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THE b-ADIC DIAPHONY OF DIGITAL $(\mathbf{T}, s)$-SEQUENCES

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ABSTRACT. The b-adic diaphony is a quantitative measure for the irregularity of distribution of a sequence in the unit cube. In this article we give an upper bound on the b-adic diaphony of digital $(\mathbf{T}, s)$-sequences over $\mathbb{Z}_b$. And we derive a condition on the quality function $\mathbf{T}$ such that the b-adic diaphony of the digital $(\mathbf{T}, s)$-sequence over $\mathbb{Z}_b$ is of order $\mathcal{O}((\log N)^{s/2}N^{-1})$. We also give a metrical result; for μ_s-almost all generators of a digital $(\mathbf{T}, s)$-sequence over $\mathbb{Z}_b$ the b-adic diaphony of the sequence is of order $\mathcal{O}((\log\log N)^2(\log N)^{3s/2}N^{-1})$.

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1. Introduction

The b-adic diaphony is a quantitative measure for the irregularity of distribution of a sequence in the s-dimensional unit cube. This notion was introduced by Hellekalek and Leeb [6] for $b = 2$ and later generalized by Grozdanov and Stoilova [5] for general integers $b \geq 2$. We recall now the definition of b-adic Walsh functions, which will be needed for the definition of the b-adic diaphony.

Let $b \geq 2$ be an integer. For a nonnegative integer k with base b representation $k = \kappa_{a-1}b^{a-1} + \cdots + \kappa_1 b + \kappa_0$, with $\kappa_i \in \{0, \ldots, b-1\}$ and $\kappa_{a-1} \neq 0$, we define the Walsh function ${}_b\mathrm{wal}_k : [0, 1) \to \mathbb{C}$ by

$${}_b\mathrm{wal}_k(x) := \mathrm{e}^{2\pi i(x_1\kappa_0+\cdots+x_a\kappa_{a-1})/b},$$

for $x \in [0, 1)$ with base b representation $x = \frac{x_1}{b} + \frac{x_2}{b^2} + \cdots$ (unique in the sense that infinitely many of the x_i must be different from $b-1$).

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For higher dimensions $s \geq 1$, $\mathbf{k} = (k_1, \ldots, k_s) \in \mathbb{N}_0^s$ and $\mathbf{x} = (x_1, \ldots, x_s) \in [0,1)^s$ we write

$$_b\mathrm{wal}_{\mathbf{k}}(\mathbf{x}) := \prod_{j=1}^{s} {}_b\mathrm{wal}_{k_j}(x_j).$$

Now we are ready to define the b-adic diaphony (see [5] or [6]).

Definition 1. Let $b \geq 2$ be an integer. The b-adic diaphony of the first N elements of a sequence $\omega = (\mathbf{x}_n)_{n \geq 0}$ in $[0,1)^s$ is defined by

$$F_{b,N}(\omega) := \left(\frac{1}{(1+b)^s - 1} \sum_{\substack{\mathbf{k} \in \mathbb{N}_0^s \\ \mathbf{k} \neq \mathbf{0}}} r_b(\mathbf{k}) \left| \frac{1}{N} \sum_{n=0}^{N-1} {}_b\mathrm{wal}_{\mathbf{k}}(\mathbf{x}_n) \right|^2 \right)^{1/2},$$

where for $\mathbf{k} = (k_1, \ldots, k_s) \in \mathbb{N}_0^s$, $r_b(\mathbf{k}) := \prod_{j=1}^{s} r_b(k_j)$ and for $k \in \mathbb{N}_0$,

$$r_b(k) := \begin{cases} 1 & \text{if } k = 0 \\ b^{-2a} & \text{if } b^a \leq k < b^{a+1} \text{ where } a \in \mathbb{N}_0. \end{cases}$$

Throughout this article we will write $a(k) = a$, if a is the unique determined integer such that $b^a \leq k < b^{a+1}$. If $b = 2$ we also speak of dyadic diaphony.

The b-adic diaphony is a quantitative measure for the irregularity of distribution of a sequence: a sequence ω in the s-dimensional unit cube is uniformly distributed modulo one if and only if $\lim_{N\to\infty} F_{b,N}(\omega) = 0$. This was shown in [6] for the case $b = 2$ and in [5] for the general case. Further it is shown in [1] that the b-adic diaphony is – up to a factor depending on b and s – the worst case error for quasi-Monte Carlo integration of functions from a certain Hilbert space $H_{\mathrm{wal},s,\gamma}$, which has been introduced in [2].

In the following let b be a prime, i.e. we can always take $\mathbb{Z}_b$ for the finite field of prime order b. We consider the b-adic diaphony of digital $(\mathbf{T}, s)$-sequences over $\mathbb{Z}_b$. Here s is the dimension of the sequence and $\mathbf{T} : \mathbb{N}_0 \to \mathbb{N}_0$ is the quality function of the sequence; lower quality functions imply stronger equidistribution properties. A special class among these functions are the digital (t, s)-sequences over $\mathbb{Z}_b$, where the quality function $\mathbf{T}$ is a constant t. Digital (t, s)-sequences were introduced by Niederreiter [8, 9]. The concept of $(\mathbf{T}, s)$-sequences was introduced by Larcher and Niederreiter in [7], as a quality function $\mathbf{T}$ is a more sensitive measure than a quality parameter t. For more information on $(\mathbf{T}, s)$-sequences see [3, Chapter 4].

In [4] the author showed a formula for the b-adic diaphony of digital $(0, s)$-sequences over $\mathbb{Z}_b$, $s = 1, \ldots, b$, and an upper bound for the b-adic diaphony of digital (t, s)-sequences over $\mathbb{Z}_b$ for primes b. In both cases we obtained for the asymptotic order

$$F_{b,N}(\omega) = \mathcal{O}\left(\frac{(\log N)^{s/2}}{N}\right) \qquad (\text{as } N \to \infty). \tag{1}$$

In this article we would like now to find in analogy to [4] an upper bound on the b-adic diaphony of digital $(\mathbf{T}, s)$-sequences over $\mathbb{Z}_b$ and give a condition on the quality function $\mathbf{T}$, so that we obtain the same order as in (1). With a weaker condition on $\mathbf{T}$ we still obtain the asymptotic order

$$(NF_{b,N}(\omega))^2 = \mathcal{O}\left(\sum_{u=1}^{\lceil \log_b N \rceil} u^{s-1} b^{2\mathbf{T}(u)}\right) \qquad (\text{as } N \to \infty).$$

Now we give a definition of digital $(\mathbf{T}, s)$-sequences over $\mathbb{Z}_b$. The quality function $\mathbf{T}$ is closely related to a quantity ρ_m, which in some sense "measures" the "linear independence" of s infinite matrices $C_1, \ldots, C_s$ (see [3, Chapter 4.4]). Let $C_1, \ldots, C_s$ be $\mathbb{N} \times \mathbb{N}$ matrices over the finite field $\mathbb{Z}_b$. For any integers $1 \leq i \leq s$ and $m \geq 1$ by $C_i^{(m)}$ we denote the left upper $m \times m$ sub-matrix of C_i. Then

$$\rho_m = \rho_m(C_1, \ldots, C_s) := \rho(C_1^{(m)}, \ldots, C_s^{(m)}),$$

where ρ is the independence parameter defined for s-tuples of $m \times m$ matrices over $\mathbb{Z}_b$, i.e. ρ is the largest integer such that for any choice $d_1, \ldots, d_s \in \mathbb{N}_0$ with $d_1 + \cdots + d_s = \rho$, the following holds:

the first d_1 row vectors of $C_1^{(m)}$ together with

the first d_2 row vectors of $C_2^{(m)}$ together with

$\vdots$

the first d_s row vectors of $C_s^{(m)}$

are linearly independent over the finite field $\mathbb{Z}_b$.

Definition 2. For $n \geq 0$ let $n = n_0 + n_1 b + n_2 b^2 + \cdots$ be the base b representation of n. For $j \in \{1, \ldots, s\}$ multiply the vector $\mathbf{n} = (n_0, n_1, \ldots)^\top$ by the matrix C_j,

$$C_j \cdot \mathbf{n} =: (x_n^j(1), x_n^j(2), \ldots)^\top \in \mathbb{Z}_b^\infty,$$

and set

$$x_n^{(j)} := \frac{x_n^j(1)}{b} + \frac{x_n^j(2)}{b^2} + \cdots .$$

Finally set $\mathbf{x}_n := (x_n^{(1)}, \ldots, x_n^{(s)})$.

The digital sequence ω constructed this way by the $\mathbb{N}\times\mathbb{N}$ matrices $C_1,\dots,C_s$ over $\mathbb{Z}_b$ is a strict $(\mathbf{T},s)$-sequence in base b with $\mathbf{T}(m)=m-\rho_m$ for all $m\in\mathbb{N}$. The matrices $C_1,\dots,C_s$ are called the generator matrices of the sequence.

Remark 1. (1) Any strict digital $(\mathbf{T},s)$-sequence over $\mathbb{Z}_b$ is a digital $(\mathbf{U},s)$-sequence over $\mathbb{Z}_b$ for all $\mathbf{U}$ with $\mathbf{U}(m)\geq\mathbf{T}(m)$ for all m.

(2) The concept of (t,s)-sequences in base b is contained in the concept of $(\mathbf{T},s)$-sequences in base b. We just have to take for $\mathbf{T}$ the constant function $\mathbf{T}(m)=t$ for all m (resp. $\mathbf{T}(m)=m$ for $m\leq t$).

For more information on digital $(\mathbf{T},s)$-sequences we refer to [3].

Definition 3. Let ω be a uniformly distributed strict digital $(\mathbf{T},s)$-sequence in base b. For $r\in\mathbb{N}_0$ we set

$$\eta(r):=\min\{m:m-\mathbf{T}(m)\geq r\}.$$

This minimum exists for all r, because $\lim_{m\to\infty}m-\mathbf{T}(m)=\infty$ if ω is uniformly distributed modulo 1 (see [3, Theorem 4.32]).

We will need the following properties of the function η:

(1) η is non-decreasing.

(2) The condition $\eta(r)>u$ is equivalent to $u-\mathbf{T}(u)<r$.

These properties follow easily from the fact that $S(m):=m-\mathbf{T}(m)$ is non-decreasing (see [3, p.133]) and from the definition of η.

Finally we need the definition of the function ψ_b.

Definition 4. Let β be an integer in $\{1,\dots,b-1\}$. For $x\in\left[\frac{j}{b},\frac{j+1}{b}\right)$, $j\in\{0,\dots,b-1\}$ we set

$$\psi_b^\beta(x):=\frac{b^2(b^2-1)}{12}\left|\frac{1}{b}\frac{z_\beta^j-1}{z_\beta-1}+z_\beta^j\left(x-\frac{j}{b}\right)\right|^2,$$

where $z_\beta=\mathrm{e}^{\frac{2\pi i}{b}\beta}={}_b\mathrm{wal}_1\left(\frac{\beta}{b}\right)$; then the function is extended to the reals by periodicity. The function ψ_b is now defined as the mean of the functions ψ_b^β:

$$\psi_b(x):=\frac{1}{b-1}\sum_{\beta=1}^{b-1}\psi_b^\beta(x).$$

We will need two facts about ψ_b:

(1) The function ψ_b is bounded (see [4, Lemma 12]),

(2) $\psi_b(x) = \frac{b^2(b^2-1)}{12}x^2$ on the interval $\left[0, \frac{1}{b}\right)$ (see [4, Lemma 11(1)]).

For further properties of the function ψ_b we refer to [4, Lemma 11, Lemma 13].

2. Results

We show now an upper bound on the b-adic diaphony of digital $(\mathbf{T}, s)$-sequences. From this we derive (under certain conditions on the quality function $\mathbf{T}$) the asymptotic order of the b-adic diaphony of these sequences. We also give a metrical result. The proofs of the results below are given in Section 3.

Theorem 1. *Let ω be a digital $(\mathbf{T}, s)$-sequence over $\mathbb{Z}_b$. For any $N \geq 1$ we have*

$$(NF_{b,N}(\omega))^2$$
$$\leq \frac{1}{(b+1)^s - 1} \frac{12}{b^3(b+1)} \sum_{w=1}^{s} \binom{s}{w} \left(\frac{b^4}{b^2-1}\right)^w \sum_{u=1}^{\infty} \psi_b\left(\frac{N}{b^u}\right) b^u \sum_{v=u-1}^{\infty} \frac{v^{w-1}}{b^v} b^{2\mathbf{T}(v)}$$
$$\leq c \sum_{u=1}^{\infty} \psi_b\left(\frac{N}{b^u}\right) b^u \sum_{v=u-1}^{\infty} \frac{v^{s-1}}{b^v} b^{2\mathbf{T}(v)},$$

where c is a constant that does not depend on N.

From the above theorem we obtain now the asymptotic behaviour of certain digital $(\mathbf{T}, s)$-sequences over $\mathbb{Z}_b$.

Corollary 1. *Let ω be a digital $(\mathbf{T}, s)$-sequence over $\mathbb{Z}_b$ satisfying the property that*

$$\sum_{v=u-1}^{\infty} \frac{v^{s-1}}{b^v} b^{2\mathbf{T}(v)} \leq c_1 \frac{u^{s-1}}{b^u} b^{2\mathbf{T}(u)} \qquad \textit{for all } u \in \mathbb{N},$$

where c_1 is a constant that does not depend on u. Then for the b-adic diaphony of the first $N \geq 2$ elements of ω we have

$$(NF_{b,N}(\omega))^2 = \mathcal{O}\left(\sum_{u=1}^{\lceil \log_b N \rceil} u^{s-1} b^{2\mathbf{T}(u)}\right) \qquad (\textit{as } N \to \infty).$$

Remark 2. (1) For $u \in \mathbb{N}, s \in \mathbb{N}_0$ we have

$$\sum_{v=u-1}^{\infty} \frac{v^s}{b^v} \leq \left(2b \sum_{v=0}^{\infty} \frac{v^s}{b^v}\right) \frac{u^s}{b^u} \leq c_1 \frac{u^s}{b^u}.$$

(2) For $\mathbf{T}(v) = t$ the above condition is satisfied and we get immediately the asymptotic order of the b-adic diaphony of digital (t, s)-sequences over $\mathbb{Z}_b$

$$F_{b,N}(\omega) = \mathcal{O}\left(\frac{(\log N)^{s/2}}{N}\right) \qquad (\text{as } N \to \infty),$$

(see also [4, Corollary 10]).

(3) For $\mathbf{T}(m) \leq \max(C, \log_b \log m)$, $C \geq 0$, the above condition is satisfied and we get

$$F_{b,N}(\omega) = \mathcal{O}\left(\frac{(\log\log N)(\log N)^{s/2}}{N}\right) \qquad (\text{as } N \to \infty).$$

In the last example we already came close to the desired asymptotic order $\mathcal{O}((\log N)^{s/2}N^{-1})$. In the next corollary we give an additional condition on $\mathbf{T}$, which guarantees such an asymptotic behaviour.

Corollary 2. *Let ω be a digital $(\mathbf{T}, s)$-sequence over $\mathbb{Z}_b$ satisfying*

(1) $\sum_{v=u-1}^{\infty} \frac{v^{s-1}}{b^v} b^{2\mathbf{T}(v)} \leq c_1 \frac{u^{s-1}}{b^u} b^{2\mathbf{T}(u)}$ *for all* $u \in \mathbb{N}$,

(2) $\frac{1}{m} \sum_{u=1}^{m} b^{2\mathbf{T}(u)} \leq c_2$ *for all* $m \in \mathbb{N}$,

where the constants c_1, c_2 do not depend on u and m, respectively. Then we have

$$F_{b,N}(\omega) = \mathcal{O}\left(\frac{(\log N)^{s/2}}{N}\right) \qquad (\text{as } N \to \infty).$$

Now we are interested in the order of the b-adic diaphony of digital $(\mathbf{T}, s)$-sequences, when the quality function $\mathbf{T}$ does not necessarily fulfill the conditions from Corollary 2 or Corollary 1, i.e. what order we can get for almost all digital $(\mathbf{T}, s)$-sequences. In the following we explain what we mean by "almost all".

Let $\mathcal{M}_s$ denote the set of all s-tuples of $\mathbb{N} \times \mathbb{N}$ matrices over $\mathbb{Z}_b$. We define the probability measure μ_s on $\mathcal{M}_s$ as the product measure induced by a certain probability measure μ on the set $\mathcal{M}$ of all infinite matrices over $\mathbb{Z}_b$. We can view $\mathcal{M}$ as the product of denumerable many copies of the sequence space $\mathbb{Z}_b^{\mathbb{N}}$ over $\mathbb{Z}_b$, and so we define μ as the product measure induced by a certain probability measure $\tilde{\mu}$ on $\mathbb{Z}_b^{\mathbb{N}}$. For $\tilde{\mu}$ we just take the measure on $\mathbb{Z}_b^{\mathbb{N}}$ induced by the equiprobability measure on $\mathbb{Z}_b$.

We use now the result from [3, Example 5.50.], that μ_s-almost all s-tuples $(C_1, \ldots, C_s) \in \mathcal{M}_s$ generate a digital $(\mathbf{T}, s)$-sequence over $\mathbb{Z}_b$ such that for some constant L we have

$$\mathbf{T}(m) \leq s \log_b m + 2 \log_b \log m + L \tag{2}$$

for all integers $m \geq 2$, to obtain the following metrical result as a consequence of Corollary 1.

Corollary 3. *μ_s-almost all s-tuples $(C_1,\dots,C_s) \in \mathcal{M}_s$ generate a digital $(\mathbf{T},s)$-sequence over $\mathbb{Z}_b$ such that*

$$F_{b,N}(\omega) = \mathcal{O}\left(\frac{(\log\log N)^2(\log N)^{3s/2}}{N}\right) \qquad (\textit{as } N \to \infty).$$

3. Proofs

In this section we provide now the proofs of the previous results from Section 2.

Proof of Theorem 1. It is enough to show Theorem 1 for strict digital $(\mathbf{T},s)$-sequences over $\mathbb{Z}_b$. If ω is not uniformly distributed modulo one the upper bound in Theorem 1 is infinite and therefore trivially fulfilled. So let in the following ω be a uniformly distributed, strict digital $(\mathbf{T},s)$-sequence over $\mathbb{Z}_b$, i.e. the function η is always well defined. The first steps of this proof are the same as in [4, Proof of Theorem 6]. So we just recall these steps without a detailed elaboration. For a point $\mathbf{x}_n$ of ω and for $\emptyset \neq \mathfrak{u} \subseteq \{1,\dots,s\}$, we define $\mathbf{x}_n^{(\mathfrak{u})}$ as the projection of $\mathbf{x}_n$ onto the coordinates in $\mathfrak{u}$. We have

$$(NF_{b,N}(\omega))^2 = \frac{1}{(b+1)^s-1} \sum_{\substack{\emptyset\neq\mathfrak{u}\subseteq\{1,\dots,s\} \\ \mathfrak{u}=\{w_1,\dots,w_{|\mathfrak{u}|}\}}} \Sigma(\mathfrak{u}),$$

where

$$\Sigma(\mathfrak{u}) := \sum_{k_{w_1}=1}^{\infty} \cdots \sum_{k_{w_{|\mathfrak{u}|}}=1}^{\infty} \left(\prod_{j\in\mathfrak{u}} \frac{1}{b^{2a(k_j)}}\right) \left|\sum_{n=0}^{N-1} {}_b\mathrm{wal}_{(k_{w_1},\dots,k_{w_{|\mathfrak{u}|}})}(\mathbf{x}_n^{(\mathfrak{u})})\right|^2.$$

For the sake of simplicity we assume in the following $\mathfrak{u} = \{1,\dots,\sigma\}$, $1 \leq \sigma \leq s$, and set $\mathbf{k}_\sigma := (k_1,\dots,k_\sigma)$, where $k_j, 1 \leq j \leq \sigma$, has b-adic expansion $k_j = \kappa_0^{(j)} + \kappa_1^{(j)} b + \cdots + \kappa_{a_j}^{(j)} b^{a_j}$, $\kappa_{a_j}^{(j)} \neq 0$. The other cases are dealt with a similar fashion. Let $C_j = (c_{v,w}^{(j)})_{v,w\geq 1}$ and let $\mathbf{c}_i^{(j)}$ be the i-th row vector of the generator matrix C_j. Define

$$u(\mathbf{k}_\sigma) := \min\left\{ l \geq 1 : \sum_{j=1}^{\sigma} (\kappa_0^{(j)} c_{1,l}^{(j)} + \cdots + \kappa_{a_j}^{(j)} c_{a_j+1,l}^{(j)}) \neq 0 \right\}$$

and

$$\boldsymbol{\beta}_{\mathbf{k}_\sigma} = (\beta_{\mathbf{k}_\sigma,0}, \beta_{\mathbf{k}_\sigma,1}, \ldots)^\top := \sum_{j=1}^{\sigma} (\kappa_0^{(j)} \mathbf{c}_1^{(j)} + \cdots + \kappa_{a_j}^{(j)} \mathbf{c}_{a_j+1}^{(j)}).$$

Since $C_1, \ldots, C_s$ generate a digital $(\mathbf{T}, s)$-sequence over $\mathbb{Z}_b$ one can verify with the same arguments as in [4, Proof of Theorem 6] that $u(\mathbf{k}_\sigma) \leq \eta\left(\sum_{j=1}^{\sigma} a_j + \sigma\right)$ $=: \eta(R_\sigma + \sigma)$, since the $\eta(R_\sigma + \sigma) \times (R_\sigma + \sigma)$ matrix

$$\begin{aligned} &\mathcal{C}(a_1, \ldots, a_\sigma) \\ &:= \begin{pmatrix} c_{1,1}^{(1)} & \ldots & c_{a_1+1,1}^{(1)} & \ldots & c_{1,1}^{(\sigma)} & \ldots & c_{a_\sigma+1,1}^{(\sigma)} \\ c_{1,2}^{(1)} & \ldots & c_{a_1+1,2}^{(1)} & \ldots & c_{1,2}^{(\sigma)} & \ldots & c_{a_\sigma+1,2}^{(\sigma)} \\ \vdots & & \vdots & & \vdots & & \vdots \\ c_{1,\eta(R_\sigma+\sigma)}^{(1)} & \ldots & c_{a_1+1,\eta(R_\sigma+\sigma)}^{(1)} & \ldots & c_{1,\eta(R_\sigma+\sigma)}^{(\sigma)} & \ldots & c_{a_\sigma+1,\eta(R_\sigma+\sigma)}^{(\sigma)} \end{pmatrix} \end{aligned}$$

has rank $R_\sigma + \sigma$. We have

$$\begin{aligned} &\Sigma(\{1, \ldots, \sigma\}) \\ &= \frac{12}{b^2(b^2-1)} \sum_{a_1=0}^{\infty} \cdots \sum_{a_\sigma=0}^{\infty} \frac{1}{b^{2R_\sigma}} \sum_{u=1}^{\eta(R_\sigma+\sigma)} \sum_{\beta=1}^{b-1} b^{2u} \psi_b^\beta \left(\frac{N}{b^u}\right) \underbrace{\sum_{k_1=b^{a_1}}^{b^{a_1+1}-1} \cdots \sum_{k_\sigma=b^{a_\sigma}}^{b^{a_\sigma+1}-1}}_{\substack{u(\mathbf{k}_\sigma)=u \\ \beta_{\mathbf{k}_\sigma, u(\mathbf{k}_\sigma)-1}=\beta}} 1\,. \end{aligned}$$

We need to evaluate the sum

$$\underbrace{\sum_{k_1=b^{a_1}}^{b^{a_1+1}-1} \cdots \sum_{k_\sigma=b^{a_\sigma}}^{b^{a_\sigma+1}-1}}_{\substack{u(\mathbf{k}_\sigma)=u \\ \beta_{\mathbf{k}_\sigma, u(\mathbf{k}_\sigma)-1}=\beta}} 1$$

for $1 \leq u \leq \eta(R_\sigma + \sigma)$ and $\beta \in \{1, \ldots, b-1\}$. This is the number of digits $\kappa_0^{(1)}, \ldots, \kappa_{a_1-1}^{(1)}, \theta_1, \ldots, \kappa_0^{(\sigma)}, \ldots, \kappa_{a_\sigma-1}^{(\sigma)}, \theta_\sigma \in \{0, \ldots, b-1\}$, $\theta_1 \neq 0, \ldots, \theta_\sigma \neq 0$, such that

$$\mathcal{C}(a_1,\ldots,a_\sigma)\begin{pmatrix}\kappa_0^{(1)}\\ \vdots\\ \kappa_{a_1-1}^{(1)}\\ \theta_1\\ \vdots\\ \kappa_0^{(\sigma)}\\ \vdots\\ \kappa_{a_\sigma-1}^{(\sigma)}\\ \theta_\sigma\end{pmatrix}=\begin{pmatrix}0\\ \vdots\\ 0\\ \beta\\ x_{u+1}\\ \vdots\\ x_{\eta(R_\sigma+\sigma)}\end{pmatrix} \tag{3}$$

for arbitrary $x_{u+1},\ldots,x_{\eta(R_\sigma+\sigma)}\in\mathbb{Z}_b$. Let now $1\le u\le\eta(R_\sigma+\sigma)$ and $\beta\in\{1,\ldots,b-1\}$ be fixed. For a fixed choice of $x_{u+1},\ldots,x_{\eta(R_\sigma+\sigma)}$ the system (3) has at most one solution. There are $b^{\eta(R_\sigma+\sigma)-u}$ possible choices for the $x_{u+1},\ldots,x_{\eta(R_\sigma+\sigma)}$. So we have

$$\underbrace{\sum_{k_1=b^{a_1}}^{b^{a_1+1}-1}\cdots\sum_{k_\sigma=b^{a_\sigma}}^{b^{a_\sigma+1}-1}}_{\substack{u(\mathbf{k}_\sigma)=u\\ \beta_{\mathbf{k}_\sigma,u(\mathbf{k}_\sigma)-1}=\beta}}1\le b^{\eta(R_\sigma+\sigma)-u}.$$

Now we have

$$\begin{aligned}
&\Sigma(\{1,\ldots,\sigma\})\\
&\le\frac{12}{b^2(b^2-1)}\sum_{a_1,\ldots,a_\sigma=0}^{\infty}\frac{1}{b^{2R_\sigma}}\sum_{u=1}^{\eta(R_\sigma+\sigma)}\sum_{\beta=1}^{b-1}b^{2u}\psi_b^{\beta}\left(\frac{N}{b^u}\right)b^{\eta(R_\sigma+\sigma)-u}\\
&=\frac{12}{b^2(b+1)}\sum_{a_1,\ldots,a_\sigma=0}^{\infty}\frac{1}{b^{2R_\sigma}}\sum_{u=1}^{\eta(R_\sigma+\sigma)}\psi_b\left(\frac{N}{b^u}\right)b^u b^{\eta(R_\sigma+\sigma)}\\
&=\frac{12}{b^2(b+1)}\sum_{u=1}^{\infty}\psi_b\left(\frac{N}{b^u}\right)\sum_{\substack{a_1,\ldots,a_\sigma=0\\ \eta(R_\sigma+\sigma)\ge u}}^{\infty}\frac{b^u b^{\eta(R_\sigma+\sigma)}}{b^{2R_\sigma}}\\
&=\frac{12}{b^2(b+1)}\sum_{u=1}^{\infty}\psi_b\left(\frac{N}{b^u}\right)\sum_{\substack{a_1,\ldots,a_\sigma=0\\ (u-1)-\mathbf{T}(u-1)<R_\sigma+\sigma}}^{\infty}\frac{b^u b^{\eta(R_\sigma+\sigma)}}{b^{2R_\sigma}}
\end{aligned}$$

$$\leq \frac{12}{b^2(b+1)} \sum_{u=1}^{\infty} \psi_b\left(\frac{N}{b^u}\right) \sum_{w=(u-1)-\mathbf{T}(u-1)-\sigma+1}^{\infty} \binom{w+\sigma-1}{\sigma-1} \frac{b^u b^{\eta(w+\sigma)}}{b^{2w}}$$

$$\leq \frac{12}{b^2(b+1)} \sum_{u=1}^{\infty} \psi_b\left(\frac{N}{b^u}\right) \sum_{v=u-1}^{\infty} b^u b^{v+1} \sum_{w=v-\mathbf{T}(v)-\sigma+1}^{(v+1)-\mathbf{T}(v+1)-\sigma} \binom{w+\sigma-1}{\sigma-1} \frac{1}{b^{2w}}$$

$$< \frac{12}{b(b+1)} \sum_{u=1}^{\infty} \psi_b\left(\frac{N}{b^u}\right) \sum_{v=u-1}^{\infty} b^u b^v \sum_{w=v-\mathbf{T}(v)-\sigma+1}^{\infty} \binom{w+\sigma-1}{\sigma-1} \frac{1}{b^{2w}}$$

$$\leq \frac{12}{b(b+1)} \sum_{u=1}^{\infty} \psi_b\left(\frac{N}{b^u}\right) \sum_{v=u-1}^{\infty} b^u b^v \frac{1}{b^{2v-2\mathbf{T}(v)-2\sigma+2}} \binom{v-\mathbf{T}(v)}{\sigma-1} \left(1-\frac{1}{b^2}\right)^{-\sigma}$$

$$\leq \frac{12}{b^3(b+1)} \left(\frac{b^4}{b^2-1}\right)^{\sigma} \sum_{u=1}^{\infty} \psi_b\left(\frac{N}{b^u}\right) b^u \sum_{v=u-1}^{\infty} \frac{v^{\sigma-1}}{b^v} b^{2\mathbf{T}(v)},$$

where we have used [3, Lemma 13.24] for the penultimate inequality. So we get

$$(NF_{b,N}(\omega))^2$$

$$= \frac{1}{(b+1)^s-1} \sum_{\substack{\emptyset \neq \mathfrak{u} \subseteq \{1,\dots,s\} \\ \mathfrak{u}=\{w_1,\dots,w_{|\mathfrak{u}|}\}}} \Sigma(\mathfrak{u})$$

$$\leq \frac{1}{(b+1)^s-1} \frac{12}{b^3(b+1)} \sum_{w=1}^{s} \binom{s}{w} \left(\frac{b^4}{b^2-1}\right)^{w} \sum_{u=1}^{\infty} \psi_b\left(\frac{N}{b^u}\right) b^u \sum_{v=u-1}^{\infty} \frac{v^{w-1}}{b^v} b^{2\mathbf{T}(v)}.$$

□

P r o o f o f C o r o l l a r y 1. For any $b^{m-1} < N \leq b^m$ we obtain out of Theorem 1 and the special form of ψ_b on $\left[0, \frac{1}{b}\right)$ that

$$(NF_{b,N}(\omega))^2$$

$$\leq c \sum_{u=1}^{\infty} \psi_b\left(\frac{N}{b^u}\right) b^u \sum_{v=u-1}^{\infty} \frac{v^{s-1}}{b^v} b^{2\mathbf{T}(v)}$$

$$\leq cc_1 \sum_{u=1}^{m} \psi_b\left(\frac{N}{b^u}\right) u^{s-1} b^{2\mathbf{T}(u)} + cc_1 \sum_{u=m+1}^{\infty} \frac{b^2(b^2-1)}{12} \frac{N^2}{b^{2u}} u^{s-1} b^{2\mathbf{T}(u)}$$

$$\leq cc_1 \sum_{u=1}^{m} \psi_b\left(\frac{N}{b^u}\right) u^{s-1} b^{2\mathbf{T}(u)} + cc_1 \frac{b^2(b^2-1)}{12} \frac{N}{b^{m+1}} b^m \sum_{u=m-1}^{\infty} \frac{u^{s-1}}{b^u} b^{2\mathbf{T}(u)}$$

$$\leq \tilde{c}_1 \sum_{u=1}^{m} u^{s-1} b^{2\mathbf{T}(u)} + \tilde{c}_2 m^{s-1} b^{2\mathbf{T}(m)}$$

$$= \mathcal{O}\left(\sum_{u=1}^{m} u^{s-1} b^{2\mathbf{T}(u)}\right),$$

where all appearing constants may depend only on b and s. □

Proof of Corollary 2. From Corollary 1 and the additional condition that $\frac{1}{m}\sum_{u=1}^{m} b^{2\mathbf{T}(u)} \leq c_2$ for all $m \in \mathbb{N}$, we get for any $b^{m-1} < N \leq b^m$

$$\begin{aligned}(NF_{b,N}(\omega))^2 &\leq \tilde{c}\sum_{u=1}^{m} u^{s-1} b^{2\mathbf{T}(u)} \\ &\leq \tilde{c}m^s \frac{1}{m}\sum_{u=1}^{m} b^{2\mathbf{T}(u)} \\ &\leq \tilde{c}c_2 m^s,\end{aligned}$$

where all appearing constants may depend only on b and s. From this it follows immediately that

$$F_{b,N}(\omega) = \mathcal{O}\left(\frac{(\log N)^{s/2}}{N}\right) \qquad (\text{as } N \to \infty).$$

□

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