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A good permutation for one-dimensional diaphony

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Abstract. In this article we focus on two aspects of one-dimensional diaphony $F(S_b^\sigma, N)$ of generalised van der Corput sequences in arbitrary bases. First we give a permutation with the best distribution behaviour concerning the diaphony known so far. We improve a result of Chaix and Faure from 1993 from a value of $1.31574\ldots$ for a permutation in base 19 to $1.13794\ldots$ for our permutation in base 57. Moreover for an infinite sequence X and its symmetric version $\widetilde{X}$, we analyse the connection between the diaphony $F(X, N)$ and the L_2-discrepancy $L_2(\widetilde{X}, N)$ using another result of Chaix and Faure. Therefore we state an idea how to get a lower bound for the diaphony of generalised van der Corput sequences in arbitrary base b.

Keywords. Diaphony, generalised van der Corput sequence.

2010 Mathematics Subject Classification. 11K06, 11K38.

1 Introduction

In the field of quasi-Monte Carlo methods various notions of discrepancy occur in the evaluation of the distribution behaviour of infinite sequences. The term low discrepancy sequence is used for sequences showing a particularly good distribution behaviour. An example for a low discrepancy sequence is the (one-dimensional) generalised van der Corput sequence whose behaviour and properties were studied in great detail by Henri Faure in many different papers. In this paper we consider two different types of discrepancy, namely the (classical) diaphony as introduced by Zinterhof [6] and the L_2-discrepancy of generalised van der Corput sequences. Main motivation is the fact that in the last 15 years computer systems have become much faster. Together with changed and improved algorithms we are able to provide permutations that beat the best known values for one-dimensional diaphony significantly. Moreover these new results let us also investigate the connection between the diaphony and the L_2-discrepancy of generalised van der Corput sequences numerically.

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2 Definitions and notations

Let $X = (x_i)_{i \geq 1}$ be an infinite sequence in the half open unit interval $[0, 1[$. For $N \geq 1$ and for reals $0 \leq \alpha < \beta \leq 1$ let $A([\alpha, \beta[, N, X)$ denote the number of indices $i \leq N$ for which $x_i \in [\alpha, \beta[$. The discrepancy function of X is defined as $E([\alpha, \beta[, N, X) = A([\alpha, \beta[, N, X) - (\beta - \alpha)N$.

Definition 2.1. For any one-dimensional, infinite sequence X the *L_2-discrepancy* of the first N points of X is defined by

$$L_2(N, X) := \left(\int_0^1 |E([0, \alpha[, N, X)|^2 d\alpha \right)^{\frac{1}{2}}.$$

Definition 2.2. For any one-dimensional, infinite sequence X the *diaphony* F of the first N points of X is defined by

$$F(N, X) := \left(2 \cdot \sum_{m=1}^{\infty} \frac{1}{m^2} \left| \sum_{n=1}^{N} \exp^{2\pi i m x_n} \right|^2 \right)^{\frac{1}{2}}.$$

Remark 2.3. Note that there is an equivalent definition for the diaphony (details in [5]):

For any one-dimensional, infinite sequence X the *diaphony* F of the first N points of X is defined by

$$F(N, X) := \left(2\pi^2 \int_0^1 \int_0^1 |E([\alpha, \beta[, N, X)|^2 d\alpha d\beta \right)^{\frac{1}{2}}.$$

Note that with this second definition it is easy to see that L_2-discrepancy and diaphony are linked in a way similar to the relation of star discrepancy and extreme discrepancy.

We call a one-dimensional sequence X a *low discrepancy sequence* if there exists a constant c such that for all N

$$F^2(N, X) \leq c \cdot \log N.$$

As a consequence computing

$$f(X) := \limsup_{N \to \infty} (F^2(N, X)/\log N)$$

enables us to look for sequences with “best distribution” behaviour.

Throughout the paper let $b \geq 2$ and $n \geq 1$ be integers. Let $\mathfrak{S}_b$ be the set of all permutations of $\{0, 1, \dots, b-1\}$. The identity in $\mathfrak{S}_b$ is always denoted by id. In all examples and concrete results we will write down the permutations in the following way: For $\sigma = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 0 & 4 & 2 & 6 & 1 & 5 & 3 & 7 \end{pmatrix}$ we will write $\sigma = (0, 4, 2, 6, 1, 5, 3, 7)$.

Definition 2.4. For a fixed base $b \geq 2$ and a permutation $\sigma \in \mathfrak{S}_b$ the *generalised van der Corput sequence* S_b^σ is defined by

$$S_b^\sigma(n) = \sum_{j=0}^{\infty} \sigma(a_j(n)) b^{-j-1},$$

where $\sum_{j=0}^{\infty} a_j(n) b^j$ is the b-adic representation of an integer $n \geq 0$.

The analysis of the diaphony of S_b^σ is based on special functions which have been first introduced by Faure in [1] and which are defined as follows:

Definition 2.5. For $\sigma \in \mathfrak{S}_b$ let $\mathcal{Z}_b^\sigma = (\sigma(0)/b, \sigma(1)/b, \dots, \sigma(b-1)/b)$. For $h \in \{0, 1, \dots, b-1\}$ and $x \in [\frac{k-1}{b}, \frac{k}{b}[$ where $k \in \{1, \dots, b\}$ we define

$$\varphi_{b,h}^\sigma(x) := \begin{cases} A([0, h/b[; k; \mathcal{Z}_b^\sigma) - hx & \text{if } 0 \leq h \leq \sigma(k-1), \\ (b-h)x - A([h/b, 1[; k; \mathcal{Z}_b^\sigma) & \text{if } \sigma(k-1) < h < b. \end{cases}$$

The function $\varphi_{b,h}^\sigma$ is extended to the reals by periodicity.

Note that $\varphi_{b,0}^\sigma = 0$ for any σ and that $\varphi_{b,h}^\sigma(0) = 0$ for any $\sigma \in \mathfrak{S}_b$ and any $h \in \{0, \dots, b-1\}$. Furthermore:

Definition 2.6. Let

$$\varphi_b^{\sigma,(r)} := \sum_{h=0}^{b-1} (\varphi_{b,h}^\sigma)^r$$

where for $r = 1$ we omit the superscript, i.e., $\varphi_b^\sigma := \varphi_b^{\sigma,(1)}$.

Chaix and Faure introduced in [4] a new class of functions based on $\varphi_b^{\sigma,(r)}$:

Definition 2.7.

$$\chi_b^\sigma := b\varphi_b^{\sigma,(2)} - (\varphi_b^\sigma)^2.$$

Proposition 2.8 (Propriété 3.3 in [4]). *The following holds:*

- *The function* φ_b^σ *is continuous, piecewise linear on the intervals* $[k/b, (k+1)/b]$ *and* $\varphi_b^\sigma(0) = \varphi_b^\sigma(1)$.
- *The functions* $\varphi_b^{\sigma,(2)}$ *and* χ_b^σ *are continuous, piecewise quadratic on the intervals* $[k/b, (k+1)/b]$ *and* $\varphi_b^{\sigma,(2)}(0) = \varphi_b^{\sigma,(2)}(1)$.

For our investigations the following is of great importance:

Proposition 2.9 (Propriété 3.5 in [4]). *For each interval* $[k/b, (k+1)/b]$ *the parabolic arcs of* χ_b^σ *are translated versions of the parabola* $y = b^2(b^2-1)x^2/12$.

Remark 2.10. Therefore if we consider two intervals of the form $I_i := [k_i/b, (k_i+1)/b]$, $0 \le k_i < b$, $i \in \{1, 2\}$, with $\chi_b^\sigma(k_1/b) = \chi_b^\sigma(k_2/b)$ and $\chi_b^\sigma((k_1+1)/b) < \chi_b^\sigma((k_2+1)/b)$ then it follows immediately that $\min \chi_b^\sigma(I_1) < \min \chi_b^\sigma(I_2)$.

The starting point for our work are two theorems that show that the diaphony of S_b^σ can be computed exactly. The first theorem states how to compute the diaphony of the first N points of S_b^σ, while the second theorem establishes a method for computing the constant which allows us to classify different permutations in arbitrary base b.

Theorem 2.11 (Theorem 4.2 in [4]). *For all* $N \ge 1$, *we have*

$$F^2(N, S_b^\sigma) = 4\pi^2 \sum_{j=1}^{\infty} \chi_b^\sigma(N b^{-j})/b^2.$$

Theorem 2.12 (Theorem 4.10 in [4]). *Let*

$$\gamma_b^\sigma := \inf_{n \ge 1} \sup_{x \in [0,1]} \left(\sum_{j=1}^{n} \chi_b^\sigma(x b^j)/n \right),$$

then

$$f(S_b^\sigma) = \limsup_{N \to \infty} (F^2(N, S_b^\sigma)/\log N) = 4\pi^2 \gamma_b^\sigma/(b^2 \log b).$$

3 A good permutation

In order to improve the best known result for diaphony obtained by Chaix and Faure in [4] in base 19, we will give a permutation in base 57 and examine its dominant intervals. According to Theorem 2.12 it is necessary to compute γ_b^σ. Therefore we introduce so called supporting functions $g_n^{b,\sigma} : [0,1] \to \mathbb{R}$, with

$$g_n^{b,\sigma}(x) := \frac{1}{n}\sum_{j=0}^{n-1} \chi_b^\sigma(xb^j).$$

Following [1] and [4] we examine intervals $I_h^n := [h/b^n, (h+1)/b^n]$, with $h \in 0, \ldots, b^n - 1$.

Definition 3.1. The interval I_h^n is called *dominated*, if there exists a set J of integers with $h \notin J$ such that $g_n^{b,\sigma}(x) \leq \max_{j \in J} g_n^{b,\sigma}((x + (j-h))/b^n)$, for all $x \in I_h^n$. Otherwise the interval is called *dominant*.

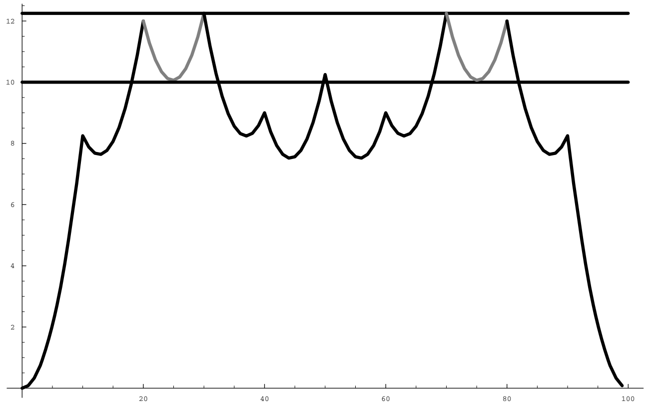

Figure 1. Dominant intervals (gray) of a χ-function in base 10.

Our algorithms enabled us to find many permutations with exceptionally good distribution behaviour. The best ones we obtained so far are in base 57, where for the following we were able to prove an exact value.

Theorem 3.2. *For* $\sigma := \sigma_{57} = (0, 24, 37, 8, 43, 18, 52, 29, 11, 48, 33, 4, 21, 40, 14, 54, 26, 45, 6, 35, 16, 50, 31, 2, 20, 39, 10, 47, 27, 55, 13, 42, 23, 3, 32, 51, 17, 36, 7, 46, 28, 56, 15, 41, 22, 5, 34, 49, 9, 25, 53, 38, 12, 30, 1, 19, 44)$ *we have*

$$f(S_{57}^\sigma) = \frac{4\pi^2 \cdot 42407}{112 \cdot (57^2 \log 57)} = 1.13794\ldots.$$

To simplify the proof we first show the following result for the supporting functions of our permutation σ_{57}:

Lemma 3.3. *For all $n \geq 2$ the functions $g_n := g_n^{57,\sigma}$ have two dominant intervals $I_{k_1^n}^n$ and $I_{k_2^n}^n$, with $k_1^n = 26 \cdot 57^{n-1} + \sum_{i=0}^{n-2} 5 \cdot 57^i$ and $k_2^n = 26 \cdot 57^{n-1} + 6 + \sum_{i=1}^{n-2} 5 \cdot 57^i$. For $n = 2$, $\max_{x \in [0,1]} g_2(x) = g_2(\frac{26 \cdot 57+7}{57^2})$ and for $n \geq 3$, $\max_{x \in [0,1]} g_n(x) = g_n(k_1^n / 57^n)$.*

Proof. Numerical investigations show that there are two dominant intervals I_{25}^1 and I_{26}^1 for $n = 1$. This can be verified easily with Remark 2.10 and Table 1.

k	0	1	2	3	4	5	6	7	8	9
$\chi_{57}^{\sigma}(\frac{k}{57})$	0	$270\frac{2}{3}$	$290\frac{2}{3}$	332	$366\frac{2}{3}$	$402\frac{2}{3}$	380	$410\frac{2}{3}$	$360\frac{2}{3}$	408
k	10	11	12	13	14	15	16	17	18	19
$\chi_{57}^{\sigma}(\frac{k}{57})$	$396\frac{2}{3}$	$410\frac{2}{3}$	410	$400\frac{2}{3}$	$400\frac{2}{3}$	422	$394\frac{2}{3}$	$428\frac{2}{3}$	396	$430\frac{2}{3}$
k	20	21	22	23	24	25	26	27	28	29
$\chi_{57}^{\sigma}(\frac{k}{57})$	$398\frac{2}{3}$	414	$394\frac{2}{3}$	$422\frac{2}{3}$	374	$\mathbf{416\frac{2}{3}}$	$\mathbf{430\frac{2}{3}}$	**428**	$422\frac{2}{3}$	$412\frac{2}{3}$
k	30	31	32	33	34	35	36	37	38	39
$\chi_{57}^{\sigma}(\frac{k}{57})$	374	$426\frac{2}{3}$	$416\frac{2}{3}$	416	$424\frac{2}{3}$	$358\frac{2}{3}$	416	$416\frac{2}{3}$	$426\frac{2}{3}$	368
k	40	41	42	43	44	45	46	47	48	49
$\chi_{57}^{\sigma}(\frac{k}{57})$	$418\frac{2}{3}$	$410\frac{2}{3}$	422	$400\frac{2}{3}$	$400\frac{2}{3}$	410	$410\frac{2}{3}$	$396\frac{2}{3}$	408	$369\frac{2}{3}$
k	50	51	52	53	54	55	56	57		
$\chi_{57}^{\sigma}(\frac{k}{57})$	$372\frac{2}{3}$	396	$384\frac{2}{3}$	$306\frac{2}{3}$	332	$282\frac{2}{3}$	$270\frac{2}{3}$	0		

Table 1. Values of χ_{57}^{σ}.

For $n = 2$ we get again two dominant intervals I_{1487}^2 and I_{1488}^2 and

$$\max_{x \in [0,1]} g_2(x) = g_2\left(\frac{1489}{57^2}\right) = 811.8481\ldots.$$

For $n = 3$ we get the two dominant intervals I_{84764}^3 and I_{84765}^3 and

$$\max_{x \in [0,1]} g_3(x) = g_3\left(\frac{84764}{57^3}\right) = 1190.1108\ldots.$$

Further numerical investigations allow us to make the following induction hypothesis: For $n \geq 3$ the indices k_1^n and k_2^n of the dominant intervals $I_{k_1^n}^n$ and $I_{k_2^n}^n$

of g_n are

$$k_1^n = 26 \cdot 57^{n-1} + \sum_{i=0}^{n-2} 5 \cdot 57^i$$

and

$$k_2^n = 26 \cdot 57^{n-1} + 6 + \sum_{i=1}^{n-2} 5 \cdot 57^i.$$

For the induction step let us suppose that our induction hypothesis holds for an arbitrary $n \geq 3$. To prove that it holds for $n+1$ we have to add $\chi_{57}^{\sigma}(xb^n)$ to g_n on $I_{k_1^n}^n$ and $I_{k_2^n}^n$ and check that g_{n+1} has $I_{k_1^{n+1}}^{n+1}$ and $I_{k_2^{n+1}}^{n+1}$ again as dominant intervals. We verified this for all subintervals of $I_{k^n}^n$ and found that those two intervals are again dominant. Moreover the same numerical investigations show that the maximum of g_{n+1} is at position $p_{n+1} = \frac{1}{57^{n+1}} \cdot (26 \cdot 57^n + \sum_{i=0}^{n-1} 5 \cdot 57^i)$. □

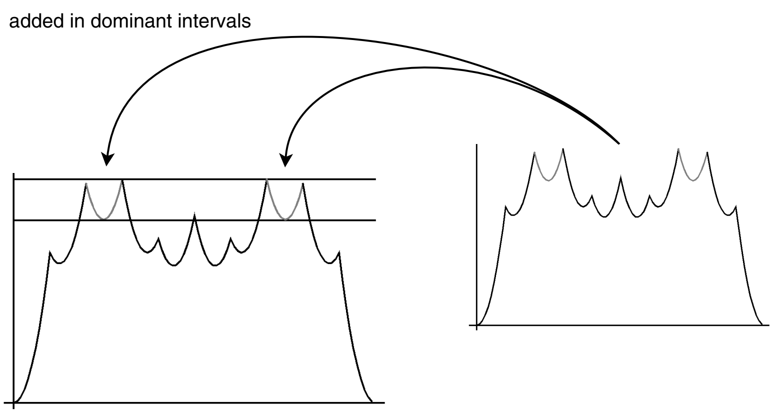

Figure 2. To determine max g_{n+1} it suffices to add the initial χ function only in the dominant intervals of g_n.

Proof of Theorem 3.2. According to Lemma 3.3 the position of the maximum of g_n is $p_n = \frac{1}{57^n}(26 \cdot 57^{n-1} + \sum_{i=0}^{n-2} 5 \cdot 57^i)$. Due to its definition g_n is the sum of n values of the χ-function divided by n. Therefore we can compute the maximum of the n-th approximation step by simply evaluating the χ-function at those positions. For interval I_5^1 we have $\chi(x) = 879396x^2 - 171000x + 8636$, for I_{26}^1 we have $\chi(x) = 879396x^2 - 817836x + 190508$. Let $f(i,n) := \sum_{j=i}^{n} \frac{5}{57^{j+1-i}}$.

Consequently

$$n \cdot g_n(p_n) = 87936 \cdot \left((p_n)^2 + \sum_{j=1}^{n-1} (f(i, n-1))^2 \right) - 817836 \cdot p_n$$
$$- 171000 \cdot \left(\sum_{j=1}^{n-1} f(i, n-1) \right) + 190508 + (n-1) \cdot 8636.$$

Easy calculation gives

$$g_n(p_n) = \frac{42407}{112} + \frac{725 \cdot 3^{1-2n} 19^{2-2n}}{112n} - \frac{25 \cdot 3^{3-2n} 19^{4-2n}}{12544n}$$
$$+ \frac{125 \cdot 3^{3-n} 19^{1-n}}{56n} - \frac{625 \cdot 3^{3-n} 19^{2-n}}{392n}$$
$$+ \frac{725 \cdot 3^{2-n} 19^{3-n}}{3136n} + \frac{680007}{12544n}.$$

For $n \to \infty$ we get

$$\lim_{n\to\infty} g_n(p_n) = \gamma_{57}^{\sigma} = \frac{42407}{112},$$

and therefore

$$f(S_{57}^{\sigma}) = \frac{4\pi^2 \cdot 42407}{112 \cdot (57^2 \log 57)} = 1.13794\ldots. \qquad \square$$

4 A lower bound and some estimations

An infinite sequence $Y = (y_n)_{n \geq 1}$ is called symmetric, if for all n

$$y_{2n} + y_{2n+1} = 1$$

holds. A symmetric sequence Y is said to be created by the sequence $X = (x_n)_{n \geq 1}$, if for all n

$$y_{2n} = x_n \quad \text{or} \quad y_{2n+1} = x_n$$

holds. We will denote this by $Y := \widetilde{X}$. Due to Theorem 4.11 in [4] the L_2-discrepancy of $\widetilde{S_b^{\sigma}}$ can be bounded by the diaphony of S_b^{σ} as follows:

$$l_2(\widetilde{S_b^{\sigma}}) := \limsup_{N\to\infty} \left(\frac{(L_2(N, \widetilde{S_b^{\sigma}}))^2}{\log N} \right) \leq \frac{f(S_b^{\sigma})}{\pi^2}.$$

In [2] Faure proved that

$$0.089 \le l_2(\widetilde{S_2^{\mathrm{id}}}) \le 0.103,$$

which is the best value for the L_2-discrepancy currently known. With our new best value for the diaphony it is interesting to investigate this connection once again. We tried to improve the value of the L_2-discrepancy by finding a permutation for the diaphony that is good enough to beat the bound of Faure. It turned out that it is not possible (at least in "small" bases) to improve the value by simply applying the above mentioned inequality.

To outline our approach let us first recall the basic facts and the according remark about χ_b^σ-functions, which were mentioned in the introduction of this paper. Assuming we have already found a permutation in base b and know that $\max \chi_b^\sigma = s$, we now want to estimate γ_b^σ (see Theorem 2.12). So we construct a function $\lambda = \lambda_{b,s}$ only depending on the base b and on s. First we define λ on the interval $[x, y] := [\frac{1}{b}\lfloor\frac{b}{2}\rfloor, \frac{1}{b}(\lfloor\frac{b}{2}\rfloor + 1)]$, by $\lambda(x) = \lambda(y) := s$ and by $\lambda(z) := Az^2 + Bz + C$ for $z \in [x, y]$. Let $A = \frac{b^2(b^2-1)}{12}$, then B and C can be computed by simply solving the linear system

$$\lambda(x) = s, \quad \lambda(y) = s.$$

We have now constructed a fictive dominant interval for χ_b^σ and can introduce new functions $\widetilde{g}_n = \widetilde{g}_n(b, s)$ similar to the already defined functions g_n to get a lower bound for γ_b^σ. In detail:

$$\widetilde{g}_n := \frac{1}{n}\Big(\lambda(z_1) + \sum_{j=2}^{n} \lambda\Big(z_{j-1} + \frac{1}{b^j}\Big\lfloor\frac{b}{2}\Big\rfloor\Big)\Big),$$

with $n \in \mathbb{N}$, $z_1 := x$ and for $j > 1$, $z_j := z_{j-1} + \frac{1}{b^j}\lfloor\frac{b}{2}\rfloor$. The fact that we always add the maximum of χ_b^σ in the middle (= the minimum) of the fictive dominant interval gives us a lower bound for γ_b^σ if the real χ_b^σ function contains a dominant interval of the form $[k_1, k_2]$ with $\chi_b^\sigma(k_1) = \chi_b^\sigma(k_2) = \max \chi_b^\sigma$.

This lower bound turned out to be sufficient for all practical investigations, also if the dominant interval of χ_b^σ is not of the above assumed form. However for theoretical correctness we assume now that the real set of dominant intervals D contains at least one interval I with a bound whose value is smaller than $\max \chi_b^\sigma$, which means that $\max_{I \in D} \min \chi_b^\sigma(I) < \min \chi_b^\sigma([x, y])$. This implies that it is theoretically possible to find a function χ_b^σ for which γ_b^σ is smaller than the value of our lower bound. To be able to guarantee that this will not happen we can compute an interval $J = J_l \cup J_r$, which has to contain $\max \chi_b^\sigma$. To get the bounds of J_l and J_r we set $m := A(x + \frac{x}{b})^2 + B(x + \frac{x}{b}) + C$ and solve the equation

$$m = Az^2 + B_I z + C_I, \tag{4.1}$$

for z (see Figure 3), where B_I, C_I denote the coefficients of χ_b^σ in the interval I that contains $\max_{I \in D} \min \chi_b^\sigma(I)$. It means if the real maximum is added left of z_1 or right of z_2 in the real dominant interval, then the resulting value has to be greater than the value in the fictive dominant interval. This information is now transformed to fit the initial χ_b^σ-function which is added (review Figure 2). As a result we get J as follows:

$$J = \left[0, x - \frac{1}{b}\left\lceil\left(x + \frac{1}{2b} - z_1\right) \cdot b^2\right\rceil\right] \cup \left[y + \frac{1}{b}\left\lceil\left(x + \frac{1}{2b} - z_2\right) \cdot b^2\right\rceil, 1\right],$$

where z_1, z_2 are the solution of equation (4.1) with $z_1 < z_2$.

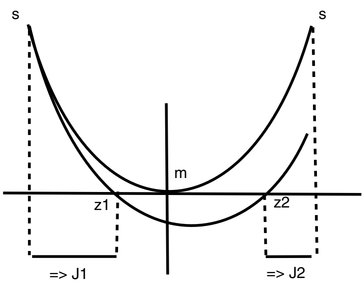

Figure 3. Main idea of the lower bound.

We can now formulate the following criterion:

Lemma 4.1. *Let a base b and a permutation σ be given and let* $\max \chi_b^\sigma \geq s$. *Then*

$$\lim_{n\to\infty} \widetilde{g}_n(b,s) \leq \gamma_b^\sigma$$

holds if there exists an interval $[k/b, (k+1)/b]$ with $\chi_b^\sigma(\frac{k}{b}) = \chi_b^\sigma(\frac{k+1}{b}) = s$ (it is sufficient that both values are $\geq$ s) or if there exists an $x \in$ argmax χ_b^σ with $x \in J$.

Remark 4.2. In practice it is possible that the requirements for the lemma are not fulfilled in some cases. However we did not find any permutation for which this lower bound did not hold. It turns out that the χ_b^σ-functions of those permutations, that do not fulfill the requirements, usually have one value $\chi_b^\sigma(z)$ with $z \in J$ which is "only a little bit" smaller than the maximum of χ_b^σ, so that the lower bound still holds.

Using various backtracking search strategies we were able to determine lowest possible values for max $\chi_b^\sigma = s$ in the bases 4 to 50. Furthermore we have good (but due to computational complexity not necessarily optimal) values for bases 51 to 72.

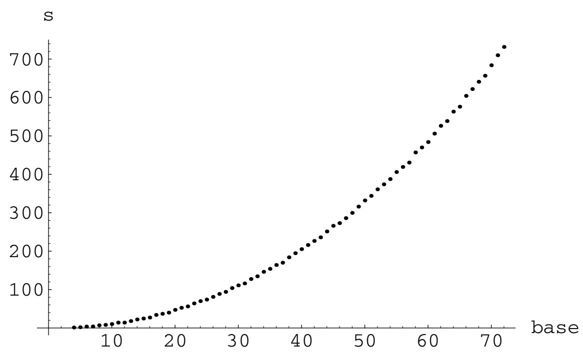

Figure 4. Smallest possible values for $s = \max \chi_b^\sigma$.

Remark 4.3. The functions $\widetilde{g}_n(b,s)$, which do not depend on a specific permutation, help investigating the connection between the L_2-discrepancy and the diaphony of $\widetilde{S_b^\sigma}$ respectively S_b^σ (see Tables 2–4). According to these values it is not possible to obtain a new best value for the L_2-discrepancy in bases $b < 51$ and it seems not possible for $b < 73$. Moreover it seems that the best values of γ_b^σ for any base $b < 73$ are on a fairly constant level which lead us to the conclusion that it is not possible to improve the value of Faure for the L_2-discrepancy stated in [2] via the diaphony for any "small" base.

Remark 4.4. The functions $\widetilde{g}_n(b,s)$ in combination with the position of the maximum of a specific function χ_b^σ support our computational search for new best values for the diaphony of $\widetilde{S_b^\sigma}$. It is a tedious and time consuming work to determine all dominant intervals for a specific function χ_b^σ and to compute γ_b^σ. As a simple heuristic rule for picking good candidates we state that the smaller the values of χ_b^σ are near the bounds of the interval [0, 1] the smaller γ_b^σ usually is.

Remark 4.5. We can, of course, provide large files containing permutations for bases $b < 73$ for which max $\chi_b^\sigma = s$ holds.

b	s	$\widetilde{g}_\infty(b,s)$	$f(\widetilde{g}_\infty(b,s),b)$	$(f(\widetilde{g}_\infty(b,s),b))/\pi^2$
4	1.25	0.972222	1.73041	0.175328
5	2	3/2	1.47176	0.14912
6	11/3	89/30	1.81571	0.18397
7	4	3	1.24212	0.125853
8	7	40/7	1.6951	0.171749
9	8	19/3	1.40486	0.142342
10	10.25	8.21296	1.40813	0.142674
11	14	23/2	1.56474	0.158541
12	14	243/22	1.21863	0.123473
13	18	29/2	1.32057	0.133802
14	22.25	18.2115	1.38996	0.140832
15	74/3	20	1.29584	0.131296
16	27.25	21.9611	1.22149	0.123762
17	34	28	1.35002	0.136786
18	37	515/17	1.27708	0.129396
19	40	65/2	1.20707	0.122302
20	47.25	38.9605	1.28358	0.130053
21	158/3	87/2	1.27906	0.129596
22	56.25	46.2103	1.21941	0.123552
23	64	53	1.26146	0.127813
24	70	1335/23	1.25178	0.126832
25	74	61	1.19703	0.121285
26	81	1674/25	1.20023	0.121609

Table 2. Best values for s for $b \leq 26$ and the according estimations.

b	s	$\widetilde{g}_\infty(b,s)$	$f(\widetilde{g}_\infty(b,s),b)$	$(f(\widetilde{g}_\infty(b,s),b))/\pi^2$
27	266/3	147/2	1.20769	0.122364
28	94	12589/162	1.17432	0.118984
29	104	173/2	1.20586	0.12218
30	111	5353/58	1.1903	0.120602
31	116	96	1.14844	0.116361
32	127	3277/31	1.17592	0.11914
33	404/3	112	1.16122	0.117656
34	146.25	122.21	1.18353	0.119917
35	154	257/2	1.16478	0.118017
36	164	9593/70	1.16493	0.118033
37	170	283/2	1.13004	0.114497
38	184	11393/74	1.15714	0.117243
39	584/3	163	1.15482	0.117008
40	205.25	171.959	1.15019	0.116539
41	216	181	1.14467	0.115979
42	680/3	23365/123	1.13742	0.115245
43	236	395/2	1.12115	0.113596
44	1005/4	36285/172	1.13679	0.115181
45	266	1343/6	1.14634	0.116149
46	273	61819/270	1.11573	0.113047
47	286	240	1.11403	0.112875
48	899/3	35491/141	1.11412	0.112884
49	316	266	1.12382	0.113867
50	332	13718/49	1.13009	0.114502

Table 3. Best values for s for $27 \leq b \leq 50$ and the according estimations.

b	s^*	$\widetilde{g}_\infty(b,s^*)$	$f(\widetilde{g}_\infty(b,s^*),b)$	$(f(\widetilde{g}_\infty(b,s^*),b))/\pi^2$
51	344	1739/6	1.11885	0.113364
52	361	93241/306	1.12591	0.114079
53	374	631/2	1.11682	0.113158
54	1163/3	103973/318	1.10969	0.112436
55	406	343	1.11705	0.113181
56	1677/4	77871/220	1.10696	0.112159
57	1292/3	363	1.09096	0.110537
58	457	66170/171	1.1184	0.113317
59	470	795/2	1.10559	0.11202
60	484	48267/118	1.09557	0.111005
61	506	857/2	1.1059	0.112051
62	2105/4	108875/244	1.11037	0.112504
63	1616/3	456	1.09475	0.110921
64	2253/4	361337/756	1.10768	0.112231
65	576	488	1.09235	0.110678
66	2417/4	133521/260	1.11089	0.112556
67	622	1057/2	1.1054	0.112001
68	641	72991/134	1.10216	0.111672
69	1970/3	1115/2	1.0918	0.110623
70	684	240931/414	1.10362	0.11182
71	710	605	1.11152	0.11262
72	2195/3	132850/213	1.11063	0.112531

Table 4. So far best (but improvable) values for s for $51 \leq b \leq 72$ and the according estimations.

5 Conclusion

In the first part of our work we present a permutation in base 57 that generates a generalised van der Corput sequence with exceptionally good distribution behaviour with respect to the diaphony. Since it is a very tedious work to determine the exact asymptotic distribution behaviour of a sequence generated by a given permutation, the second part of our paper states a method how this computations can be approximated respectively simplified. The so computed lower bounds are then used to examine an inequality stated by Faure. We find that this inequality that links the diaphony of generalised van der Corput sequences with the L_2-discrepancy of its symmetric version can not be used to improve the best known numerical results for the L_2-discrepancy of these sequences. In a forthcoming work we will try to find algorithms for constructing good permutations like Faure did in [3]. Moreover we will use a different approach to get a lower bound for any generalised van der Corput sequence in arbitrary base b as well as to determine and count classes of permutations that show similar distribution behaviour with respect to the diaphony.

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