... discrete graph on k vertices $\quad |E(G_{n,k})| = \binom{n}{2} - \binom{k}{2}$

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$\binom{n}{2} - \binom{k+1}{2} < m \le \binom{n}{2} - \binom{k}{2} \qquad m' := \binom{n}{2} - \binom{k}{2} - m \le k-1$

m' ... the number of edges to be deleted from $G_{n,k}$

$0 \le r := \left\lceil \frac{m'}{n-k} \right\rceil \quad \exists$ a vertex in $V(K_{n-k})$ with $\ge r$ edges deleted

$$c := \begin{cases} 1, & 0 \le m' \le \lfloor \frac{n-k}{2} \rfloor \text{ or } m' = (n-k-1)^2 \\ 2, & \text{otherwise} \end{cases}$$

Sparse graphs (continued)

Theorem

If $n \geq 49$, $m \in \{n-1, \ldots, \binom{n}{2} - \binom{\lceil n/2 \rceil}{2}\}$, $\mathcal{C} \subseteq \mathcal{P} \subseteq \mathcal{D}_{1+}$ *and integers* k, m', r, c *are defined as before, then* $w(n, m, \mathcal{P}) = 2n - k - r - c$.

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$w(n, m, \mathcal{C}) \geq 2n - k - r - c$ by constructions
$V(D_k) = A = \{a_1, \ldots, a_k\}$, $V(K_{n-k}) = B = \{b_1, \ldots, b_{n-k}\}$

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$V(D_k) = A = \{a_1, \ldots, a_k\}$, $V(K_{n-k}) = B = \{b_1, \ldots, b_{n-k}\}$

$m' \leq \lfloor \frac{n-k}{2} \rfloor \Rightarrow 0 \leq r \leq 1$, $c = 1$
M' a matching of size m' in $G_{n,k}\langle B \rangle$ with $M' \neq \emptyset \Rightarrow b_1 \in V(M')$
$G_{n,k} - M' \in \mathcal{C}(n, m)$ degrees in A: $n - k$ B: $n - 1 - r \rightarrow n - 1$
$w(G_{n,k} - M') = w(a_1 b_1) = (n - k) + (n - 1 - r) = 2n - k - r - c$

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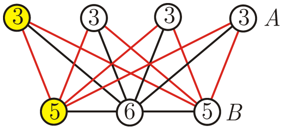

optimum graph G with parameters $n = 7$, $m = 14$, $k = 4$, $m' = 1$, $r = 1$, $c = 1$, $w(G) = 7$

Sparse constructions

$\lfloor \frac{n-k}{2} \rfloor < m' \leq k-1$, $m' \neq (n-k-1)^2 \Rightarrow c = 2$
$E_{m'} := \{a_i b_i : i = 1, \ldots, m'\}$ indices modulo $n - k \leq k$

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degrees of vertices in $G_{n,k} - E_{m'} \in \mathcal{C}(n,m)$:
A ... $n-k-1$, $n-k$ B ... $n-1-\lceil \frac{m'}{n-k} \rceil$, $n-1-\lfloor \frac{m'}{n-k} \rfloor$
$w(G_{n,k} - E_{m'}) = w(a_2 b_1) = (n-k-1) + (n-1-\lceil \frac{m'}{n-k} \rceil) =$
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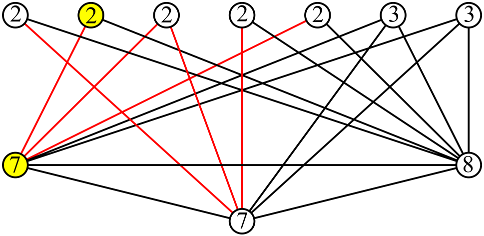

optimum graph G
$n = 10,\ m = 19,\ k = 7,$
$m' = 5,\ r = 2,\ c = 2,$
$w(G) = 9$

Sparse constructions (continued)

$$m' = (n-k-1)^2 \Rightarrow c = 1$$
$$r = \lceil \tfrac{m'}{n-k} \rceil = \lceil \tfrac{(n-k)^2 - 2(n-k)+1}{n-k} \rceil = \lceil n-k-2+\tfrac{1}{n-k} \rceil = n-k-1$$

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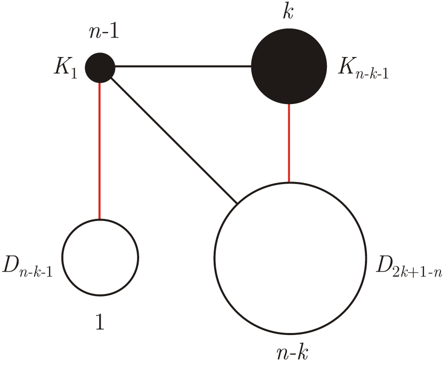

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$w(G) = 2n - k - r - c = n$

$w(G) = (n-1)+1 = k+(n-k)$

deleted all edges between $V(D_{n-k-1})$ and $V(K_{n-k-1})$

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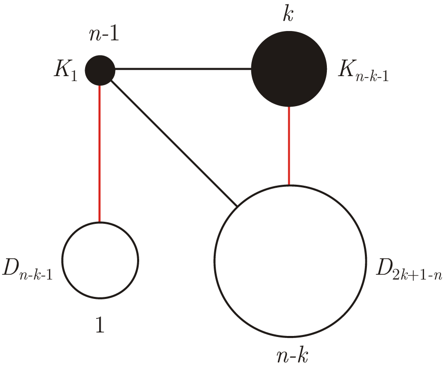

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$w(n, m, \mathcal{D}_{1+}) \leq 2n-k-r-c$ ... crucial part of Theorem
different analysis for $r \leq 6$ and $r \geq 7$

Dense graphs

$\frac{2m}{n}$ ... average degree of a graph in $\mathcal{I}(n,m)$ $d := \lfloor \frac{2m}{n} \rfloor \leq n-1$
$\exists$ a graph $G \in \mathcal{C}(n,m)$ with $\delta(G) = d \Rightarrow w(n,m,\mathcal{C}) \geq 2d$

$$e := \begin{cases} 0, & m = \binom{n}{2} - 1 \\ 0, & d \leq n-3 \wedge (\exists q \in \{0,\dots,d-1\}\ 2m \equiv q \pmod{n}) \\ 1, & \text{otherwise} \end{cases}$$

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If $n \geq 15$, $\binom{n}{2} - \binom{\lceil n/2 \rceil}{2} + 1 \leq m \leq \binom{n}{2}$, $\mathcal{C} \subseteq \mathcal{P} \subseteq \mathcal{D}_{1+}$ and integers d, e are defined as above, then $w(n, m, \mathcal{P}) = 2d + e$.

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$w(n,m,\mathcal{D}_{1+}) \leq 2d + e$ ... easier than for sparse graphs

$w(n,m,\mathcal{C}) \geq 2d + e$: optimum graph by constructions depending on the parity of n and d

Dense construction

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first case: $dn \equiv 0 \pmod{2} \Rightarrow d(2d + 2 - n) \equiv 0 \pmod{2}$
let G_1 be a $(2d + 2 - n)$-regular graph with $|V(G_1)| = d$
easy: $2(2d + 2 - n) \geq n > d \Rightarrow G_1$ is Hamiltonian

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$G_2 := D_{n-d}$ ... 0-regular graph with $|V(G_2)| = n - d$
$G := G_1 \oplus G_2 \quad 2|E(G)| = d(2d+2-n) + 2d(n-d) = d(n+2)$

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degrees in G for $V(G_1)$: $(2d+2-n) + (n-d) = d+2$
degrees in G for $V(G_2)$: $0 + d = d$

Dense construction (continued)

$C = (x_1, x_2, \ldots, x_{d-1}, x_d, x_0)$ a Hamiltonian cycle in G_1
$G \to \tilde{G} := G - \{x_{2i-1}x_{2i} : i \in \{1, \ldots, s\}\}$
$V(\tilde{G})| = n$, $E(\tilde{G}) = |E(G)| - s = m$, $\tilde{G} \in \mathcal{C}(n, m)$

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degrees in $\tilde{G}$: $\deg_{\tilde{G}}(x_i) = \deg_G(x_i) - 1 = d + 1$, $i = 1, \ldots, 2s \leq d$
$\deg_{\tilde{G}}(x_i) = \deg_G(x_i) = d + 2$, $i = 2s + 1, \ldots, d$
degrees for $V(G_2)$ remain d

Dense construction (continued)

$C = (x_1, x_2, \dots, x_{d-1}, x_d, x_0)$ a Hamiltonian cycle in G_1
$G \to \tilde{G} := G - \{x_{2i-1}x_{2i} : i \in \{1, \dots, s\}\}$
$V(\tilde{G})| = n$, $E(\tilde{G}) = |E(G)| - s = m$, $\tilde{G} \in \mathcal{C}(n,m)$

degrees in $\tilde{G}$: $\deg_{\tilde{G}}(x_i) = \deg_G(x_i) - 1 = d+1$, $i = 1, \dots, 2s \le d$
$\deg_{\tilde{G}}(x_i) = \deg_G(x_i) = d+2$, $i = 2s+1, \dots, d$
degrees for $V(G_2)$ remain d

$v \in V(G_2) \Rightarrow w_{\tilde{G}}(x_i v) \ge (d+1) + d = 2d+1$, $i = 1, \dots, d$
$x_i x_j \in E(\tilde{G}) \Rightarrow w_{\tilde{G}}(x_i x_j) \ge 2(d+1) = 2d+2$
$w(\tilde{G}) \ge 2d+1$

Dense construction (continued)

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second case: $d \equiv n \equiv 1 \pmod 2$ $\to$ similar construction
G_1 a $(2d + 3 - n)$-regular graph with $|V(G_1)| = d$

Thank you.