Cuts and trees

Bernhard Krön

University of Vienna

joint with M.J. Dunwoody (Univ. of Southampton, UK)

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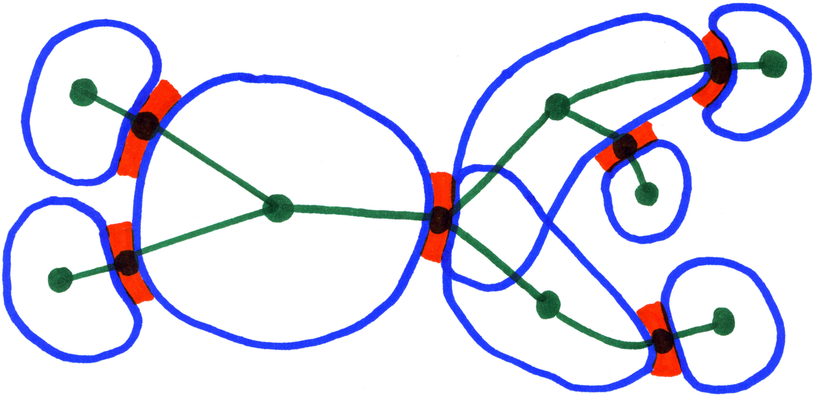
We cannot decompose 3-connected graphs into 4-connected blocks like that.

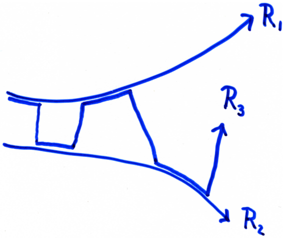
tree decomposition → tree



ends of graphs

graph $X = (VX, EX)$ ray R 

$R_1 \approx R_2 \iff \exists R_3$ such that $|R_3 \cap R_1| = |R_3 \cap R_2| = \infty.$



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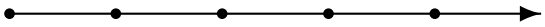
graph $X = (VX, EX)$ ray R 

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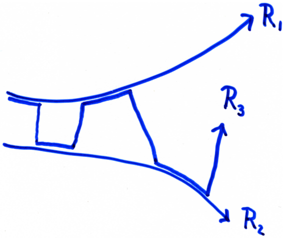

Equivalence classes are the ends of X.

This definition is due to Halin (1964).

$C \subset VX, \quad NC = \{x \in VX \setminus C \mid x \sim C\}, \quad C^* = VX \setminus (C \cup NC)$

write $R \rightsquigarrow C$ if $|C \setminus R| < \infty$.

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Equivalent definitition:

$R_1 \not\approx R_2$ if $\exists C$ s.t. $|NC| < \infty$, $R_1 \rightsquigarrow C$ and $R_2 \rightsquigarrow C^*$

That is, large sets C and C^* are separated by the small set NC.

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Freudenthal's original definition from 1931 for locally compact, connected space with a countable base:
small = compact, large = not compact, open, connected

In 1945 he studied ends of locally finite graphs.

Cayley graphs

$G = \langle S \rangle$ finitely generated,
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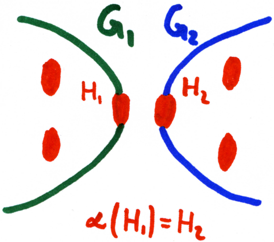
$VX = G$

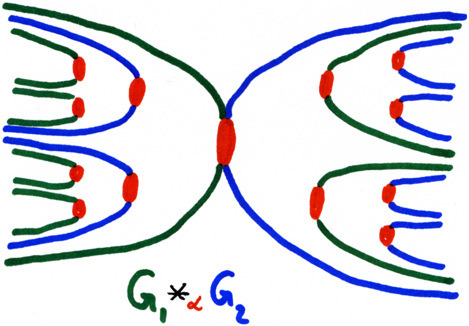
$EX: \quad x \sim y \iff \exists s \in S$ such that $xs = y$

or $\quad x \sim y \iff x^{-1}y \in S$

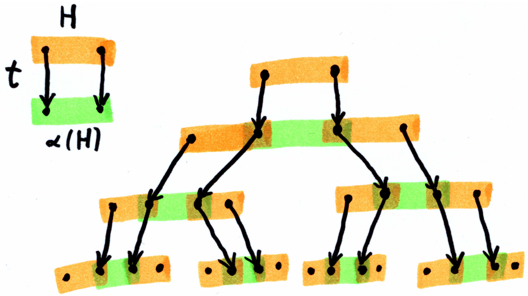
If $S = S^{-1} = \{s^{-1} \mid s \in S\}$ then X is undirected.

free amalgamated products





HNN-extensions



Stallings' structure theorem

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G splits over $H < G$ if

$G = G_1 *_\alpha G_2$ or $G = G_1 *^\alpha$ for some $\alpha : H \to H'$.

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Theorem (Stallings' structure theorem, 1971)

G finitely generated. Then

G has more than one end $\iff$ G splits over finite subgroup

Dicks and Dunwoody's theory

Combinatorial (graph theoretic) proof of Stallings' theorem by Dunwoody:

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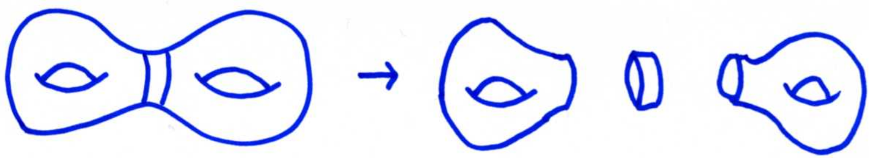
Fully developed theory in the book
M.J. Dunwoody, W. Dicks "Groups acting on Graphs", 1989.

historic root = Seifert - van Kampen theorem



$X = U \cup V$ $\quad U \quad U \cap V \quad V$

$$\pi_1(X,x) \cong \pi_1(U,x) *_{\pi_1(U \cap V, x)} \pi_1(V,x), \qquad x \in U \cap V$$

Axioms - vertex cuts

Let X be a connected graph.

Sets $\mathcal{C}$ and $\mathcal{S}$ of non-empty sets of vertices are called cut-system and separator-system, respectively, if they satisfy the following.

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Note: $\mathcal{S}$ is determined by $\mathcal{C}$, but $\mathcal{C}$ is usually not determined by $\mathcal{S}$.

A set is called large if it contains a cut.

Examples of cut systems

Example

X *1-connected*

$\mathcal{S}$ = *1-element separators (cut-points)*

$\mathcal{C}$ = *components obtained by removing a separator*

Examples of cut systems

for us the most important example:

Example

X connected

$\mathcal{S}$ = finite sets which separate ends

$\mathcal{C}$ = those components in the complement of a separator, which contain a ray (i.e., an end).

Examples of cut systems

more general:

Example

X connected, $M \subset VX$

$\mathcal{S}$ = finite subsets of M which separate ends

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The following is the setting in the book of Dicks and Dunwoody.

Example

edge cuts: put an additional vertex on each edge and let M be the set of these new vertices.

vertex cuts - edge cuts

E edge cut $\Rightarrow$ E vertex cut

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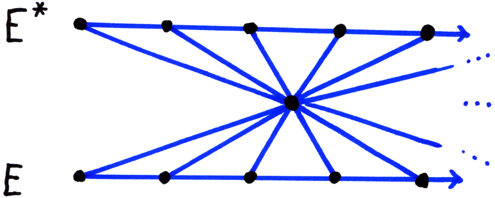

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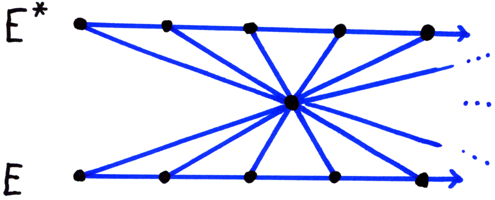

E vertex cut $\not\Leftarrow$ E edge cut

X locally finite: E vertex cut $\iff$ E edge cut

What about finite graphs?

$Y \subset VX$ is called k-inseparable if $|Y| > k$ and for all $S \subset VX$, $|S| = k$, the set Y is contained in the union of S and one component of $VX \setminus S$. In other words: No k-set S separates Y.

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Example

Let κ be the minimal integer (if it exists) such that there is a κ-set S which separates two κ-inseparable sets Y_1 and Y_2 in the sense that there are components C_1 and C_2 of $VX \setminus S$, $C_1 \neq C_2$ and

$$Y_1 \subset C_1 \cup S \quad \text{and} \quad Y_2 \subset C_2 \cup S.$$

$\mathcal{S}$ = κ-sets which separate two κ-inseparable sets

$\mathcal{C}$ = those components in the complement a separator, which contain a κ-inseparable set.

minimal cuts

Let $\mathcal{C}$ be a cut system and let κ be the minimal cardinality of a separator. A cut whose boundary has κ elements is called a minimal cut.

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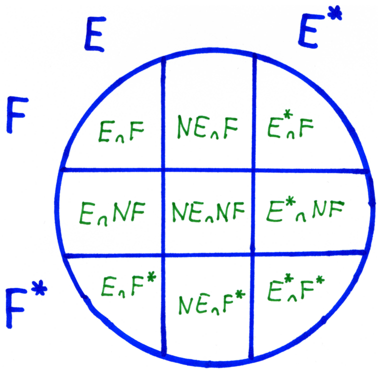
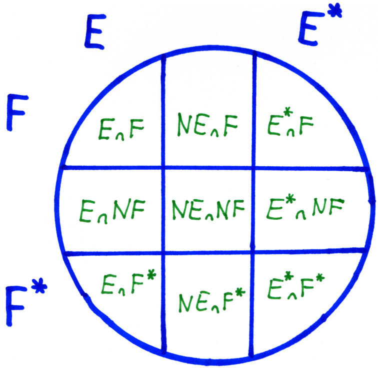
Theorem (Dunwoody, Krön)

The set of minimal cuts is a cut system.

nestedness



4 corners, 4 links, 1 center.



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isolated corner = not large and both adjacent links are empty

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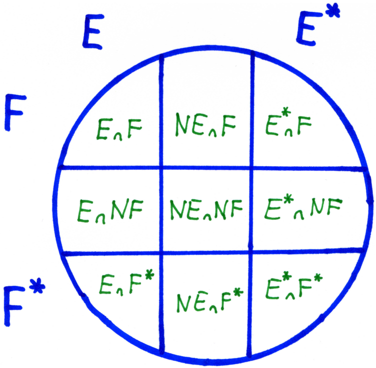

4 corners, 4 links, 1 center.

isolated corner = not large and both adjacent links are empty

Sets of vertices E and F are nested if they have an isolated corner.

Main theorem

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Every cut system has a minimal cut which is nested with all other cuts.

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The existence of so-called "structure cuts" (or D-cuts) follows as corollary.

See Dunwoody "Cutting up graphs" 1982,

Dicks and Dunwoody "Groups acting on Graphs" 1989.

Counterexample

In the papers on edge cuts, nestedness of cuts E and F means that one of following inclusions hold

$$E \subset F \quad \text{or} \quad E \subset VX \setminus F \quad \text{or} \quad VX \setminus E \subset F \quad \text{or} \quad VX \setminus E \subset VX \setminus F.$$

Equivalently,

$$E \subset F \quad \text{or} \quad F \subset E \quad \text{or} \quad E \cap F = \emptyset \quad \text{or} \quad E \cup F = VX.$$

There are graphs with more than one end, where there is no automorphism invariant cut system which satisfies the above condition for all pairs of cuts. We have to use our weaker concept of nestedness which uses isolated corners.

Generalized structure trees

Let G be a group acting on X and let $\mathcal{C}$ be a G-set.

Let $\mathcal{N}$ be the set of minimal cuts which are nested with all other cuts.

Then $\mathcal{N}$ is a G-set. And there is a tree T with

$$ET = \mathcal{N} \quad \text{and} \quad VT = \mathcal{S} \cup \mathcal{N} / \sim$$

where $\sim$ is an equivalence relation on $\mathcal{N}$. The equivalence classes correspond to the blocks between the separators.

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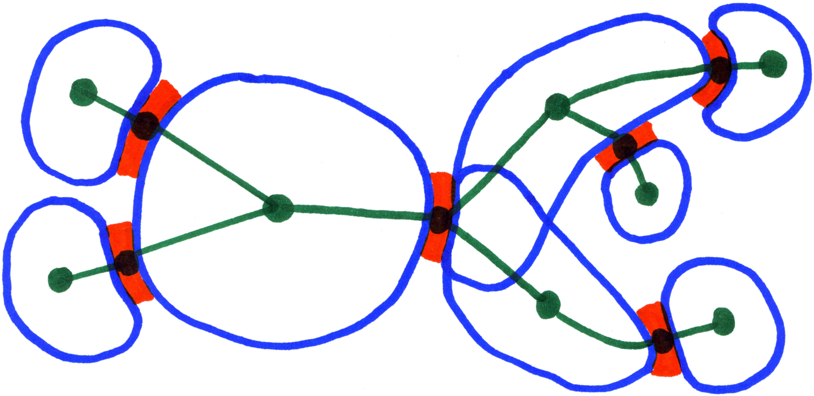
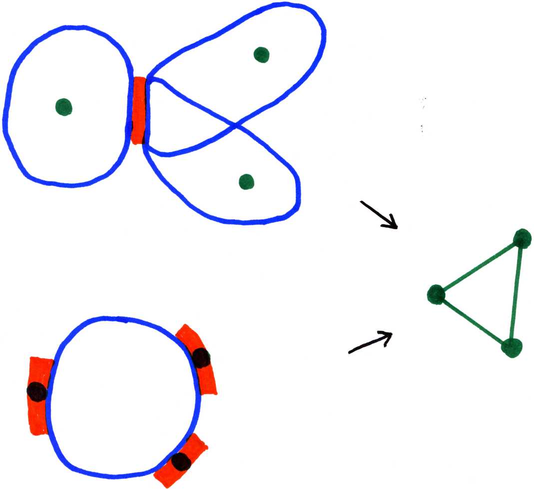
Then G acts on T such that the stabilizers of the edges of T are the stabilizers of cuts in $\mathcal{N}$.



Why two types of vertices?



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Let G be a finitely generated group. Then

G has more than one end $\iff$

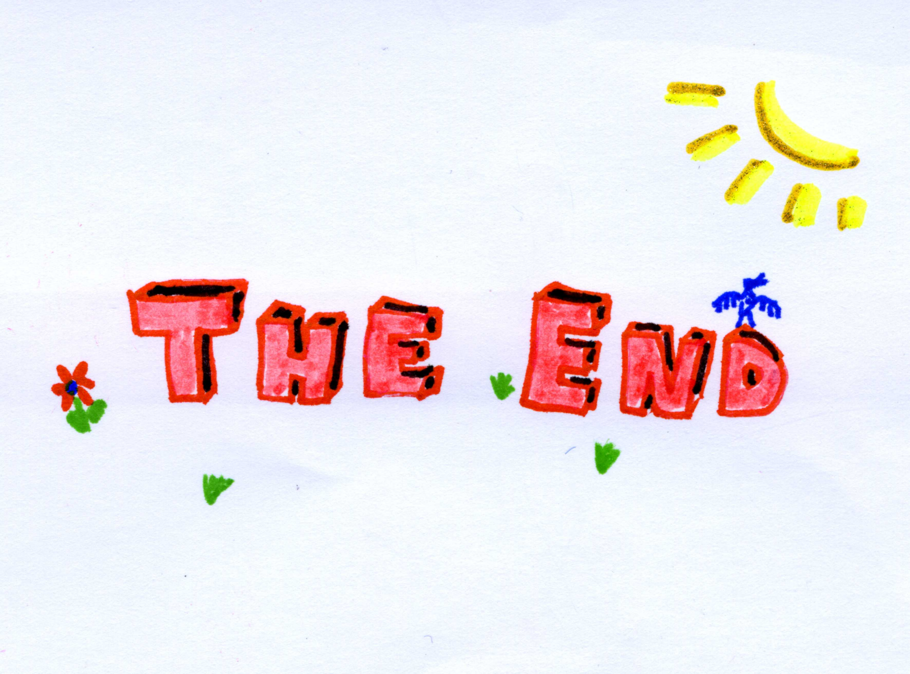
G splits over a finite subgroup.

Theorem (Dunwoody, Krön)

Let G be any group. Then

G has a Cayley-graph with more than one end $\iff$

G splits over a finite subgroup.


THE END