Lattice-Nyström Method for Fredholm Integral Equations of the Second Kind

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Dedicated to Henryk Woźniakowski on the occasion of his 60th birthday.

Abstract

We consider Fredholm integral equations of the second kind of the form $f(\boldsymbol{x}) = g(\boldsymbol{x}) + \int k(\boldsymbol{x}-\boldsymbol{y}) f(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}$, where g and k are given functions from weighted Korobov spaces. These spaces are characterized by a smoothness parameter $\alpha > 1$ and weights $\gamma_1 \geq \gamma_2 \geq \cdots$. The weight γ_j moderates the behavior of the functions with respect to the jth variable. We approximate f by the Nyström method using n rank-1 lattice points. The combination of convolution and lattice group structure means that the resulting linear system can be solved in $O(n \log n)$ operations.

We analyze the worst case error measured in sup norm across functions g in the unit ball and a class of functions k in weighted Korobov spaces. We show that the generating vector of the lattice rule can be constructed component-by-component to achieve the optimal rate of convergence $O(n^{-\alpha/2+\delta})$, $\delta > 0$, with the implied constant independent of the dimension d under an appropriate condition on the weights. This construction makes use of an error criterion similar to the worst case integration error in weighted Korobov spaces, and the computational cost is only $O(n \log n\, d)$ operations.

We also study the notion of QMC-Nyström tractability: tractability means that the smallest n needed to reduce the worst case error (or normalized error) to ε is bounded polynomially in ε^{-1} and d; strong tractability means that the bound is independent of d. We prove that strong QMC-Nyström tractability in the absolute sense holds iff $\sum_{j=1}^{\infty} \gamma_j < \infty$, and QMC-Nyström tractability holds in the absolute sense iff $\limsup_{d\to\infty} \sum_{j=1}^{d} \gamma_j / \log(d+1) < \infty$.

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1 Introduction

We study certain *Fredholm integral equations of the second kind*,

$$f(\boldsymbol{x}) \;=\; g(\boldsymbol{x}) + \int_{[0,1]^d} \kappa(\boldsymbol{x},\boldsymbol{y}) f(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}, \tag{1}$$

where the kernel κ is assumed to be of the form $\kappa(\boldsymbol{x},\boldsymbol{y}) = k(\boldsymbol{x}-\boldsymbol{y})$, with $k(\boldsymbol{x})$ having period one in each component of $\boldsymbol{x}$. Further, we assume that g and k belong to a *weighted Korobov space* H (and hence are continuous on $[0,1]^d$). The general Fredholm integral equation problem has been analyzed in many papers under many different settings, usually without the convolution assumption, see for example [5–9,15,17–19] and the references therein. The weighted Korobov spaces have also been considered in many papers, see for example [16]. These spaces are characterized by a smoothness parameter $\alpha > 1$ and weights $1 \ge \gamma_1 \ge \gamma_2 \ge \cdots > 0$, where γ_j moderates the behavior of the functions with respect to the jth variable; a small γ_j means that the functions depend weakly on the jth variable. More general weights are considered in [4].

We approximate f using the *Nyström method* based on *quasi-Monte Carlo* (*QMC*) *rules*, that is, equal-weight integration rules. Let $\boldsymbol{t}_1,\ldots,\boldsymbol{t}_n$ be points in $[0,1]^d$. Our approximation of f is given by

$$f_n(\boldsymbol{x}) \;:=\; g(\boldsymbol{x}) + \frac{1}{n}\sum_{i=1}^{n} \kappa(\boldsymbol{x},\boldsymbol{t}_i) f_n(\boldsymbol{t}_i), \tag{2}$$

where the function values $f_n(\boldsymbol{t}_1),\ldots,f_n(\boldsymbol{t}_n)$ are obtained by solving the linear system

$$f_n(\boldsymbol{t}_j) \;=\; g(\boldsymbol{t}_j) + \frac{1}{n}\sum_{i=1}^{n} \kappa(\boldsymbol{t}_j,\boldsymbol{t}_i) f_n(\boldsymbol{t}_i), \quad j=1,\ldots,n. \tag{3}$$

We shall refer to our method formally as the *QMC-Nyström method*. Further assumptions on the kernel κ (or equivalently, the function k), the value n, and the points $\boldsymbol{t}_1,\ldots,\boldsymbol{t}_n$ are needed to ensure the stability and the existence of a unique solution for (3). The details are given in the next section.

We analyze the *worst case error* of the QMC-Nyström method, which is essentially the worst possible error $f - f_n$, measured in sup norm, across functions g in the unit ball and a class of functions k in a weighted Korobov space; the precise definition is given in the next section. In particular, we seek a good *lattice point set* $\boldsymbol{t}_1,\ldots,\boldsymbol{t}_n$ which leads to as small a worst case error as possible; hence the name *lattice-Nyström method*. A rank-1 lattice rule is a QMC

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rule with points given by $\boldsymbol{t}_i = \{i\boldsymbol{z}/n\}$, $i = 1, 2, \ldots, n$. Here $\boldsymbol{z}$ is known as the *generating vector*, which is an integer vector having no factor in common with n, and the braces around a vector indicate that each component of the vector is to be replaced by its fractional part. In analogy to known results on lattice rules for the integration problem in weighted Korobov spaces (see for example [3,10,16]), we prove in Theorem 5 that, for a sufficiently large n, a generating vector $\boldsymbol{z}$ can be constructed *component-by-component* for the integral equation problem such that the worst case error achieves the *optimal rate of convergence*

$$O(n^{-\alpha/2+\delta}), \qquad \delta > 0,$$

in weighted Korobov spaces. Moreover, the implied constant in the big-O notation can be bounded polynomially in d or even independently of d provided that the weights γ_j satisfy certain conditions.

The group structure of lattice points, together with the convolution assumption $\kappa(\boldsymbol{x}, \boldsymbol{y}) = k(\boldsymbol{x} - \boldsymbol{y})$, means that

$$\kappa(\boldsymbol{t}_j, \boldsymbol{t}_i) \;=\; k(\boldsymbol{t}_j - \boldsymbol{t}_i) \;=\; k(\boldsymbol{t}_{(j-i) \bmod n}), \qquad \text{with} \quad \boldsymbol{t}_0 := \boldsymbol{t}_n.$$

Thus the approximation (2) requires a total of $N = 2\,n$ function evaluations, that is, n evaluations of the function g and n evaluations of the function k at the lattice points. It also means that the linear system (3) can be solved using Fast Fourier Transform, with only $O(n \log n)$ operations. This has been studied in [20].

We also study *tractability* and *strong tractability* of the QMC-Nyström method in the *absolute* and/or *normalized* sense. Roughly speaking, tractability in the absolute sense means that the minimal value of n needed in the QMC-Nyström method to reduce the worst case error to $\varepsilon \in (0, 1)$ is bounded polynomially in d and ε^{-1}; strong tractability means that the bound is independent of d. We show in Theorem 6 that *strong QMC-Nyström tractability in the absolute sense* holds iff

$$\sum_{j=1}^{\infty} \gamma_j \;<\; \infty, \tag{4}$$

and *QMC-Nyström tractability in the absolute sense* holds iff

$$\limsup_{d\to\infty} \frac{\sum_{j=1}^{d} \gamma_j}{\log(d+1)} \;<\; \infty. \tag{5}$$

(Strong) tractability in the normalized sense is defined in terms of the normalized error with respect to the initial error. The conditions (4) and (5) are also sufficient conditions for (strong) QMC-Nyström tractability in the normalized sense, but we were unable to prove that they are also necessary.

This paper is organized as follows. In Section 2 we formulate the problem and we define the worst case error criterion and the notion of QMC-Nyström tractability. Section 3 contains the main results of this paper. We obtain worst case error bounds and derive necessary and/or sufficient conditions for QMC-Nyström tractability. We also prove that the generating vector for a lattice rule can be constructed component-by-component to achieve the optimal rate of convergence. Finally in Section 4 we give some additional remarks.

2 Problem Formulation

2.1 Preliminaries

Let $D = [0,1]^d$, and let $C = C(D)$ denote the class of continuous functions on D equipped with the sup norm $\|f\|_{\sup} = \sup_{\boldsymbol{x}\in D} |f(\boldsymbol{x})|$. For the space of bounded linear operators from C to C, we equip it with the usual induced operator norm $\|T\| = \|T\|_{C\to C} = \sup_{\|f\|_{\sup}\le 1} \|Tf\|_{\sup}$. In particular, for a given kernel $\kappa \in C\times C$ we are interested in the integral operator $K : C \to C$,

$$Kf \;=\; \int_D \kappa(\cdot,\boldsymbol{y}) f(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}, \quad \text{with} \quad \|K\| \;=\; \max_{\boldsymbol{x}\in D} \int_D |\kappa(\boldsymbol{x},\boldsymbol{y})|\,\mathrm{d}\boldsymbol{y},$$

and the corresponding discrete operator $K_n : C \to C$,

$$K_n f \;=\; \frac{1}{n}\sum_{i=1}^n \kappa(\cdot,\boldsymbol{t}_i) f(\boldsymbol{t}_i), \quad \text{with} \quad \|K_n\| \;=\; \max_{\boldsymbol{x}\in D} \frac{1}{n}\sum_{i=1}^n |\kappa(\boldsymbol{x},\boldsymbol{t}_i)|,$$

where $\boldsymbol{t}_1,\ldots,\boldsymbol{t}_n \in D$. The operator K is compact and the sequence $\{K_n\}$ is *collectively compact*, see Anselone [1]. Throughout this paper we consider kernels of the form $\kappa(\boldsymbol{x},\boldsymbol{y}) = k(\boldsymbol{x}-\boldsymbol{y})$ with $k \in C$ periodic. Thus

$$\|K\| \;=\; \int_D |k(\boldsymbol{y})|\,\mathrm{d}\boldsymbol{y} \;\le\; \|k\|_{\sup} \quad \text{and} \quad \|K_n\| \;=\; \max_{\boldsymbol{x}\in D} \frac{1}{n}\sum_{i=1}^n |k(\boldsymbol{x}-\boldsymbol{t}_i)| \;\le\; \|k\|_{\sup},$$

where the inequalities become equalities when k is a constant function.

Let $H = H^{(d)}_{\boldsymbol{\gamma},\alpha}(D)$ denote a weighted Korobov space, where $\boldsymbol{\gamma} = (\gamma_j)_{j\ge 1}$ is a sequence of positive weights and $\alpha > 1$ is a smoothness parameter. For any

$$f(\boldsymbol{x}) \;=\; \sum_{\boldsymbol{h}\in\mathbb{Z}^d} \hat{f}(\boldsymbol{h})\, e^{2\pi\mathrm{i}\boldsymbol{h}\cdot\boldsymbol{x}}, \quad \text{with} \quad \hat{f}(\boldsymbol{h}) \;=\; \int_D f(\boldsymbol{x})\, e^{-2\pi\mathrm{i}\boldsymbol{h}\cdot\boldsymbol{x}}\,\mathrm{d}\boldsymbol{x},$$

the norm of f in H is given by

$$\|f\|_H \;=\; \left(\sum_{\boldsymbol{h}\in\mathbb{Z}^d} |\hat{f}(\boldsymbol{h})|^2 r_\alpha(\boldsymbol{\gamma},\boldsymbol{h})\right)^{1/2},$$

where

$$r_\alpha(\boldsymbol{\gamma},\boldsymbol{h}) \;=\; \prod_{j=1}^d r_\alpha(\gamma_j,h_j), \quad \text{with} \quad r_\alpha(\gamma,h) \;=\; \begin{cases} 1 & \text{if } h=0, \\ \gamma^{-1}|h|^\alpha & \text{otherwise.} \end{cases}$$

Additionally, we assume that $1 \geq \gamma_1 \geq \gamma_2 \geq \cdots > 0$ and thus $r_\alpha(\boldsymbol{\gamma},\boldsymbol{h}) \geq 1$ for all $\boldsymbol{h} \in \mathbb{Z}^d$. Using the Cauchy-Schwarz inequality, we have for all $f \in H$

$$\begin{aligned} \|f\|_{\sup} \;\leq\; \sum_{\boldsymbol{h}\in\mathbb{Z}^d} |\hat{f}(\boldsymbol{h})| \;\leq\; \left(\sum_{\boldsymbol{h}\in\mathbb{Z}^d} |\hat{f}(\boldsymbol{h})|^2 r_\alpha(\boldsymbol{\gamma},\boldsymbol{h})\right)^{1/2} \left(\sum_{\boldsymbol{h}\in\mathbb{Z}^d} \frac{1}{r_\alpha(\boldsymbol{\gamma},\boldsymbol{h})}\right)^{1/2} \\ = \|f\|_H \prod_{j=1}^d \left(1+2\zeta(\alpha)\gamma_j\right)^{1/2}, \end{aligned} \tag{6}$$

where $\zeta(x) := \sum_{h=1}^\infty h^{-x}$ denotes the Riemann Zeta function. Thus H is embedded in C. Furthermore, the inequalities in (6) become equalities when f is a multiple of the function $\sum_{\boldsymbol{h}\in\mathbb{Z}^d} e^{2\pi \mathrm{i}\boldsymbol{h}\cdot\boldsymbol{x}}/r_\alpha(\boldsymbol{\gamma},\boldsymbol{h})$.

2.2 Fredholm integral equations and the Nyström method

Given $g,k \in H$, we study the solution $S(g,k) := f$ of the Fredholm integral equation (1), which we express as

$$f \;=\; g + Kf,$$

or as $(I-K)f = g$, where $I : C \to C$ denotes the identity operator $If = f$. Assuming that the operator $(I-K)^{-1}$ exists, by the Fredholm alternative we have $\|(I-K)^{-1}\| < \infty$, and

$$f \;=\; (I-K)^{-1} g.$$

Since $(I-K)^{-1} e^{2\pi\mathrm{i}\boldsymbol{h}\cdot\boldsymbol{x}} = e^{2\pi\mathrm{i}\boldsymbol{h}\cdot\boldsymbol{x}}/(1-\hat{k}(\boldsymbol{h}))$, we have

$$\|(I-K)^{-1}\| \;\geq\; \frac{1}{|1-\hat{k}(\boldsymbol{h})|} \qquad \forall\, \boldsymbol{h} \in \mathbb{Z}^d, \tag{7}$$

which guarantees that $\hat{k}(\boldsymbol{h}) \neq 1$. Because $k \in H$, we have $\hat{k}(\boldsymbol{h}) \to 0$ for large $\boldsymbol{h}$, and thus (7) allows us to deduce that

$$\|(I-K)^{-1}\| \;\geq\; 1.$$

Since $\kappa(\boldsymbol{x},\boldsymbol{y}) = k(\boldsymbol{x}-\boldsymbol{y})$, it is easily shown that

$$(Kf)(\boldsymbol{x}) \;=\; \sum_{\boldsymbol{h}\in\mathbb{Z}^d} \hat{k}(\boldsymbol{h})\hat{f}(\boldsymbol{h})\, e^{2\pi\mathrm{i}\boldsymbol{h}\cdot\boldsymbol{x}},$$

implying $\hat{f}(\boldsymbol{h}) = \hat{g}(\boldsymbol{h}) + \hat{k}(\boldsymbol{h})\hat{f}(\boldsymbol{h})$ for all $\boldsymbol{h} \in \mathbb{Z}^d$. Thus we have

$$\hat{f}(\boldsymbol{h}) = \frac{\hat{g}(\boldsymbol{h})}{1 - \hat{k}(\boldsymbol{h})}. \tag{8}$$

Hence one way to approximate f is to use approximations of the Fourier coefficients of g and k. This will be studied in a separate paper. Moreover we have

$$\|f\|_H = \left(\sum_{\boldsymbol{h} \in \mathbb{Z}^d} \left| \frac{\hat{g}(\boldsymbol{h})}{1 - \hat{k}(\boldsymbol{h})} \right|^2 r_\alpha(\boldsymbol{\gamma}, \boldsymbol{h}) \right)^{1/2} \leq \|(I - K)^{-1}\| \|g\|_H, \tag{9}$$

where the inequality becomes equality when g and k are both constant functions. We emphasize that the norm of $(I - K)^{-1}$ in (9) is the operator norm in C, not in H.

Using the QMC-Nyström method, we approximate f by the algorithm $A_n(g, k) := f_n$, with f_n given by (2), or alternatively expressed,

$$f_n = g + K_n f_n,$$

where the function values $f_n(\boldsymbol{t}_1), \ldots, f_n(\boldsymbol{t}_n)$ are to be obtained by solving the linear system (3). Suppose that

$$\Delta_n := \|(I - K)^{-1}\| \|(K - K_n)K_n\| < 1,$$

then the operator $(I - K_n)^{-1}$ exists and

$$\|(I - K_n)^{-1}\| \leq \frac{1 + \|(I - K)^{-1}\| \|K_n\|}{1 - \Delta_n},$$

see [1]. Then f_n is well defined and we have

$$f_n = (I - K_n)^{-1} g.$$

Note that $\Delta_n < 1$ is essentially a condition on the value of n and the quality of the points $\boldsymbol{t}_1, \ldots, \boldsymbol{t}_n$. Provided that $\|Kf - K_n f\| \to 0$ for all $f \in C$, the collective compactness of $\{K_n\}$ yields $\|(K - K_n)K_n\| \to 0$. More details can be found in [1].

2.3 Error formulation

We are ready to define the integral equation problem on H. Let

$$\beta > 0 \quad \text{and} \quad \mu > 1$$

be fixed. Recall that

$$S(g,k) \;=\; (I-K)^{-1}g \qquad \text{and} \qquad A_n(g,k) \;=\; (I-K_n)^{-1}g.$$

We define the *worst case error* of a QMC-Nyström method by

$$e_{n,d}(A_n) \;:=\; \sup_{\substack{\|g\|_H \le 1 \\ \|k\|_H \le \beta,\, \|(I-K)^{-1}\| \le \mu}} \|S(g,k) - A_n(g,k)\|_{\sup},$$

that is, we are interested in a class of problems where $k \in H$ satisfies

$$\|k\|_H \;\le\; \beta \qquad \text{and} \qquad 1 \;\le\; \|(I-K)^{-1}\| \;\le\; \mu. \tag{10}$$

Due to linearity in g, we have for all $g \in H$ and all k satisfying (10) that

$$\|S(g,k) - A_n(g,k)\|_{\sup} \;\le\; e_{n,d}(A_n)\, \|g\|_H.$$

However, a similar result does not hold for k. Note that the constants β and μ in (10) are mutually independent, in that for appropriate choices of k either $\|k\|_H$ or $\|(I-K)^{-1}\|$ can be arbitrarily large while the other is bounded.

The *initial error* associated with the zero algorithm $A_0 \equiv 0$ is defined as

$$e_{0,d} \;:=\; \sup_{\substack{\|g\|_H \le 1 \\ \|k\|_H \le \beta,\, \|(I-K)^{-1}\| \le \mu}} \|S(g,k)\|_{\sup}.$$

For $\varepsilon \in (0,1)$, we are interested in the smallest value of n for which either $e_{n,d}(A_n) \le \varepsilon$, which corresponds to tractability in the absolute sense, or $e_{n,d}(A_n) \le \varepsilon e_{0,d}$, which corresponds to tractability in the normalized sense.

First we define tractability in the absolute sense. For $\varepsilon \in (0,1)$ and $d \ge 1$, let

$$n^{\mathrm{abs}}(\varepsilon,d) \;:=\; \min\{n : \exists \text{ QMC-Nyström method } A_n \text{ with } e_{n,d}(A_n) \;\le\; \varepsilon\}.$$

The integral equation problem is said to be *QMC-Nyström tractable in the absolute sense* iff there exist nonnegative constants C, p and q independent of ε and d such that

$$n^{\mathrm{abs}}(\varepsilon,d) \;\le\; C\,\varepsilon^{-p}\,d^{q} \qquad \forall\,\varepsilon \in (0,1) \quad \forall\,d \ge 1.$$

The problem is said to be *strongly QMC-Nyström tractable in the absolute sense* iff the above condition holds with $q = 0$.

Tractability and strong tractability in the normalized sense can be defined in a similar way, with $n^{\mathrm{abs}}(\varepsilon,d)$ replaced by

$$n^{\mathrm{nor}}(\varepsilon,d) \;:=\; \min\{n : \exists \text{ QMC-Nyström method } A_n \text{ with } e_{n,d}(A_n) \;\le\; \varepsilon\, e_{0,d}\}.$$

Note that the *cost* (in terms of the number of function evaluations of g and k) for the lattice-Nyström method is $N = 2\,n$, while for the general QMC-Nyström method it can be $N = n^2 + n$.

3 Error Analysis

3.1 Initial error

For all g satisfying $\|g\|_H \le 1$ and all k satisfying (10), it follows from (6) and (9) that

$$\|S(g,k)\|_{\sup} \;\le\; \|(I-K)^{-1}\|\|g\|_H \prod_{j=1}^{d} \left(1+2\zeta(\alpha)\gamma_j\right)^{1/2} \;\le\; \mu \prod_{j=1}^{d} \left(1+2\zeta(\alpha)\gamma_j\right)^{1/2}.$$

This provides an upper bound on the initial error $e_{0,d}$. Note that this upper bound does not depend on β.

To obtain a lower bound on the initial error, we consider specific functions g and k. Let $k \equiv c$, with

$$0 \;<\; c \;:=\; \min\left(\beta, 1-\tfrac{1}{\mu}\right) \;<\; 1.$$

Then $\|k\|_H = c \le \beta$ and $\|(I-K)^{-1}\| = 1/(1-c) \le \mu$. We define

$$g(\boldsymbol{x}) \;:=\; \frac{1}{G}\left(\sum_{\boldsymbol{h}\in\mathbb{Z}^d} \frac{e^{2\pi \mathrm{i}\boldsymbol{h}\cdot\boldsymbol{x}}}{r_\alpha(\boldsymbol{\gamma},\boldsymbol{h})} - c\right), \quad \text{with} \quad G \;:=\; \prod_{j=1}^{d} \left(1+2\zeta(\alpha)\gamma_j\right)^{1/2}.$$

Then $\|g\|_H \le 1$, and it is not hard to see that

$$\frac{\hat{g}(\boldsymbol{h})}{1-\hat{k}(\boldsymbol{h})} \;=\; \frac{1}{G\, r_\alpha(\boldsymbol{\gamma},\boldsymbol{h})} \qquad \forall\, \boldsymbol{h} \in \mathbb{Z}^d.$$

Thus for this choice of g and k, it follows from (8) that

$$\|S(g,k)\|_{\sup} \;=\; \left\| \frac{1}{G} \sum_{\boldsymbol{h}\in\mathbb{Z}^d} \frac{e^{2\pi \mathrm{i}\boldsymbol{h}\cdot\boldsymbol{x}}}{r_\alpha(\boldsymbol{\gamma},\boldsymbol{h})} \right\|_{\sup} \;=\; \prod_{j=1}^{d} \left(1+2\zeta(\alpha)\gamma_j\right)^{1/2}.$$

Hence we have a lower bound on the initial error with the same dependence on d as the upper bound obtained before. In other words, we know exactly how the initial error increases with d.

This lower bound does not give an indication of the dependence on β and μ. A different lower bound can be obtained by choosing $g \equiv k \equiv c$ with c defined as above. In this case, $\|S(g,k)\|_{\sup} = c/(1-c)$.

Our analysis leads to the following result.

Lemma 1 *Let $c := \min(\beta, 1 - 1/\mu)$. The initial error satisfies*

$$\max\left(\frac{c}{1-c}, \prod_{j=1}^{d}(1+2\zeta(\alpha)\gamma_j)^{1/2}\right) \le e_{0,d} \le \mu \prod_{j=1}^{d}(1+2\zeta(\alpha)\gamma_j)^{1/2}.$$

Note that if $\beta \ge 1$ then $c/(1-c) = \mu - 1$. In this case, we see that the initial error increases linearly with μ.

3.2 Lower bound on the worst case error

Again we consider a constant function $k \equiv c$, with $c := \min(\beta, 1 - 1/\mu)$. Then $\|k\|_H \le \beta$ and $\|(I-K)^{-1}\| \le \mu$. Moreover, for any g it is easy to show that

$$f = (I-K)^{-1}g = g + \frac{c}{1-c}\int_D g(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$$

and

$$f_n = (I-K_n)^{-1}g = g + \frac{c}{1-c}\frac{1}{n}\sum_{i=1}^{n} g(\boldsymbol{t}_i).$$

Thus it follows by definition that

$$\begin{aligned} e_{n,d}(A_n) &\ge \sup_{\|g\|_H \le 1} \|S(g,k) - A_n(g,k)\|_{\sup} \\ &= \frac{c}{1-c}\sup_{\|g\|_H\le 1}\left|\int_D g(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} - \frac{1}{n}\sum_{i=1}^{n} g(\boldsymbol{t}_i)\right| \\ &= \frac{c}{1-c}\, e_{n,d}^{\mathrm{wor-int}}(\boldsymbol{t}_1,\ldots,\boldsymbol{t}_n), \end{aligned} \tag{11}$$

where $e_{n,d}^{\mathrm{wor-int}}(\boldsymbol{t}_1,\ldots,\boldsymbol{t}_n)$ denotes the *worst case integration error* in H using quadrature points $\boldsymbol{t}_1,\ldots,\boldsymbol{t}_n$.

It is known from [16] that in weighted Korobov spaces we have

$$e_{n,d}^{\mathrm{wor-int}}(\boldsymbol{t}_1,\ldots,\boldsymbol{t}_n) \ge e_{n,1}^{\mathrm{wor-int}}\left(0, \tfrac{1}{n}, \tfrac{2}{n}, \ldots, \tfrac{n-1}{n}\right) = \left(\frac{2\zeta(\alpha)\gamma_1}{n^\alpha}\right)^{1/2}.$$

This rate of convergence of $O(n^{-\alpha/2})$ is optimal for the integration problem in weighted Korobov spaces (see also Sharygin's lower bound [9]). In fact, it was proved in [3,10] that a generating vector $\boldsymbol{z}$ for a rank-1 lattice rule can be constructed component-by-component to achieve the rate of convergence $O(n^{-\alpha/2+\delta})$, $\delta > 0$. Not surprisingly, this is also the optimal rate of convergence for the integral equation problem. Later we will show that a generating vector

$\boldsymbol{z}$ for a rank-1 lattice rule can be constructed component-by-component, based on a different error criterion, to achieve this optimal rate of convergence.

In terms of the dependence on d, it was shown in [16] that

$$e_{n,d}^{\mathrm{wor-int}}(\boldsymbol{t}_1,\ldots,\boldsymbol{t}_n) \,\geq\, \left(\frac{1}{n}\prod_{j=1}^{d}\left(1+2\zeta(\alpha)\omega_\alpha\gamma_j\right)-1\right)^{1/2},$$

where $\omega_\alpha := \min(1, 1/(2\gamma_1|\theta_{\min}|)) \leq 1$, with $-1 < \theta_{\min} < -1 + 2^{-\alpha}$ denoting the minimum of the function $\theta(x) = \sum_{h=1}^{\infty}\cos(2\pi hx)/h^\alpha$, see [2] or [4, Eq. (26)]. Moreover, it was proved in [16] that the integration problem in weighted Korobov spaces is *strongly QMC tractable* iff (4) holds, and *QMC tractable* iff (5) holds. Note that since the initial integration error is exactly 1, there is no need to distinguish between tractability in the normalized sense and tractability in the absolute sense.

We summarize the lower bounds in the following lemma.

Lemma 2 *Let $c := \min(\beta, 1 - 1/\mu)$. The worst case error for the QMC-Nyström method satisfies*

$$e_{n,d}(A_n) \,\geq\, \frac{c}{1-c}\max\left(\frac{2\zeta(\alpha)\gamma_1}{n^\alpha}, \frac{1}{n}\prod_{j=1}^{d}\left(1+2\zeta(\alpha)\omega_\alpha\gamma_j\right)-1\right)^{1/2},$$

where $\omega_\alpha \leq 1$ is some constant independent of n and d.

For tractability in the absolute sense, we see from the relationship (11) that (4) and (5) are necessary conditions for strong QMC-Nyström tractability and QMC-Nyström tractability, respectively. Later we will see that these conditions are also sufficient for tractability in the absolute sense.

Unfortunately, we cannot obtain necessary conditions for tractability in the normalized sense because the d-dependence in the lower bound is too weak compared with the bounds on the initial error. Indeed, since $\omega_\alpha \leq 1$, we cannot see how the normalized error $e_{n,d}(A_n)/e_{0,d}$ increases with d.

3.3 Upper bound on the worst case error

By subtracting $(I - K_n)f_n = g$ from $(I - K_n)f = (I - K)f + (K - K_n)f = g + (K - K_n)f$, we obtain

$$f - f_n \,=\, (I - K_n)^{-1}(K - K_n)f.$$

Thus

$$\|S(g,k) - A_n(g,k)\|_{\sup} \;=\; \|f - f_n\|_{\sup} \;\le\; \|(I-K_n)^{-1}\| \|(K-K_n)f\|_{\sup}.$$

Recall that

$$\|(I-K_n)^{-1}\| \;\le\; \frac{1 + \|(I-K)^{-1}\| \|K_n\|}{1-\Delta_n},$$

when $\Delta_n := \|(I-K)^{-1}\| \|(K-K_n)K_n\| < 1$. We can bound $\|K_n\|$ as follows

$$\|K_n\| \;\le\; \|k\|_{\sup} \;\le\; \|k\|_H \prod_{j=1}^{d} (1 + 2\zeta(\alpha)\gamma_j)^{1/2}.$$

Hence we can write

$$\|f - f_n\|_{\sup} \;\le\; \frac{1 + \|(I-K)^{-1}\| \|k\|_H \prod_{j=1}^{d}(1+2\zeta(\alpha)\gamma_j)^{1/2}}{1 - \|(I-K)^{-1}\| \|(K-K_n)K_n\|} \|(K-K_n)f\|_{\sup}.$$

The term $\|(K-K_n)K_n\|$ controls whether or not $\Delta_n < 1$, while $\|(K-K_n)f\|_{\sup}$ determines the rate of convergence. It remains to obtain bounds on these two terms.

Let $\boldsymbol{t}_1, \ldots, \boldsymbol{t}_n$ be rank-1 lattice points generated by $\boldsymbol{z}$, that is, $\boldsymbol{t}_i = \{i\boldsymbol{z}/n\}$ where $\{x\} = x - \lfloor x \rfloor$. We have

$$\begin{aligned}
((K-K_n)f)(\boldsymbol{x}) \;&=\; \int_D k(\boldsymbol{x}-\boldsymbol{y}) f(\boldsymbol{y}) \,\mathrm{d}\boldsymbol{y} - \frac{1}{n}\sum_{i=1}^{n} k(\boldsymbol{x}-\boldsymbol{t}_i) f(\boldsymbol{t}_i) \\
&=\; - \sum_{\substack{\boldsymbol{h}\in\mathbb{Z}^d\setminus\{\boldsymbol{0}\} \\ \boldsymbol{h}\cdot\boldsymbol{z}\equiv 0 \;(\bmod\; n)}} \hat{F}_{\boldsymbol{x}}(\boldsymbol{h}),
\end{aligned}$$

where $F_{\boldsymbol{x}}(\boldsymbol{y}) := k(\boldsymbol{x}-\boldsymbol{y})f(\boldsymbol{y})$, and

$$\begin{aligned}
\hat{F}_{\boldsymbol{x}}(\boldsymbol{h}) \;&=\; \int_D k(\boldsymbol{x}-\boldsymbol{y}) f(\boldsymbol{y}) \, e^{-2\pi \mathrm{i} \boldsymbol{h}\cdot\boldsymbol{y}} \,\mathrm{d}\boldsymbol{y} \\
&=\; \sum_{\boldsymbol{\ell}\in\mathbb{Z}^d} \sum_{\boldsymbol{p}\in\mathbb{Z}^d} \hat{k}(\boldsymbol{\ell}) \hat{f}(\boldsymbol{p}) \, e^{2\pi\mathrm{i}\boldsymbol{\ell}\cdot\boldsymbol{x}} \int_D e^{2\pi\mathrm{i}(\boldsymbol{p}-\boldsymbol{\ell}-\boldsymbol{h})\cdot\boldsymbol{y}} \,\mathrm{d}\boldsymbol{y} \\
&=\; \sum_{\boldsymbol{\ell}\in\mathbb{Z}^d} \hat{k}(\boldsymbol{\ell}) \hat{f}(\boldsymbol{h}+\boldsymbol{\ell}) \, e^{2\pi\mathrm{i}\boldsymbol{\ell}\cdot\boldsymbol{x}}.
\end{aligned}$$

Thus it follows from the Cauchy-Schwarz inequality that

$$
\begin{aligned}
\|(K-K_n)f\|_{\sup} &= \sup_{\boldsymbol{x}\in D}\Bigg|\sum_{\substack{\boldsymbol{h}\in\mathbb{Z}^d\setminus\{\boldsymbol{0}\}\\ \boldsymbol{h}\cdot\boldsymbol{z}\equiv 0 \ (\bmod\ n)}}\sum_{\boldsymbol{\ell}\in\mathbb{Z}^d}\hat{k}(\boldsymbol{\ell})\hat{f}(\boldsymbol{h}+\boldsymbol{\ell})\,e^{2\pi\mathrm{i}\boldsymbol{\ell}\cdot\boldsymbol{x}}\Bigg| \\
&\le \sum_{\substack{\boldsymbol{h}\in\mathbb{Z}^d\setminus\{\boldsymbol{0}\}\\ \boldsymbol{h}\cdot\boldsymbol{z}\equiv 0 \ (\bmod\ n)}}\sum_{\boldsymbol{\ell}\in\mathbb{Z}^d}|\hat{k}(\boldsymbol{\ell})||\hat{f}(\boldsymbol{h}+\boldsymbol{\ell})| \qquad (12)\\
&\le \sum_{\boldsymbol{\ell}\in\mathbb{Z}^d}\left[|\hat{k}(\boldsymbol{\ell})|\left(\sum_{\substack{\boldsymbol{h}\in\mathbb{Z}^d\setminus\{\boldsymbol{0}\}\\ \boldsymbol{h}\cdot\boldsymbol{z}\equiv 0 \ (\bmod\ n)}}|\hat{f}(\boldsymbol{h}+\boldsymbol{\ell})|^2 r_\alpha(\boldsymbol{\gamma},\boldsymbol{h}+\boldsymbol{\ell})\right)^{1/2}\right.\\
&\qquad\qquad \left.\times\left(\sum_{\substack{\boldsymbol{h}\in\mathbb{Z}^d\setminus\{\boldsymbol{0}\}\\ \boldsymbol{h}\cdot\boldsymbol{z}\equiv 0 \ (\bmod\ n)}}\frac{1}{r_\alpha(\boldsymbol{\gamma},\boldsymbol{h}+\boldsymbol{\ell})}\right)^{1/2}\right]\\
&\le \|f\|_H\left(\sum_{\boldsymbol{\ell}\in\mathbb{Z}^d}|\hat{k}(\boldsymbol{\ell})|^2 r_\alpha(\boldsymbol{\gamma},\boldsymbol{\ell})\right)^{1/2}\left(\sum_{\boldsymbol{\ell}\in\mathbb{Z}^d}\frac{1}{r_\alpha(\boldsymbol{\gamma},\boldsymbol{\ell})}\sum_{\substack{\boldsymbol{h}\in\mathbb{Z}^d\setminus\{\boldsymbol{0}\}\\ \boldsymbol{h}\cdot\boldsymbol{z}\equiv 0 \ (\bmod\ n)}}\frac{1}{r_\alpha(\boldsymbol{\gamma},\boldsymbol{h}+\boldsymbol{\ell})}\right)^{1/2}\\
&\le \|(I-K)^{-1}\|\|g\|_H\|k\|_H S_{n,d}(\boldsymbol{z}),
\end{aligned}
$$

where in the last step we used (9) and the definition

$$
S_{n,d}(\boldsymbol{z}) := \left(\sum_{\substack{\boldsymbol{h}\in\mathbb{Z}^d\setminus\{\boldsymbol{0}\}\\ \boldsymbol{h}\cdot\boldsymbol{z}\equiv 0 \ (\bmod\ n)}}\sum_{\boldsymbol{\ell}\in\mathbb{Z}^d}\frac{1}{r_\alpha(\boldsymbol{\gamma},\boldsymbol{\ell})\,r_\alpha(\boldsymbol{\gamma},\boldsymbol{h}+\boldsymbol{\ell})}\right)^{1/2}. \qquad (13)
$$

Using a similar argument to that above, we obtain

$$
\begin{aligned}
&\|(K-K_n)K_n\|\\
&= \sup_{\boldsymbol{x}\in D}\frac{1}{n}\sum_{j=1}^n\left|\int_D k(\boldsymbol{x}-\boldsymbol{y})k(\boldsymbol{y}-\boldsymbol{t}_j)\,\mathrm{d}\boldsymbol{y}-\frac{1}{n}\sum_{i=1}^n k(\boldsymbol{x}-\boldsymbol{t}_i)k(\boldsymbol{t}_i-\boldsymbol{t}_j)\right|\\
&= \sup_{\boldsymbol{x}\in D}\frac{1}{n}\sum_{j=1}^n\Bigg|\sum_{\substack{\boldsymbol{h}\in\mathbb{Z}^d\setminus\{\boldsymbol{0}\}\\ \boldsymbol{h}\cdot\boldsymbol{z}\equiv 0 \ (\bmod\ n)}}\sum_{\boldsymbol{\ell}\in\mathbb{Z}^d}\hat{k}(\boldsymbol{\ell})\hat{k}(\boldsymbol{h}+\boldsymbol{\ell})\,e^{2\pi\mathrm{i}\boldsymbol{\ell}\cdot\boldsymbol{x}}e^{-2\pi\mathrm{i}(\boldsymbol{h}+\boldsymbol{\ell})\cdot\boldsymbol{t}_j}\Bigg|\\
&\le \sum_{\substack{\boldsymbol{h}\in\mathbb{Z}^d\setminus\{\boldsymbol{0}\}\\ \boldsymbol{h}\cdot\boldsymbol{z}\equiv 0 \ (\bmod\ n)}}\sum_{\boldsymbol{\ell}\in\mathbb{Z}^d}|\hat{k}(\boldsymbol{\ell})||\hat{k}(\boldsymbol{h}+\boldsymbol{\ell})|\\
&\le \|k\|_H^2 S_{n,d}(\boldsymbol{z}).
\end{aligned}
$$

Therefore when k satisfies (10) we have

$$
\Delta_n \le \|(I-K)^{-1}\|\|k\|_H^2 S_{n,d}(\boldsymbol{z}) \le \mu\beta^2 S_{n,d}(\boldsymbol{z}).
$$

To ensure that $\Delta_n < 1$, it is sufficient to demand that $S_{n,d}(\boldsymbol{z}) < 1/(\mu\beta^2)$. When this holds, we have

$$\begin{aligned}\|(I-K_n)^{-1}\| &\le \frac{1+\|(I-K)^{-1}\|\|k\|_H\prod_{j=1}^d(1+2\zeta(\alpha)\gamma_j)^{1/2}}{1-\Delta_n}\\ &\le \frac{1+\mu\beta\prod_{j=1}^d(1+2\zeta(\alpha)\gamma_j)^{1/2}}{1-\mu\beta^2 S_{n,d}(\boldsymbol{z})}\\ &\le \frac{1+\mu\beta}{1-\mu\beta^2 S_{n,d}(\boldsymbol{z})}\prod_{j=1}^d(1+2\zeta(\alpha)\gamma_j)^{1/2}.\end{aligned}$$

Thus for g satisfying $\|g\|_H \le 1$ and k satisfying (10), we have

$$\begin{aligned}\|f-f_n\|_{\sup} &\le \|(I-K_n)^{-1}\|\|(I-K)^{-1}\|\|g\|_H\|k\|_H S_{n,d}(\boldsymbol{z})\\ &\le \frac{(1+\mu\beta)\mu\beta\, S_{n,d}(\boldsymbol{z})}{1-\mu\beta^2\, S_{n,d}(\boldsymbol{z})}\prod_{j=1}^d(1+2\zeta(\alpha)\gamma_j)^{1/2}.\end{aligned}$$

We summarize this discussion in the following lemma.

Lemma 3 *Suppose there exists an integer vector $\boldsymbol{z}$ for which $S_{n,d}(\boldsymbol{z})$ defined in* (13) *satisfies*

$$S_{n,d}(\boldsymbol{z}) < \frac{1}{\mu\beta^2}.$$

Then the worst case error for the lattice-Nyström method satisfies

$$e_{n,d}(A_n) \le \frac{(1+\mu\beta)\mu\beta\, S_{n,d}(\boldsymbol{z})}{1-\mu\beta^2\, S_{n,d}(\boldsymbol{z})}\prod_{j=1}^d(1+2\zeta(\alpha)\gamma_j)^{1/2}.$$

We need $S_{n,d}(\boldsymbol{z}) < 1/(\mu\beta^2)$ to control the denominator in the error bound (to ensure that $\Delta_n < 1$). As long as $S_{n,d}(\boldsymbol{z})$ converges with n, this condition can be trivially fulfilled with a large enough n. On the other hand, the $S_{n,d}(\boldsymbol{z})$ in the numerator determines the rate of convergence of the worst case error.

3.4 Component-by-component construction of $\boldsymbol{z}$

Here we present an algorithm for constructing a generating vector $\boldsymbol{z}$ that leads to the optimal rate of convergence $O(n^{-\alpha/2+\delta})$, $\delta > 0$. For simplicity, we restrict ourselves to n being a prime number. In this case, the components of the generating vector $\boldsymbol{z}$ can be restricted to the set $\{1, 2, \ldots, n-1\}$.

Algorithm 1 *Let n be a prime number.*

1. *Set $z_1 = 1$.*

2. *For $s = 2, 3, \ldots, d$, with $z_1, z_2, \ldots, z_{s-1}$ already chosen and fixed, find $z_s \in \{1, 2, \ldots, n-1\}$ to minimize $S_{n,s}(z_1, \ldots, z_{s-1}, z_s)$.*

Lemma 4 *Let n be prime and let $\boldsymbol{z}^* \in \{1, 2, \ldots, n-1\}^d$ be constructed by Algorithm 1. Then*

$$S_{n,d}(\boldsymbol{z}^*) \;\le\; \frac{1}{\delta\, n^{1/(2\lambda)}} \prod_{j=1}^{d} \Big(1 + 2(1+\delta^\lambda)^{1/2}\zeta(\alpha\lambda)\gamma_j^\lambda\Big)^{1/\lambda}$$

for all $\lambda \in (1/\alpha, 1]$ and $\delta \in (0, 2^{-3\alpha}]$.

Proof. The proof of this lemma is long and tedious, and is therefore deferred to the Appendix. □

We now obtain a sufficient condition on n to ensure that $S_{n,d}(\boldsymbol{z}) < 1/(\mu\beta^2)$. It is enough to choose n such that the upper bound in Lemma 4 with $\lambda = 1$ and $\delta = 2^{-3\alpha}$ is no greater than, say, $1/(2\mu\beta^2)$. In other words, if

$$n \;\ge\; (2\mu\beta^2)^2\, 2^{6\alpha} \prod_{j=1}^{d} \Big(1 + 2(1+2^{-3\alpha})^{1/2}\zeta(\alpha)\gamma_j\Big)^2, \tag{14}$$

then $S_{n,d}(\boldsymbol{z}) \le 1/(2\mu\beta^2)$, and we conclude from Lemmas 3 and 4 that

$$e_{n,d}(A_n) \;\le\; \frac{2(1+\mu\beta)\mu\beta}{\delta\, n^{1/(2\lambda)}} \prod_{j=1}^{d} \Big(1 + 2(1+\delta^\lambda)^{1/2}\zeta(\alpha\lambda)\gamma_j^\lambda\Big)^{1/\lambda} \big(1 + 2\zeta(\alpha)\gamma_j\big)^{1/2}$$

for all $\lambda \in (1/\alpha, 1]$ and $\delta \in (0, 2^{-3\alpha}]$. Taking $\lambda = 1/(\alpha - 2\delta)$ with $\delta \le \min(2^{-3\alpha}, (\alpha-1)/2))$, we see that $e_{n,d}(A_n) = O(n^{-\alpha/2+\delta})$. Comparing this with the first lower bound in Lemma 2, we see that this is the *optimal* rate of convergence.

Using the property

$$\prod_{j=1}^{d}(1+x_j) \;=\; \exp\left(\sum_{j=1}^{d}\log(1+x_j)\right) \;\le\; \exp\left(\sum_{j=1}^{d} x_j\right) \;=\; (d+1)^{\sum_{j=1}^{d} x_j/\log(d+1)} \tag{15}$$

for all $x_j > 0$, we see that the requirement (14) on n does not grow with d if (4) holds, and it grows only polynomially with d when (5) holds. The conditions (4) and/or (5) are also sufficient to ensure that $e_{n,d}(A_n)$ does not grow faster than polynomially with d. However, we will need to assume stronger conditions on the weights if we want to have the optimal rate of convergence at the same time.

Theorem 5 *Suppose n is a prime number satisfying* (14)*. Then the generating vector $\boldsymbol{z}^*$ constructed by Algorithm 1 achieves the optimal rate of conver-*

gence, with

$$e_{n,d}(A_n) \,\le\, C_{d,\delta}\, n^{-\alpha/2+\delta} \qquad \textit{and} \qquad \frac{e_{n,d}(A_n)}{e_{0,d}} \,\le\, \widetilde{C}_{d,\delta}\, n^{-\alpha/2+\delta},$$

for all $\delta \in (0, \min(2^{-3\alpha}, (\alpha-1)/2)]$*, where* $C_{d,\delta}$ *and* $\widetilde{C}_{d,\delta}$ *are independent of* n*, but depend on* δ *and* d*. Additionally, if*

$$\sum_{j=1}^{\infty} \gamma_j^{1/(\alpha-2\delta)} \,<\, \infty,$$

then the numbers $C_{d,\delta}$ *and* $\widetilde{C}_{d,\delta}$*, and the requirement* (14) *on* n*, can be bounded independently of* d*.*

To implement Algorithm 1, we need a computable expression for $S_{n,d}(\boldsymbol{z})$. We can write

$$\begin{aligned} S_{n,d}^2(\boldsymbol{z}) \,&=\, -\prod_{j=1}^{d}\left(1+2\zeta(2\alpha)\gamma_j^2\right) + \frac{1}{n}\prod_{j=1}^{d}\left(1+2\zeta(\alpha)\gamma_j\right)^2 \\ &\quad + \frac{1}{n}\sum_{k=1}^{n-1}\prod_{j=1}^{d}\left(1+\gamma_j \sum_{h\in\mathbb{Z}\setminus\{0\}} \frac{e^{2\pi \mathrm{i} khz_j/n}}{|h|^{\alpha}}\right)^2. \end{aligned} \tag{16}$$

This expression is very similar to the squared worst case integration error (see for example [16]). If α is an even integer, then the inner sum over h can be computed via

$$\sum_{h\in\mathbb{Z}\setminus\{0\}} \frac{e^{2\pi \mathrm{i} khz_j/n}}{|h|^{\alpha}} \,=\, \frac{(2\pi)^{\alpha}}{(-1)^{\alpha/2+1}\alpha!} B_{\alpha}\left(\left\{\tfrac{kz_j}{n}\right\}\right),$$

where B_α is the Bernoulli polynomial of degree α.

Following [14] and using the Fast Fourier Transform, the component-by-component construction based on the quantity $S_{n,d}(\boldsymbol{z})$ requires only $O(n \log n\, d)$ operations. In other words, the computational cost is no worse than that for the integration problem.

3.5 Tractability

First we analyze tractability in the absolute sense. For $\varepsilon \in (0,1)$, we want to find the smallest n for which $e_{n,d}(A_n) \le \varepsilon$. From Lemma 3 we see that it is sufficient to insist that

$$S_{n,d}(\boldsymbol{z}) \,\le\, \frac{1}{\varepsilon^{-1}(1+\mu\beta)\mu\beta\, \prod_{j=1}^{d}(1+2\zeta(\alpha)\gamma_j)^{1/2} + \mu\beta^2}, \tag{17}$$

the right-hand side of which is less than $1/(\mu\beta^2)$. Using Lemma 4, we see that Algorithm 1 will generate a vector $\boldsymbol{z}$ satisfying (17) if we demand that

$$n \;\ge\; \mathrm{pr}\left(\min_{\substack{\lambda\in(1/\alpha,1]\\ \delta\in(0,2^{-3\alpha}]}} \left[\frac{1}{\delta^{2\lambda}} \prod_{j=1}^{d} \left(1+2(1+\delta^\lambda)^{1/2}\zeta(\alpha\lambda)\gamma_j^\lambda\right)^2 \right.\right.$$
$$\left.\left. \times \left(\varepsilon^{-1}(1+\mu\beta)\mu\beta \prod_{j=1}^{d}(1+2\zeta(\alpha)\gamma_j)^{1/2} + \mu\beta^2\right)^{2\lambda}\right]\right), \quad (18)$$

where $\mathrm{pr}(x)$ denotes the smallest prime number greater than or equal to x. Hence we conclude that

$n^{\mathrm{abs}}(\varepsilon, d)$ is less than or equal to the right-hand side of (18).

Note that $\mathrm{pr}(x) \le 2x$, since there is a prime number in the interval $[k, 2k]$ for any positive integer k. (This is known as "Bertrand's postulate", proved by Chebyshev in 1850.)

On the other hand, the second lower bound in Lemma 2 implies

$$n^{\mathrm{abs}}(\varepsilon, d) \;\ge\; \frac{1}{1+\varepsilon^2(1/c-1)^2} \prod_{j=1}^{d} (1+2\zeta(\alpha)\omega_\alpha\gamma_j)\,.$$

Similarly, for tractability in the normalized sense we obtain

$$n^{\mathrm{nor}}(\varepsilon, d) \;\le\; \mathrm{pr}\left(\min_{\substack{\lambda\in(1/\alpha,1]\\ \delta\in(0,2^{-3\alpha}]}} \left[\frac{1}{\delta^{2\lambda}} \prod_{j=1}^{d} \left(1+2(1+\delta^\lambda)^{1/2}\zeta(\alpha\lambda)\gamma_j^\lambda\right)^2 \right.\right.$$
$$\left.\left. \times \left(\varepsilon^{-1}(1+\mu\beta)\mu\beta + \mu\beta^2\right)^{2\lambda}\right]\right).$$

However, we were unable to derive a lower bound on $n^{\mathrm{nor}}(\varepsilon, d)$ because our lower bound on $e_{n,d}(A_n)$ was too weak compared to the initial error $e_{0,d}$.

Using again (15) and the additional property that $\log(1+x) \ge \log(1+x^*)x/x^*$ for all $x \le x^*$, we arrive at the following theorem.

Theorem 6 *Consider the Fredholm integral equation problem defined as in Section 2.*

(a) *The problem is strongly QMC-Nyström tractable in the absolute sense iff*

$$\sum_{j=1}^{\infty} \gamma_j \;<\; \infty, \quad (4)$$

and it is QMC-Nyström tractable in the absolute sense iff

$$L \;:=\; \limsup_{d\to\infty} \frac{\sum_{j=1}^d \gamma_j}{\log(d+1)} \;<\; \infty. \tag{5}$$

These conditions are also sufficient for strong QMC-Nyström tractability and QMC-Nyström tractability in the normalized sense.

(b) *If* (4) *holds and additionally* $\sum_{j=1}^\infty \gamma_j^\lambda < \infty$ *for some* $\lambda \in (1/\alpha, 1]$*, then*

$$n^{\rm abs}(\varepsilon,d) \;=\; O(\varepsilon^{-2\lambda}) \qquad \textit{and} \qquad n^{\rm nor}(\varepsilon,d) \;=\; O(\varepsilon^{-2\lambda}),$$

with the implied factors independent of ε *and* d*.*

(c) *If* (4) *does not hold but* (5) *holds, then*

$$n^{\rm abs}(\varepsilon,d) \;=\; O(\varepsilon^{-2}\, d^{\,q_1}) \qquad \textit{and} \qquad n^{\rm nor}(\varepsilon,d) \;=\; O(\varepsilon^{-2}\, d^{\,q_2}),$$

with the implied factors independent of ε *and* d*, and with* q_1 *and* q_2 *arbitrarily close to*

$$6\,\zeta(\alpha)\,L \qquad \textit{and} \qquad 4\,\zeta(\alpha)\,L,$$

respectively.

Note that Part (b) is obtained by taking any δ, say, $\delta = 2^{-3\alpha}$, and Part (c) is obtained with $\lambda = 1$ and with δ approaching 0.

4 Additional remarks

4.1 Generating vectors constructed for integration

Since the optimal rate of convergence $O(n^{-\alpha/2+\delta})$, $\delta > 0$, for the integral equation problem is the same as that for the integration problem, a natural question to ask is: can we use the generating vector already constructed for the integration problem? We came up with two approaches for estimating the resulting worst case error, but both with some undesirable effects. These are discussed below.

Since $(K-K_n)f(\boldsymbol{x})$ is essentially the integration error of the function $F_{\boldsymbol{x}}(\boldsymbol{y}) := k(\boldsymbol{x}-\boldsymbol{y})f(\boldsymbol{y})$, we can write

$$\|K-K_n\|_{\rm sup} \;\le\; \sup_{\boldsymbol{x}\in D} \|F_{\boldsymbol{x}}\|_H \, e^{\rm wor-int}_{n,d}(\boldsymbol{z}),$$

where $e^{\rm wor-int}_{n,d}(\boldsymbol{z})$ denotes the worst case integration error for a lattice rule with generating vector $\boldsymbol{z}$. We know that $\boldsymbol{z}$ can be constructed to achieve the optimal rate of convergence. However, from [13] (Appendix 2: Korobov spaces are algebras) we see that

$$\|F_{\boldsymbol{x}}\|_H \;\le\; 2^{\,d\max(1,\alpha/2)} \prod_{j=1}^{d} \left(1+2\zeta(\alpha)\gamma_j\right)^{1/2} \|k\|_H \|f\|_H .$$

This exponential dependence on d means that tractability is out of the question.

Alternatively, we can estimate the expression (12) as follows

$$\begin{aligned}
&\sum_{\substack{\boldsymbol{h}\in\mathbb{Z}^d\setminus\{\boldsymbol{0}\}\\ \boldsymbol{h}\cdot\boldsymbol{z}\equiv 0\,(\bmod\ n)}} \sum_{\boldsymbol{\ell}\in\mathbb{Z}^d} |\hat{k}(\boldsymbol{\ell})||\hat{f}(\boldsymbol{h}+\boldsymbol{\ell})| \\
&\le \sum_{\substack{\boldsymbol{h}\in\mathbb{Z}^d\setminus\{\boldsymbol{0}\}\\ \boldsymbol{h}\cdot\boldsymbol{z}\equiv 0\,(\bmod\ n)}} \left(\sum_{\boldsymbol{\ell}\in\mathbb{Z}^d} |\hat{k}(\boldsymbol{\ell})|^2 r_\alpha(\boldsymbol{\gamma},\boldsymbol{\ell})\right)^{1/2} \left(\sum_{\boldsymbol{\ell}\in\mathbb{Z}^d} \frac{|\hat{f}(\boldsymbol{h}+\boldsymbol{\ell})|^2}{r_\alpha(\boldsymbol{\gamma},\boldsymbol{\ell})}\right)^{1/2} \\
&\le \|k\|_H \sum_{\substack{\boldsymbol{h}\in\mathbb{Z}^d\setminus\{\boldsymbol{0}\}\\ \boldsymbol{h}\cdot\boldsymbol{z}\equiv 0\,(\bmod\ n)}} \left[\left(\sum_{\boldsymbol{\ell}\in\mathbb{Z}^d} |\hat{f}(\boldsymbol{h}+\boldsymbol{\ell})|^2 r_\alpha(\boldsymbol{\gamma},\boldsymbol{h}+\boldsymbol{\ell})\right)^{1/2} \right. \\
&\qquad\qquad\qquad \left. \times \left(\max_{\boldsymbol{\ell}\in\mathbb{Z}^d} \frac{1}{\sqrt{r_\alpha(\boldsymbol{\gamma},\boldsymbol{\ell})\, r_\alpha(\boldsymbol{\gamma},\boldsymbol{h}+\boldsymbol{\ell})}}\right)\right] \\
&\le \|f\|_H \|k\|_H \prod_{j=1}^{d} \max(1,2^\alpha\gamma_j)^{1/2} \sum_{\substack{\boldsymbol{h}\in\mathbb{Z}^d\setminus\{\boldsymbol{0}\}\\ \boldsymbol{h}\cdot\boldsymbol{z}\equiv 0\,(\bmod\ n)}} \frac{1}{r_{\alpha/2}(\boldsymbol{\gamma}^{1/2},\boldsymbol{h})},
\end{aligned} \tag{19}$$

where in the final step we made use of the estimate obtained in [13],

$$r_\alpha(\boldsymbol{\gamma},\boldsymbol{\ell})\, r_\alpha(\boldsymbol{\gamma},\boldsymbol{h}+\boldsymbol{\ell}) \;\ge\; \frac{r_\alpha(\boldsymbol{\gamma},\boldsymbol{h})}{\prod_{j=1}^{d}\max(1,2^\alpha\gamma_j)} \qquad \forall\,\boldsymbol{\ell}\in\mathbb{Z}^d.$$

Observe that the sum in (19) is exactly the squared worst case integration error of a lattice rule in the weighted Korobov space with α replaced by $\alpha/2$ and γ_j replaced by $\gamma_j^{1/2}$. Thus we know that a generating vector can be constructed such that this sum is of order $O(n^{-\alpha/2+\delta})$, $\delta>0$. In other words, the rate of convergence is right, and the dependence on d can be controlled by the weights, but we would require $\alpha>2$ to begin with.

In [11] and [12], functions from weighted Korobov spaces were approximated by truncated Fourier series, with vectors $\boldsymbol{h}$ from the set

$$\mathcal{A}(d,M) := \{\boldsymbol{h} \in \mathbb{Z}^d : r_\alpha(\boldsymbol{\gamma},\boldsymbol{h}) \leq M\}.$$

Since $M/r_\alpha(\boldsymbol{\gamma},\boldsymbol{h}) \geq 1$ for all $\boldsymbol{h} \in \mathcal{A}(d,M)$, the quantity $E_{n,d}(\boldsymbol{z})$ studied in [11] and [12] can be bounded above by

$$\begin{aligned}
E_{n,d}(\boldsymbol{z}) &:= \sum_{\boldsymbol{h}\in\mathcal{A}(d,M)} \sum_{\substack{\boldsymbol{\ell}\in\mathbb{Z}^d\setminus\{\boldsymbol{0}\}\\ \boldsymbol{\ell}\cdot\boldsymbol{z}\equiv 0 \pmod n}} \frac{1}{r_\alpha(\boldsymbol{\gamma},\boldsymbol{h}+\boldsymbol{\ell})} \\
&\leq \sum_{\boldsymbol{h}\in\mathcal{A}(d,M)} \frac{M}{r_\alpha(\boldsymbol{\gamma},\boldsymbol{h})} \sum_{\substack{\boldsymbol{\ell}\in\mathbb{Z}^d\setminus\{\boldsymbol{0}\}\\ \boldsymbol{\ell}\cdot\boldsymbol{z}\equiv 0 \pmod n}} \frac{1}{r_\alpha(\boldsymbol{\gamma},\boldsymbol{h}+\boldsymbol{\ell})} \\
&\leq M \sum_{\boldsymbol{h}\in\mathbb{Z}^d} \frac{1}{r_\alpha(\boldsymbol{\gamma},\boldsymbol{h})} \sum_{\substack{\boldsymbol{\ell}\in\mathbb{Z}^d\setminus\{\boldsymbol{0}\}\\ \boldsymbol{\ell}\cdot\boldsymbol{z}\equiv 0 \pmod n}} \frac{1}{r_\alpha(\boldsymbol{\gamma},\boldsymbol{h}+\boldsymbol{\ell})} = M\, S_{n,d}^2(\boldsymbol{z}).
\end{aligned}$$

Note that $S_{n,d}^2(\boldsymbol{z})$ is much easier to work with than $E_{n,d}(\boldsymbol{z})$, because it is given explicitly by (16), and there is no need to analyze the set $\mathcal{A}(d,M)$. The component-by-component construction is independent of M, and the computational cost is much cheaper.

Furthermore, the vectors obtained by minimizing $S_{n,d}^2(\boldsymbol{z})$ lead to the same n-dependence in the worst case and average case approximation error bounds as those obtained by minimizing $E_{n,d}(\boldsymbol{z})$. Hence this new quantity should be used not only for the integral equation problem, but also for the approximation problem discussed in [11] and [12].

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Appendix: proof of Lemma 4

We prove the result by induction on d, following closely the argument used in the proof of Lemma 6 in [11]. It can easily be checked that the result holds for $d = 1$.

Suppose the result has been shown for d. By separating the $h_{d+1} = 0$ and $h_{d+1} \neq 0$ terms in (13), we can write

$$S^2_{n,d+1}(\boldsymbol{z}, z_{d+1}) \;=\; \left(1 + 2\zeta(2\alpha)\gamma^2_{d+1}\right) S^2_{n,d}(\boldsymbol{z}) \;+\; \theta(\boldsymbol{z}, z_{d+1}),$$

where

$$\theta(\boldsymbol{z}, z_{d+1}) \;=\; \sum_{\ell_{d+1}\in\mathbb{Z}} \sum_{\substack{h_{d+1}=-\infty \\ h_{d+1}\neq 0}}^{\infty} \Bigg[\frac{1}{r_\alpha(\gamma_{d+1}, \ell_{d+1})} \frac{1}{r_\alpha(\gamma_{d+1}, \ell_{d+1} + h_{d+1})} \times \sum_{\boldsymbol{\ell}\in\mathbb{Z}^d} \sum_{\substack{\boldsymbol{h}\in\mathbb{Z}^d \\ \boldsymbol{h}\cdot\boldsymbol{z}\equiv -h_{d+1}z_{d+1} \pmod{n}}} \frac{1}{r_\alpha(\boldsymbol{\gamma}, \boldsymbol{\ell})} \frac{1}{r_\alpha(\boldsymbol{\gamma}, \boldsymbol{\ell} + \boldsymbol{h})} \Bigg].$$

A similar expression already appeared in the proof of Lemma 6 in [11] (but with the role of $\boldsymbol{\ell}$ and $\boldsymbol{h}$ interchanged). For $\lambda \in (1/\alpha, 1]$, we follow closely the argument used in [11], including the use of Jensen's inequality, to arrive at

$$\left[\theta(\boldsymbol{z}, z_{d+1}^*)\right]^\lambda \;\le\; \frac{1}{n-1}\sum_{z_{d+1}=1}^{n-1}[\theta(\boldsymbol{z}, z_{d+1})]^\lambda \;\le\; \Theta(\boldsymbol{z}),$$

with

$$\begin{aligned}\Theta(\boldsymbol{z}) \;&=\; \frac{\overline{G}-G}{n-1}\sum_{l_{d+1}\in\mathbb{Z}}\sum_{\substack{h_{d+1}=-\infty\\ h_{d+1}\neq 0}}^{\infty}\frac{1}{r_{\alpha\lambda}(\gamma_{d+1}^\lambda, l_{d+1})}\frac{1}{r_{\alpha\lambda}(\gamma_{d+1}^\lambda, l_{d+1}+h_{d+1})}\\ &\quad+\frac{nG-\overline{G}}{n-1}\sum_{l_{d+1}\in\mathbb{Z}}\sum_{\substack{h_{d+1}=-\infty\\ h_{d+1}\equiv 0\,(\mathrm{mod}\,n)\\ h_{d+1}\neq 0}}^{\infty}\frac{1}{r_{\alpha\lambda}(\gamma_{d+1}^\lambda, l_{d+1})}\frac{1}{r_{\alpha\lambda}(\gamma_{d+1}^\lambda, l_{d+1}+h_{d+1})},\end{aligned}$$

where

$$G \;:=\; \sum_{\boldsymbol{\ell}\in\mathbb{Z}^d}\sum_{\substack{\boldsymbol{h}\in\mathbb{Z}^d\\ \boldsymbol{h}\cdot\boldsymbol{z}\equiv 0\,(\mathrm{mod}\,n)}}\frac{1}{r_{\alpha\lambda}(\boldsymbol{\gamma}^\lambda, \boldsymbol{\ell})}\frac{1}{r_{\alpha\lambda}(\boldsymbol{\gamma}^\lambda, \boldsymbol{\ell}+\boldsymbol{h})} \;\le\; \prod_{j=1}^{d}\left(1+2\zeta(\alpha\lambda)\gamma_j^\lambda\right)^2 \;=:\; \overline{G}.$$

We have

$$\begin{aligned}W_1 \;:=\; &\sum_{l_{d+1}\in\mathbb{Z}}\sum_{\substack{h_{d+1}\in\mathbb{Z}\\ h_{d+1}\neq 0}}\frac{1}{r_{\alpha\lambda}(\gamma_{d+1}^\lambda, l_{d+1})}\frac{1}{r_{\alpha\lambda}(\gamma_{d+1}^\lambda, l_{d+1}+h_{d+1})}\\ &=\left(1+2\zeta(\alpha\lambda)\gamma_{d+1}^\lambda\right)^2-\left(1+2\zeta(2\alpha\lambda)\gamma_{d+1}^{2\lambda}\right)\\ &\le 2^2\zeta(\alpha\lambda)\gamma_{d+1}^\lambda+2^2[\zeta(\alpha\lambda)]^2\gamma_{d+1}^{2\lambda},\end{aligned}$$

$$\begin{aligned}W_2 \;:=\; &\sum_{\substack{l_{d+1}\in\mathbb{Z}\\ l_{d+1}\equiv 0\,(\mathrm{mod}\,n)}}\sum_{\substack{h_{d+1}\in\mathbb{Z}\\ h_{d+1}\equiv 0\,(\mathrm{mod}\,n)\\ h_{d+1}\neq 0}}\frac{1}{r_{\alpha\lambda}(\gamma_{d+1}^\lambda, l_{d+1})}\frac{1}{r_{\alpha\lambda}(\gamma_{d+1}^\lambda, l_{d+1}+h_{d+1})}\\ &=\left(1+\frac{2\zeta(\alpha\lambda)\gamma_{d+1}^\lambda}{n^{\alpha\lambda}}\right)^2-\left(1+\frac{2\zeta(2\alpha\lambda)\gamma_{d+1}^{2\lambda}}{n^{2\alpha\lambda}}\right)\\ &\le\frac{2^2\zeta(\alpha\lambda)\gamma_{d+1}^\lambda}{n}+\frac{2^2[\zeta(\alpha\lambda)]^2\gamma_{d+1}^{2\lambda}}{n},\end{aligned}$$

and

$$
\begin{aligned}
W_3 &:= \sum_{\substack{l_{d+1}\in\mathbb{Z}\\ l_{d+1}\not\equiv 0 \,(\mathrm{mod}\; n)}} \sum_{\substack{h_{d+1}\in\mathbb{Z}\\ h_{d+1}\equiv 0 \,(\mathrm{mod}\; n)\\ h_{d+1}\neq 0}} \frac{1}{r_{\alpha\lambda}(\gamma_{d+1}^{\lambda}, l_{d+1})} \frac{1}{r_{\alpha\lambda}(\gamma_{d+1}^{\lambda}, l_{d+1}+h_{d+1})} \\
&= \gamma_{d+1}^{2\lambda} \sum_{\substack{k=-(n-1)/2\\ k\neq 0}}^{(n-1)/2} \left[\left(\sum_{l\in\mathbb{Z}} \frac{1}{|ln+k|^{\alpha\lambda}} \right)^2 - \sum_{l\in\mathbb{Z}} \frac{1}{|ln+k|^{2\alpha\lambda}} \right] \\
&\le \gamma_{d+1}^{2\lambda} \sum_{\substack{k=-(n-1)/2\\ k\neq 0}}^{(n-1)/2} \left[\left(\frac{1}{|k|^{\alpha\lambda}} + \sum_{\substack{l\in\mathbb{Z}\\ l\neq 0}} \frac{1}{|ln|^{\alpha\lambda} \left|1+\frac{k}{ln}\right|^{\alpha\lambda}} \right)^2 - \frac{1}{|k|^{2\alpha\lambda}} \right] \\
&\le \gamma_{d+1}^{2\lambda} \sum_{\substack{k=-(n-1)/2\\ k\neq 0}}^{(n-1)/2} \left(\frac{1}{|k|^{\alpha\lambda}} \frac{2^{\alpha\lambda+2}\zeta(\alpha\lambda)}{n^{\alpha\lambda}} + \frac{2^{2\alpha\lambda+2}[\zeta(\alpha\lambda)]^2}{n^{2\alpha\lambda}} \right) \\
&\le \frac{2^{\alpha\lambda+4}[\zeta(\alpha\lambda)]^2 \gamma_{d+1}^{2\lambda}}{n}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\Theta(\boldsymbol{z}) &= \frac{\overline{G}-G}{n-1} W_1 + \frac{nG-\overline{G}}{n-1}(W_2+W_3) \\
&\le \frac{2^2\zeta(\alpha\lambda)\gamma_{d+1}^{\lambda} + 2^2[\zeta(\alpha\lambda)]^2\gamma_{d+1}^{2\lambda}}{n-1}\, \overline{G}\left(1-\frac{1}{n}\right) + \frac{nG-\overline{G}}{n-1} \frac{2^{\alpha\lambda+4}[\zeta(\alpha\lambda)]^2\gamma_{d+1}^{2\lambda}}{n} \\
&\le \left(2^2\zeta(\alpha\lambda)\gamma_{d+1}^{\lambda} + 2^2(1+2^{\alpha\lambda+2})[\zeta(\alpha\lambda)]^2\gamma_{d+1}^{2\lambda} \right) \frac{1}{n} \prod_{j=1}^{d} \left(1+2\zeta(\alpha\lambda)\gamma_j^{\lambda}\right)^2.
\end{aligned}
$$

Combining all the estimates together and making use of the induction hypothesis, the desired result then follows from

$$
\begin{aligned}
&\left(1+2\zeta(2\alpha)\gamma_{d+1}^{2}\right) + \delta\left(2^2\zeta(\alpha\lambda)\gamma_{d+1}^{\lambda} + 2^2(1+2^{\alpha\lambda+2})[\zeta(\alpha\lambda)]^2\gamma_{d+1}^{2\lambda}\right)^{1/\lambda} \\
&\le \left(1+2\zeta(\alpha\lambda)\gamma_{d+1}^{\lambda} + 2^2\delta^{\lambda}\zeta(\alpha\lambda)\gamma_{d+1}^{\lambda} + 2^2\delta^{\lambda}(1+2^{\alpha\lambda+2})[\zeta(\alpha\lambda)]^2\gamma_{d+1}^{2\lambda}\right)^{1/\lambda} \\
&\le \left(1+2^2\zeta(\alpha\lambda)\gamma_{d+1}^{\lambda} + 2^2(1+\delta^{\lambda})[\zeta(\alpha\lambda)]^2\gamma_{d+1}^{2\lambda}\right)^{1/\lambda} \\
&\le \left(1+2(1+\delta^{\lambda})^{1/2}\zeta(\alpha\lambda)\gamma_{d+1}^{\lambda}\right)^{2/\lambda},
\end{aligned}
$$

where we have used Jensen's inequality, $\delta^{\lambda} \le 1/2$ and $\delta^{\lambda}2^{\alpha\lambda+2} \le 1$ (since $\delta \le 2^{-3\alpha}$). This completes the proof of Lemma 4.