In some computations at the end we use the gfun-package.
It is just needed to guess simple functional equations for generating functions and manipulate their coefficient.
This is not needed in the remainder of this worksheet.

For the latest version see Salvy's homepage http://perso.ens-lyon.fr/bruno.salvy/software/the-gfun-package/.
For these computations gfun verison 3.91 was used.

```
> #libname := "<path>\\gfun.mla", libname;    #set the path to
  gfun.mla
  libname := "D:\\lib\\gfun.mla", "D:\\lib",
  "/usr/people/mwallner/lib/gfun.mla", "/usr/people/mwallner/lib",
  libname;
  with(plots):with(gfun): gfun:-version;
```

libname := "D:\lib\gfun.mla", "D:\lib", "/usr/people/mwallner/lib/gfun.mla", "/usr/people/mwallner/lib", "C:\Program Files\Maple 2020\lib"

$$3.98 \quad \mathbf{(1)}$$

Recurrences (code from the Maple session accompanying [Elvey Price, Fang, Wallner 2021]; see https://dmg.tuwien.ac.at/mwallner/)

Maximal number of computed terms

```
> NN:=100;
```

$$NN := 100 \quad \mathbf{(1.1)}$$

Relaxed binary trees

Compute the number up to size NN

```
> for n from 0 to NN do
  for m from 0 to NN do
     rr2[n,m] := 0:
  end:
  end:

  #initial conditions
  for n from 0 to NN do
     rr2[n,0] := 1:
  end:

  for n from 1 to NN do
     for m from 1 to n do
        rr2[n,m] := rr2[n,m-1]+(m+1)*rr2[n-1,m];
     end do:
  end do:
```

print the array

```
> for m from 7 to 0 by -1 do
  for n from 0 to 10 do
  printf("%10.0f ",rr2[n,m]);
```

```
end;
printf("\n");
end;
```

```
            0           0           0           0           0
0           0     311250     6173791    86626584  1035808538
            0           0           0           0           0
0       18628     311250     3683791    37236256   342795866
            0           0           0           0           0
1363    18628     180854     1505041    11449719    82142074
            0           0           0           0         127
1363    10450      69086      419917     2419473    13443760
            0           0           0          16         127
728      3635      16836       74487      319888     1346395
            0           0           3          16          63
220       723       2296        7143       21940       66843
            0           1           3           7          15
31         63        127         255         511        1023
            1           1           1           1           1
1           1           1           1           1           1
```

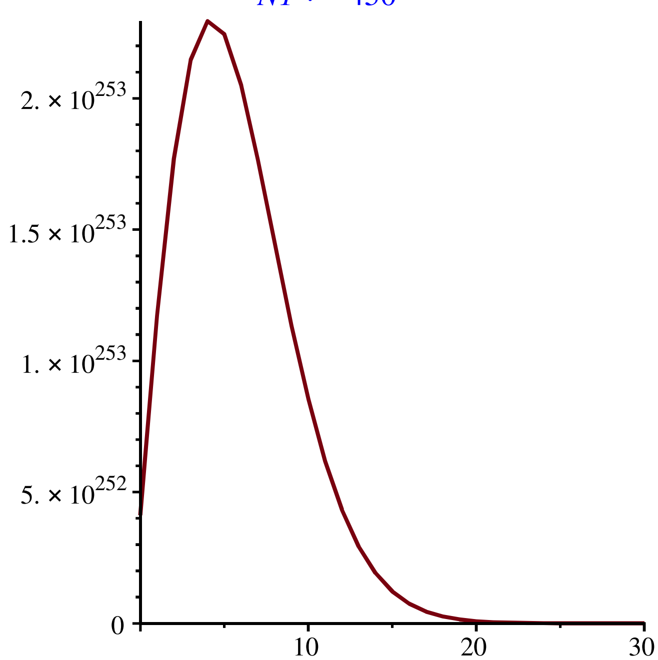
https://oeis.org/????
Number of deterministic completely defined initially connected acyclic automata with 2 inputs and n transient unlabeled states (and a unique absorbing state).

```
> seq(rr2[n,n],n=0..min(floor(NN),12));
```

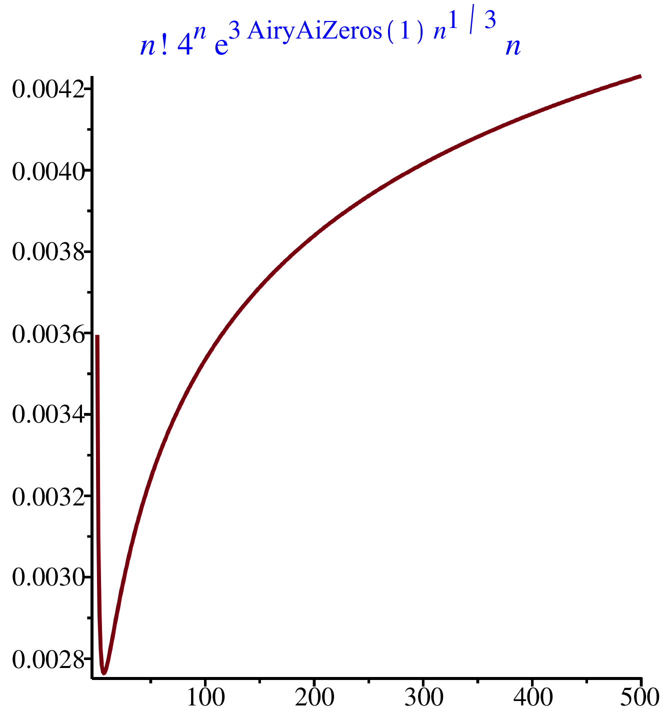
1, 1, 3, 16, 127, 1363, 18628, 311250, 6173791, 142190703, 3737431895, 110577492346, 3641313700916 **(1.1.1)**

```
> N1 := 450;
plot([seq([i,evalf(rr2[N1+i,N1-i]/factorial(N1+i))],i=0..30)]);
```

$N1 := 450$




```
> n:='n':
factorial(n)*4^n*exp(3*AiryAiZeros(1)*n^(1/3))*n^(1);
plot([seq([i,(evalf(subs(n=i,%)))/rr2[i,i]],i=1..500)]);
```

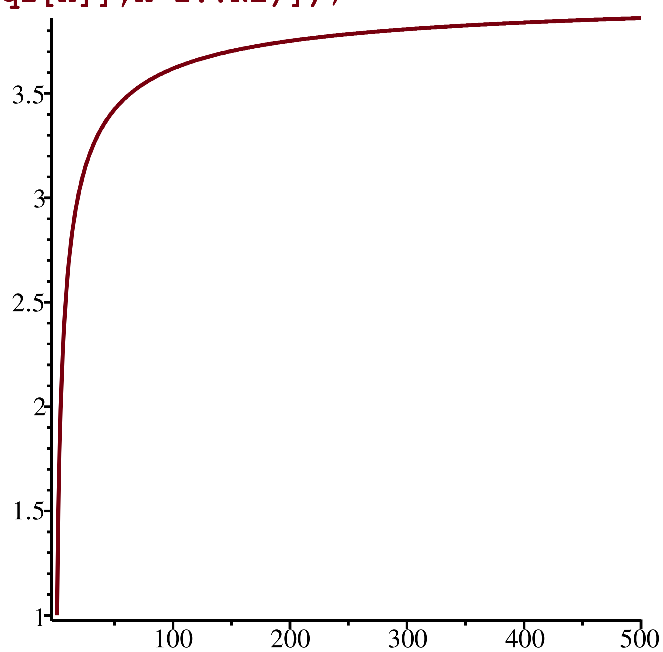



```
> N2 := floor(NN);
```

$$N2 := 500 \tag{1.1.2}$$

```
> for n from 0 to N2 do
    r2n[n] := rr2[n,n];
  end:
> for n from 1 to N2 do
    q1[n] := r2n[n]/r2n[n-1]/n;
  end:
```

```
> plot([seq([n,q1[n]],n=1..N2)]);
```



```
> evalf(q1[N2]);
  expgrowth2 := 4:
  expgrowth2,evalf(%);
```

3.861137885

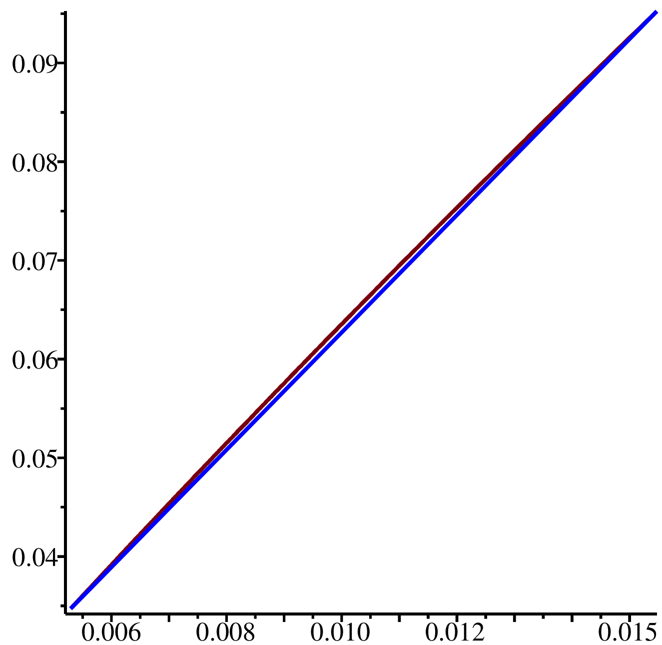
4, 4. **(1.1.3)**

```
> for n from 1 to N2 do
     q2[n] := q1[n]/expgrowth2;
  end:
```

we seem to have sigma=1/3

```
> N0 := 100:
  sigma := 1/3;
  P1 := plot([seq([sigma*n^(sigma-1),1-q2[n]],n=N0..N2)]):
  P2 := plot([[sigma*N0^(sigma-1),1-q2[N0]], [sigma*N2^(sigma-1),1-
  q2[N2]]],color=blue):
  display(P1,P2);
```

$$\sigma := \frac{1}{3}$$



Relaxed ternary trees

Compute the number up to size NN

```
> for n from 0 to NN do
  for m from 0 to max(ceil(n/2),30) do
     rr3[n,m] := 0:
  end:
  end:

  #initial conditions
  for n from 0 to NN do
     rr3[n,0] := 1:
  end:

  for n from 1 to NN do
```

```
    for m from 1 to floor(n/2) do
       rr3[n,m] := rr3[n,m-1]+(m+1)*rr3[n-1,m];
    end do:
 end do:
```

print the array

```
> for m from 7 to 0 by -1 do
  for n from 0 to 10 do
  printf("%10.0f ",rr3[n,m]);
  end;
  printf("\n");
  end;
```

```
         0          0          0          0          0
0          0          0          0          0          0
         0          0          0          0          0
0          0          0          0          0          0
         0          0          0          0          0
0          0          0          0          0     408354
         0          0          0          0          0
0          0          0       5711      56355     408354
         0          0          0          0          0
0        139       1036       5711      27800     126579
         0          0          0          0          7
36         139        480       1567       4956      15379
         0          0          1          3          7
15          31         63        127        255        511
         1          1          1          1          1
1           1          1          1          1          1
```

https://oeis.org/A082162
Number of deterministic completely defined initially connected acyclic automata with 3 inputs and n transient unlabeled states (and a unique absorbing state).

```
> seq(rr3[2*n,n],n=0..min(floor(NN/2),12));
```

1, 1, 7, 139, 5711, 408354, 45605881, 7390305396, 1647470410551, 485292763088275, 183049273155939442, 86211400693272461866, 49662741844330581221741 **(1.2.1)**

from below

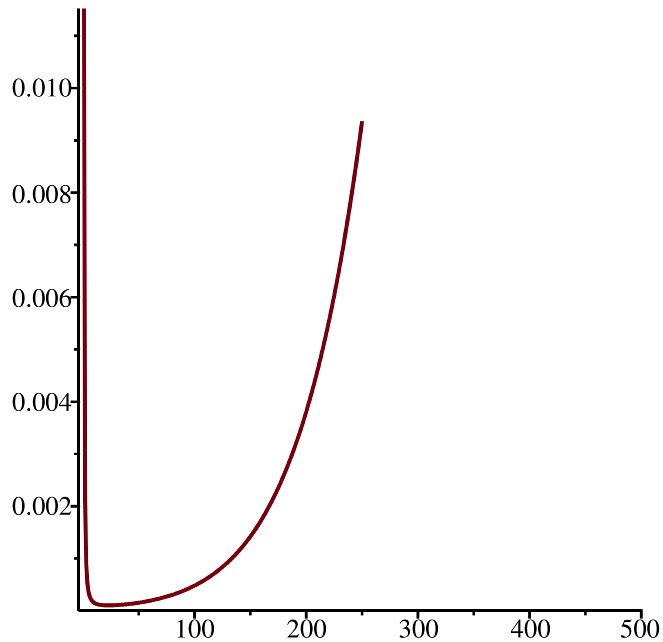
```
> n:='n':
  mu := exp((3*AiryAiZeros(1)*2^(2/3)*K^(1/3))/(2*(K - 1)^(1/3)));
```

$$\mu := e^{\frac{3\,\mathrm{AiryAiZeros}(1)\,2^{2/3}\,K^{1/3}}{2\,(K-1)^{1/3}}} \qquad \textbf{(1.2.2)}$$

could this asymptotics work? just guess on the exponential growth, at the moment it still seems a bit weird

```
> n:='n':
  factorial(n)^2*(3^3/2^2)^n*subs(K=3,mu)^(n^(1/3))*n^(-2);
  plot([seq([i,(evalf(subs(n=i,%)))/rr3[2*i,i]],i=1..500)]);
```

$$\frac{n!^2\left(\frac{27}{4}\right)^n\left(e^{\frac{3\,\mathrm{AiryAiZeros}(1)\,2^{1/3}\,3^{1/3}}{2}}\right)^{n^{1/3}}}{n^2}$$

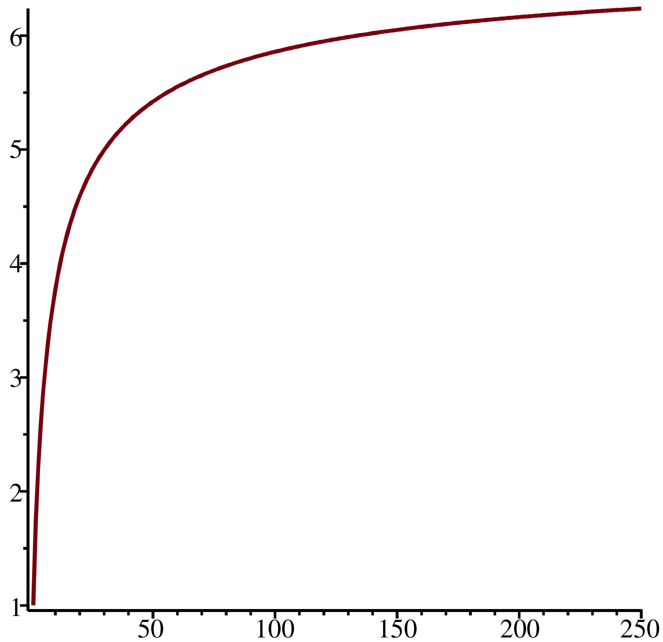



```
> #N1 := floor(NN/2);
  #plot([seq([i,evalf(rr3[2*N1+i,N1-i]/factorial(2*N1+i))],i=0..20)
  ]);
```

```
> N2 := floor(NN/2);
```

$$N2 := 250 \quad (1.2.3)$$

```
> for n from 0 to N2 do
     r3n[n] := rr3[2*n,n];
  end:
> for n from 1 to N2 do
     q1[n] := r3n[n]/r3n[n-1]/n^2;
  end:
> plot([seq([n,q1[n]],n=1..N2)]);
```




```
> evalf(q1[N2]);
  expgrowth3 := 27/4:
  expgrowth3,evalf(%);
```

$$6.237844002$$

$$\frac{27}{4}, 6.750000000 \quad (1.2.4)$$

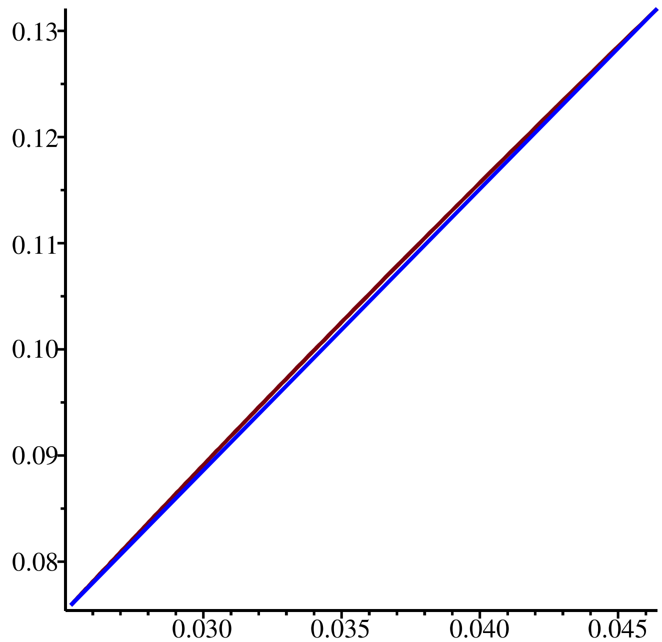
```
> #evalf(expgrowth3/4);
```

```
> for n from 1 to N2 do
      q2[n] := q1[n]/expgrowth3;
  end:
```

we seem to have sigma=1/3

```
> N0 := 100:
  sigma := 1/3;
  P1 := plot([seq([n^(sigma-1),1-q2[n]],n=N0..N2)]):
  P2 := plot([[N0^(sigma-1),1-q2[N0]], [N2^(sigma-1),1-q2[N2]]],
  color=blue):
  display(P1,P2);
```

$$\sigma := \frac{1}{3}$$



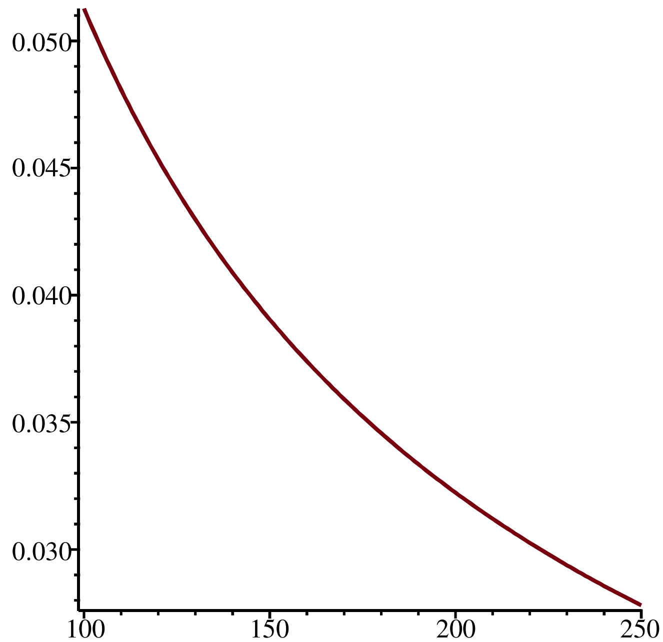

```
> q1[2];
```

$$\frac{7}{4} \tag{1.2.5}$$

```
> for n from 3 to N2 do
     #q2[n] := (q1[n]-1)/(q1[n-1]-1);
     q2[n] := log(q1[n])/log(q1[n-1]);
     #q2[n] := log(1+10*n^(sigma-1))/log(1+10*(n-1)^(sigma-1));
  end:
```

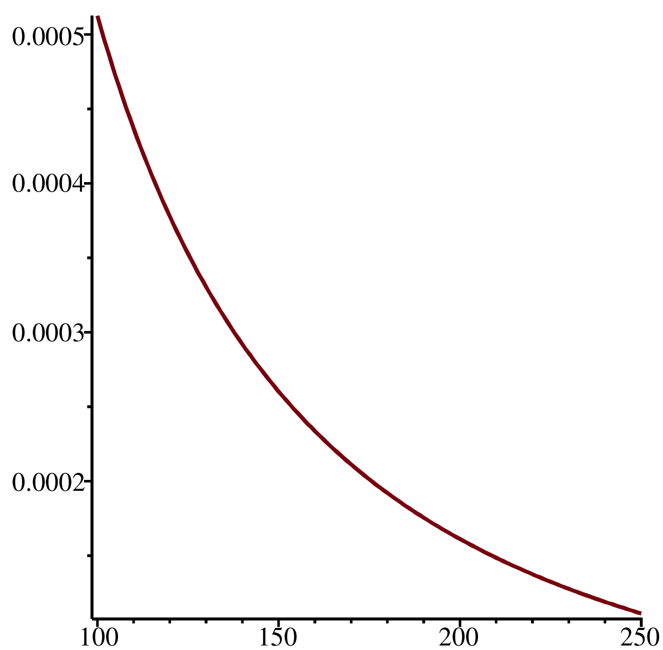
```
> N0 := 100:
  sigma := 1/3;
  plot([seq([n,n*(q2[n]-1)],n=N0..N2)]);
```

$$\sigma := \frac{1}{3}$$




```
> N0 := 100:
  sigma := 1/3;
  plot([seq([n,log(q2[n])],n=N0..N2)]);
```

$$\sigma := \frac{1}{3}$$



Relaxed 4-ary trees

Compute the number up to size NN

```
> for n from 0 to NN do
  for m from 0 to NN do
```

```
    rr4[n,m] := 0:
  end:
  end:

  #initial conditions
  for n from 0 to NN do
    rr4[n,0] := 1:
  end:

  for n from 1 to NN do
    for m from 1 to floor(n/3) do
      rr4[n,m] := rr4[n,m-1]+(m+1)*rr4[n-1,m];
    end do:
  end do:
```

print the array

```
> for m from 7 to 0 by -1 do
  for n from 0 to 10 do
  printf("%10.0f ",rr4[n,m]);
  end;
  printf("\n");
  end;
         0          0          0          0          0
0          0          0          0          0          0
         0          0          0          0          0
0          0          0          0          0          0
         0          0          0          0          0
0          0          0          0          0          0
         0          0          0          0          0
0          0          0          0          0          0
         0          0          0          0          0
0          0          0          0       1000       7255
         0          0          0          0          0
0         15         76        291       1000       3255
         0          0          0          1          3
7         15         31         63        127        255
         1          1          1          1          1
1          1          1          1          1          1
```

https://oeis.org/A102102
deterministic completely defined initially connected acyclic automata with 4 inputs and n transient unlabeled states (and a unique absorbing state) with a(0)=1

```
> seq(rr4[3*n,n],n=0..min(floor(NN/3),12));
```

1, 1, 15, 1000, 189035, 79278446, 63263422646, 86493299281972, 187766975052827491, 611024291011881918991, 2849262494779035461688236, 18362167739517547774072439880, 158759599858376078627687256207242 **(1.3.1)**

Relaxed 5-ary trees

Compute the number up to size NN

```
> for n from 0 to NN do
  for m from 0 to NN do
```

```
    rr5[n,m] := 0:
  end:
  end:

  #initial conditions
  for n from 0 to NN do
     rr5[n,0] := 1:
  end:

  for n from 1 to NN do
     for m from 1 to floor(n/4) do
        rr5[n,m] := rr5[n,m-1]+(m+1)*rr5[n-1,m];
     end do:
  end do:
```

print the array

```
> for m from 7 to 0 by -1 do
  for n from 0 to 10 do
  printf("%10.0f ",rr5[n,m]);
  end;
  printf("\n");
  end;
```

```
         0           0           0           0           0
0          0           0           0           0           0
         0           0           0           0           0
0          0           0           0           0           0
         0           0           0           0           0
0          0           0           0           0           0
         0           0           0           0           0
0          0           0           0           0           0
         0           0           0           0           0
0          0           0           0           0           0
         0           0           0           0           0
0          0           0          31         156         595
         0           0           0           0           1
3          7          15          31          63         127
         1           1           1           1           1
1          1           1           1           1           1
```

not in the OEIS

```
> seq(rr5[4*n,n],n=0..min(floor(NN/4),12));
```

1, 1, 31, 6631, 5470431, 12703473581, 68149976969707, 737746252883320473, 1460751186893228l551079, 491034897540036851729430160, 26433393473562746491056524678041, 2174200324312090709906465225146948928, 262905488055517484872519169314942799083971 **(1.4.1)**

Weighted Dyck meanders, i.e., rescaled recurrence (code from the Maple session accompanying [Elvey Price, Fang, Wallner 2021]; see https://dmg.tuwien.ac.at/mwallner/)

k-ary

```
> K := 3;
```

$K := 3$ **(1.5.1)**

```
> NN;
```

1000 **(1.5.2)**

compute the weighted Dyck meanders of size up to NN

```
> for n from 0 to NN do
  for m from -1 to NN do
  dd[n,m] := 0:
  end:end:

  dd[0,0] := 1:
  for n from 1 to NN do
     for m from 0 to n do
        dd[n,m] := (1+(n-m)/K)/((m+(K-1)*n)/K)*dd[n-1,m-1]+dd[n-1,
  m+K-1];
     end do:
  end do:
```

print the array

```
> for m from 5 to 0 by -1 do
  for n from 0 to 10 do
  printf("%10.0f ",dd[n,m]*factorial((m+(K-1)*n)/K));
  end;
  printf("\n");
  end;
```

```
         0          0          0          0          0
1          0          0         63          0          0
         0          0          0          0          1
0          0         31          0          0       1567
         0          0          0          1          0
0         15          0          0        480          0
         0          0          1          0          0
7          0          0        139          0          0
         0          1          0          0          3
0          0         36          0          0       1036
         1          0          0          1          0
0          7          0          0        139          0
```

check
K=3: http://oeis.org/A082162

```
> seq(factorial(((K-1)*n)/K)*dd[n,0],n=0..min(NN,10*K),K);
```

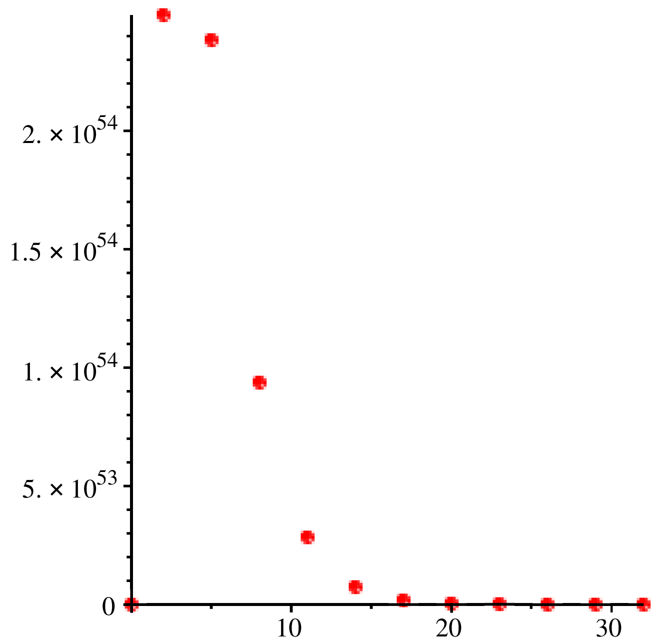
1, 1, 7, 139, 5711, 408354, 45605881, 7390305396, 1647470410551, 485292763088275, 183049273155939442 **(1.5.3)**

```
> ((K-1)*N)/K;
```

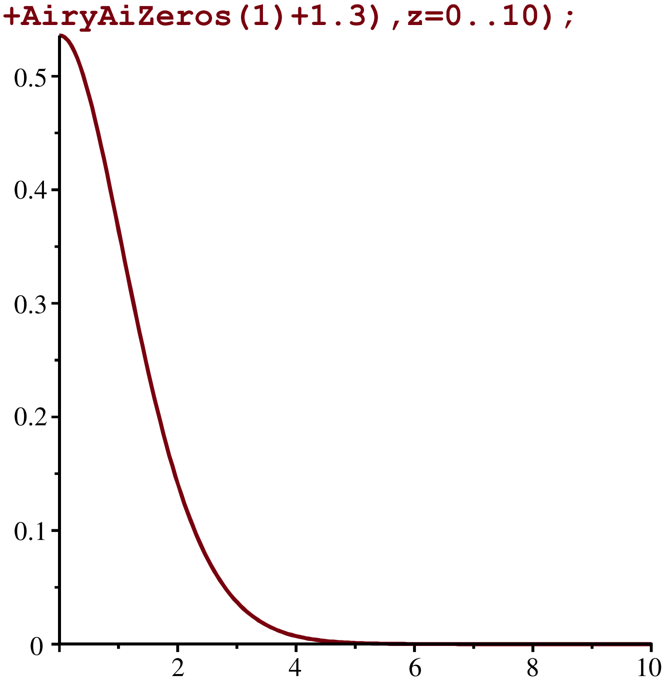
$$\frac{2N}{3}$$ **(1.5.4)**

```
> N1 := K*floor(NN/K);
  pointplot([[0,0],seq([i+K-1,dd[N1,i]],i=0..30,K)],color=red,
  symbol=solidcircle,symbolsize=15);
```

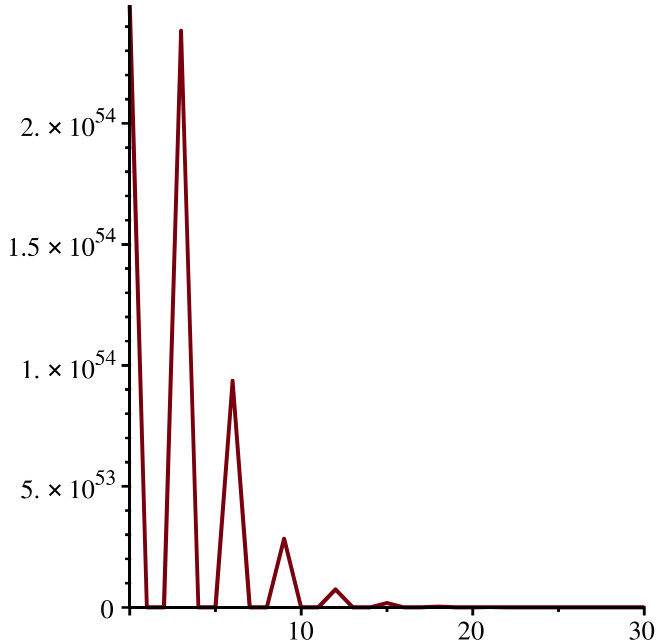
$N1 := 999$




```
> plot(AiryAi(z+AiryAiZeros(1)+1.3),z=0..10);
```




```
> plot([seq([i,dd[N1,i]],i=0..30)]);
```




```
> for n from 0 to NN do
  for m from -1 to NN do
  dd[n,m] := 0:
  end:end:

  dd[0,0] := 1:
  for n from 1 to NN do
     for m from 0 to n do
        #dd[n,m] := (1+(n-m)/K)/((m+(K-1)*n)/K)/(1/(K-1))*dd[n-1,
  m-1]+dd[n-1,m+K-1];
        dd[n,m] := (1-K*(m-K+1)/((K-1)*n+m))*dd[n-1,m-1]+dd[n-1,m+
  K-1];
     end do:
  end do:
```

print the array

```
> for m from 5 to 0 by -1 do
  for n from 0 to 10 do
  printf("%10.0f ",dd[n,m]*factorial((m+(K-1)*n)/K)*(1/(K-1))^((m+
  (K-1)*n)/K));
  end;
  printf("\n");
  end;
         0          0          0          0          0
1          0          0         63          0          0
         0          0          0          0          1
0          0         31          0          0       1567
         0          0          0          1          0
0         15          0          0        480          0
         0          0          1          0          0
7          0          0        139          0          0
         0          1          0          0          3
0          0         36          0          0       1036
         1          0          0          1          0
0          7          0          0        139          0
```

```
> seq(factorial(((K-1)*n)/K)*(1/(K-1))^((0+(K-1)*n)/K)*dd[n,0],n=0.
  .min(NN,10*K),K);
```
1, 1, 7, 139, 5711, 408354, 45605881, 7390305396, 1647470410551, 485292763088275, 183049273155939442 **(1.5.5)**

Weighted Dyck meanders, i.e., rescaled recurrence a bit different rescaling (code from the Maple session accompanying [Elvey Price, Fang, Wallner 2021]; see https://dmg.tuwien.ac.at/mwallner/)

k-ary

```
> K := 3;
```
$K := 3$ **(1.6.1)**

```
> NN:=1000;
```
$NN := 1000$ **(1.6.2)**

compute the weighted Dyck meanders of size up to NN

```
> for n from 0 to NN do
  for m from -1 to NN do
  dd[n,m] := 0:
  end:end:

  dd[0,0] := 1:
  for n from 1 to NN do
     for m from 0 to n do
        dd[n,m] := (1-K*(m-K+1)/((m+(K-1)*n)))*dd[n-1,m-1]+dd[n-1,
  m+K-1];
     end do:
  end do:
```

print the array

```
> for m from 5 to 0 by -1 do
  for n from 0 to 10 do
  printf("%10.0f ",dd[n,m]*factorial((m+(K-1)*n)/K)*(1/(K-1))^(
  (K-1)*n/K));
  end;
  printf("\n");
  end;
```

```
         0          0          0          0          0
3          0          0        200          0          0
         0          0          0          0          3
0          0         78          0          0       3949
         0          0          0          2          0
0         30          0          0        960          0
         0          0          2          0          0
11          0          0        221          0          0
         0          1          0          0          4
0          0         45          0          0       1305
         1          0          0          1          0
0          7          0          0        139          0
```

check
K=3: http://oeis.org/A082162

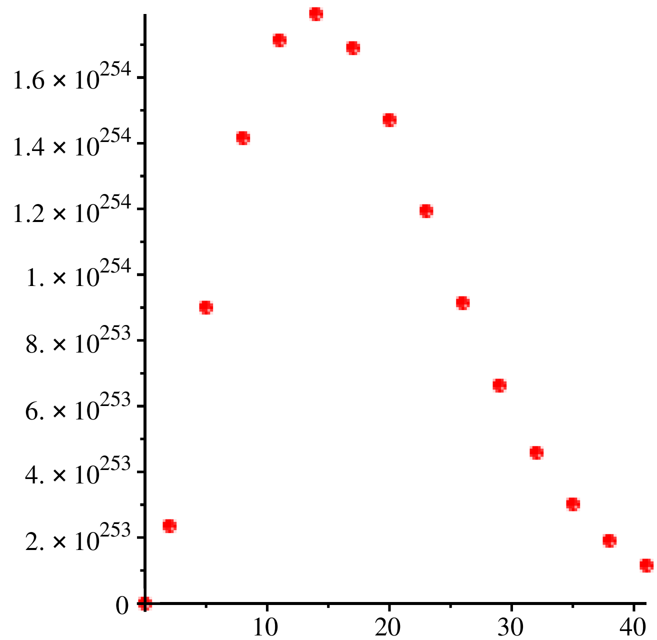
```
> seq(factorial(((K-1)*n/K))*(1/(K-1))^((K-1)*n/K)*dd[n,0],n=0..min
  (NN,10*K),K);
```

1, 1, 7, 139, 5711, 408354, 45605881, 7390305396, 1647470410551, 485292763088275, 183049273155939442 **(1.6.3)**

new rescaling much better than old one

```
> N1 := K*floor(NN/K);
  #N1 := 50*K;
  pointplot([[0,0],seq([i+K-1,dd[N1,i]*(1/(K-1))^((K-1)*N1/K)*
  factorial((i+(K-1)*N1)/K)/factorial(ceil(((K-1)*N1+i)/2/K))
  /factorial(ceil((((K-1)*N1+i)/K-1)/2))],i=0..40,K)],color=red,
  symbol=solidcircle,symbolsize=15);
```

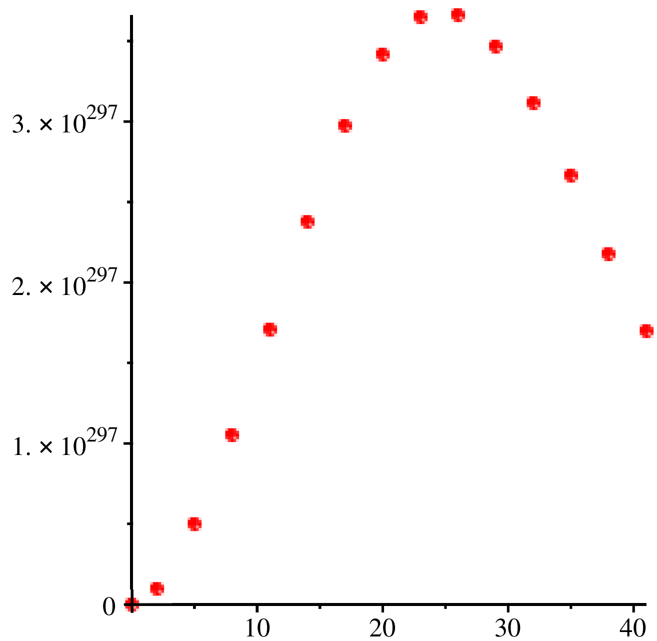
$N1 := 999$



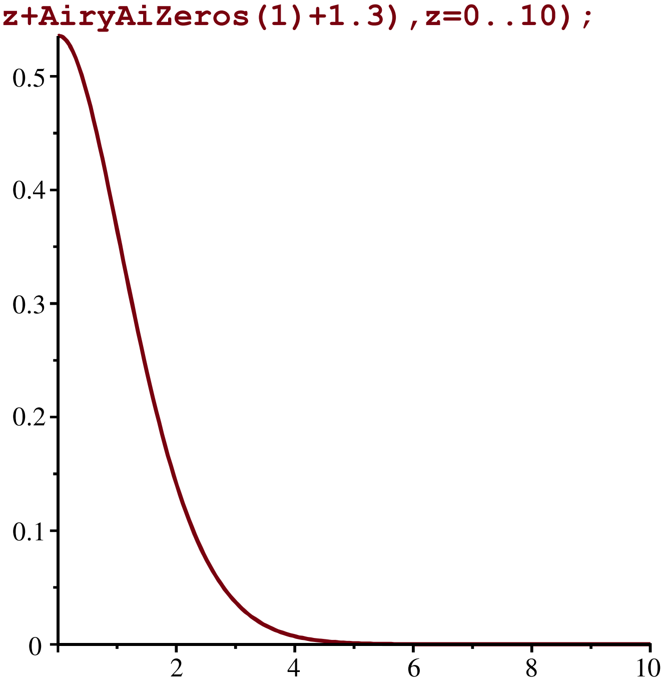

```
> N1 := K*floor(NN/K);
  N1 := 290*K;
  pointplot([[0,0],seq([i+K-1,dd[N1,i]*(1/(K-1))^((K-1)*N1/K)*
  factorial((i+(K-1)*N1)/K)/factorial(ceil(((K-1)*N1+i)/K))^2*ceil(
  ((K-1)*N1+i)/K)^(ceil(((K-1)*N1+i)/K))],i=0..40,K)],color=red,
  symbol=solidcircle,symbolsize=15);
```

$N1 := 999$

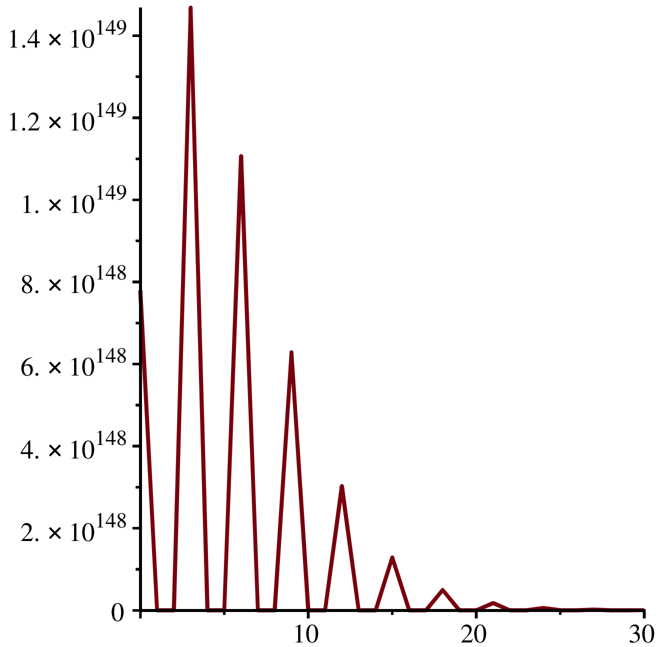
$N1 := 870$




```
> pplot(AiryAi(z+AiryAiZeros(1)+1.3),z=0..10);
```




```
> plot([seq([i,dd[N1,i]],i=0..30)]);
```




```
> for n from 0 to NN do
  for m from -1 to NN do
  dd[n,m] := 0:
  end:end:

  dd[0,0] := 1:
  for n from 1 to NN do
     for m from 0 to n do
        #dd[n,m] := (1+(n-m)/K)/((m+(K-1)*n)/K)/(1/(K-1))*dd[n-1,
  m-1]+dd[n-1,m+K-1];
        dd[n,m] := (1-K*(m-K+1)/((K-1)*n+m))*dd[n-1,m-1]+dd[n-1,m+
  K-1];
     end do:
  end do:
```

print the array

```
> for m from 5 to 0 by -1 do
  for n from 0 to 10 do
  printf("%10.0f ",dd[n,m]*factorial((m+(K-1)*n)/K)*(1/(K-1))^((m+
  (K-1)*n)/K));
  end;
  printf("\n");
  end;
```

```
         0          0          0          0          0
1          0          0         63          0          0
         0          0          0          0          1
0          0         31          0          0       1567
         0          0          0          1          0
0         15          0          0        480          0
         0          0          1          0          0
7          0          0        139          0          0
         0          1          0          0          3
0          0         36          0          0       1036
         1          0          0          1          0
0          7          0          0        139          0
```

```
> seq(factorial(((K-1)*n)/K)*(1/(K-1))^((0+(K-1)*n)/K)*dd[n,0],n=0.
  .min(NN,10*K),K);
```
1, 1, 7, 139, 5711, 408354, 45605881, 7390305396, 1647470410551, 485292763088275, 183049273155939442 **(1.6.4)**

Weighted Dyck meanders, i.e., rescaled recurrence a bit different rescaling again (code from the Maple session accompanying [Elvey Price, Fang, Wallner 2021]; see https://dmg.tuwien.ac.at/mwallner/)

k-ary

```
> K := 3;
```
$$K := 3 \quad \textbf{(1.7.1)}$$

```
> NN:=500;
```
$$NN := 500 \quad \textbf{(1.7.2)}$$

compute the weighted Dyck meanders of size up to NN

```
> for i from 0 to NN do
  aa[0,i]:=0:
  for j from -1 to NN do
  aa[i,j] := 0:
  end:end:

  aa[0,0] := 1:

  for i from 1 to NN do
     for j from 0 to i do
        aa[i,j] := (i-j+K)/((K-1)*i+j)*aa[i-1,j-1]+aa[i-1,j+K-1];
     end do:
  end do:
> nij := (i,j) -> ((K-1)*i+j)/K;
```
$$nij := (i, j) \mapsto \frac{(K-1)\cdot i + j}{K} \quad \textbf{(1.7.3)}$$

print the array

```
> for j from 5 to 0 by -1 do
  for i from 0 to 10 do
  printf("%10.0f ",aa[i,j]*factorial(nij(i,j)));
  end;
  printf("\n");
  end;
```
```
          0          0          0          0          0
1          0          0         63          0          0
          0          0          0          0          1
0          0         31          0          0       1567
          0          0          0          1          0
0         15          0          0        480          0
```

```
            0           0           1           0           0
7             0           0         139           0           0
            0           1           0           0           3
0             0          36           0           0        1036
            1           0           0           1           0
0             7           0           0         139           0
```

check
K=3: http://oeis.org/A082162

```
> seq(factorial(nij(i,0))*aa[i,0],i=0..min(NN,10*K),K);
```

1, 1, 7, 139, 5711, 408354, 45605881, 7390305396, 1647470410551, 485292763088275, 183049273155939442 **(1.7.4)**

```
>
  for i from 450 to 450 do
  seq(aa[i,j]-(dd[i,j]*(1/(K-1))^(nij(i,j))),j=1..N1,K);
  end;
```

0, 0 **(1.7.5)**

new rescaling much better than old one

```
> #N1 := K*floor(NN/K);
  N1 := 2*K;
  ([[0,0],seq([j+K-1,aa[N1,j]*factorial(nij(N1,j))/factorial(ceil
  (nij(N1,j)/2))/factorial(ceil((nij(N1,j)-1)/2)) ],j=0..30,K)],
  color=red,symbol=solidcircle,symbolsize=15);
  ([[0,0],seq([j+K-1,dd[N1,j]*(1/(K-1))^((K-1)*N1/K)*factorial(nij
  (N1,j))/factorial(ceil(nij(N1,j)/2))/factorial(ceil((nij(N1,j)-1)
  /2)) ],j=0..30,K)],color=red,symbol=solidcircle,symbolsize=15);
```

$$N1 := 6$$

$$\left[[0,0], \left[2, \frac{7}{4}\right], \left[5, \frac{5}{4}\right], \left[8, \frac{1}{36}\right], [11,0], [14,0], [17,0], [20,0], [23,0], [26,0], [29,0], [32,0]\right], color = red, symbol = solidcircle, symbolsize = 15$$

$$\left[[0,0], \left[2, \frac{7}{4}\right], \left[5, \frac{5}{2}\right], \left[8, \frac{1}{9}\right], [11,0], [14,0], [17,0], [20,0], [23,0], [26,0], [29,0], [32,0]\right], color = red, symbol = solidcircle, symbolsize = 15 \quad \textbf{(1.7.6)}$$

```
> seq(aa[N1,j],j=0..10,K);
  seq(dd[N1,j]*(1/(K-1))^(nij(N1,j)),j=0..10,K);
```

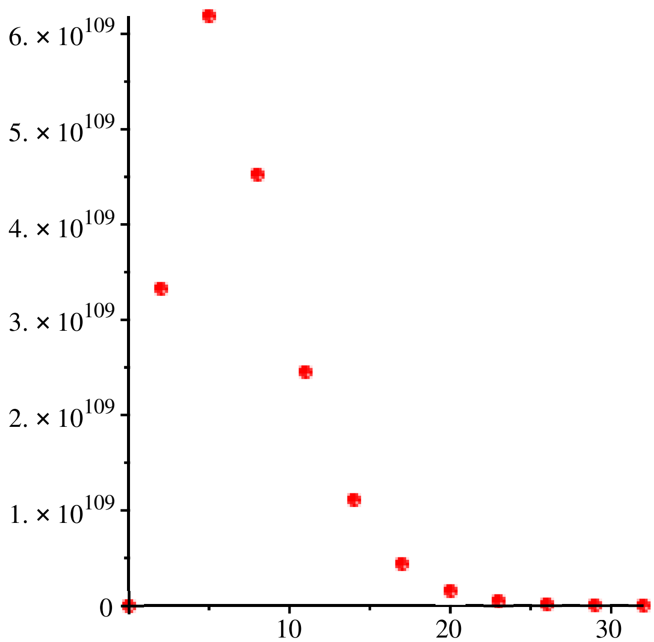
$$\frac{7}{24}, \frac{1}{8}, \frac{1}{720}, 0$$

$$\frac{7}{24}, \frac{1}{8}, \frac{1}{720}, 0 \qquad (1.7.7)$$

new rescaling much better than old one

```
> #N1 := K*floor(NN/K);
  N1 := 150*K;
  pointplot([[0,0],seq([j+K-1,aa[N1,j]*factorial(nij(N1,j))/(
  factorial(nij(N1,j))/2^nij(N1,j) ) ],j=0..30,K)],color=red,
  symbol=solidcircle,symbolsize=15);
```

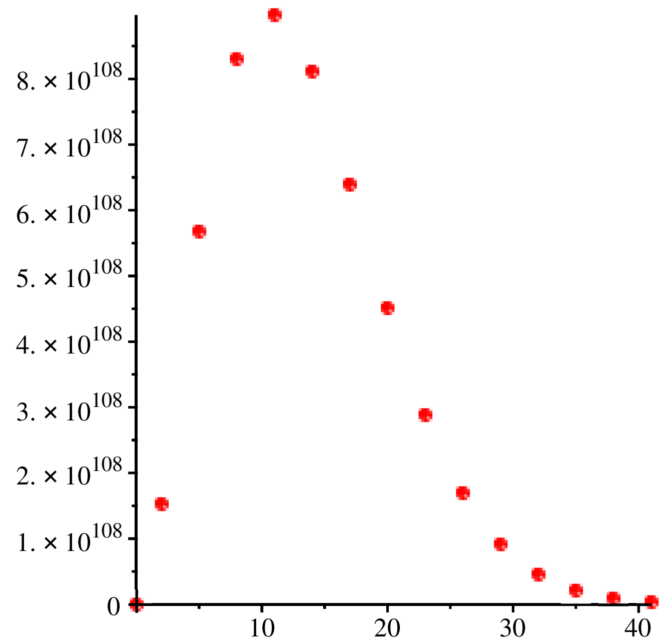
$$N1 := 450$$




```
> N1 := K*floor(NN/K);
  N1 := 150*K;
  pointplot([[0,0],seq([i+K-1,dd[N1,i]*(1/(K-1))^((K-1)*N1/K)*
  factorial(nij(N1,i))/factorial(ceil(nij(N1,i)/2))/factorial(ceil(
  (nij(N1,i)-1)/2))],i=0..40,K)],color=red,symbol=solidcircle,
  symbolsize=15);
```

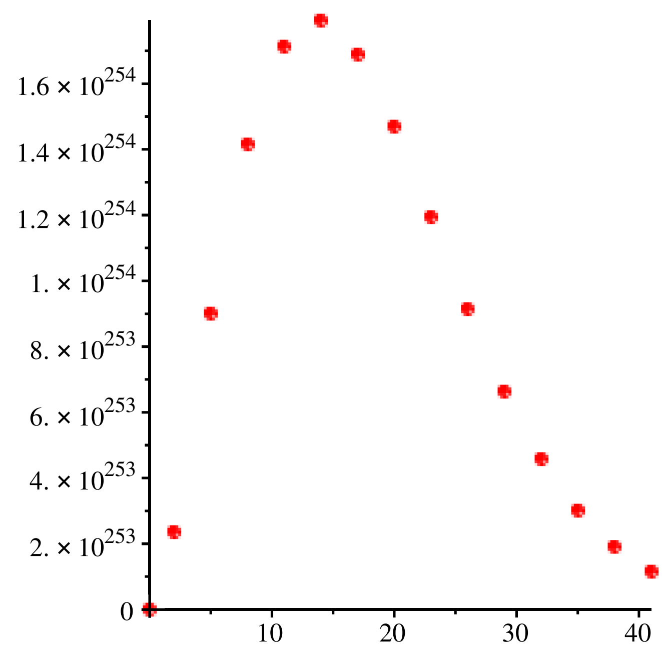
$$N1 := 999$$

$$N1 := 450$$




```
> N1 := K*floor(NN/K);
  #N1 := 50*K;
  pointplot([[0,0],seq([i+K-1,dd[N1,i]*(1/(K-1))^((K-1)*N1/K)*
  factorial((i+(K-1)*N1)/K)/factorial(ceil(((K-1)*N1+i)/2/K))
  /factorial(ceil((((K-1)*N1+i)/K-1)/2))],i=0..40,K)],color=red,
  symbol=solidcircle,symbolsize=15);
```

$$N1 := 999$$



Weighted Dyck meanders, i.e., rescaled recurrence a bit different rescaling again

(code from the Maple session accompanying [Elvey Price, Fang, Wallner 2021]; see https://dmg.tuwien.ac.at/mwallner/)

k-ary

```
> K := 3;
```

$$K := 3 \tag{1.8.1}$$

```
> NN:=1000;
```

$$NN := 1000 \tag{1.8.2}$$

compute the weighted Dyck meanders of size up to NN

```
> for i from 0 to NN do
  bb[0,i]:=0:
  for j from -1 to NN do
  bb[i,j] := 0:
  end:end:

  bb[0,0] := 1:

  for i from 1 to NN do
     for j from 0 to i do
        #bb[i,j] := (K-1)*(i-j+K)/((K-1)*i+j)*bb[i-1,j-1] + bb[i-1,
  j+K-1];
        bb[i,j] := 2*(i-j+3)/(2*i+j)*bb[i-1,j-1] + bb[i-1,j+2];
     end do:
  end do:
> nij := (i,j) -> ((K-1)*i+j)/K;
```

$$nij := (i, j) \mapsto \frac{(K-1)\cdot i+j}{K} \tag{1.8.3}$$

print the array

```
> for j from 5 to 0 by -1 do
  for i from 0 to 10 do
  printf("%10.0f ",bb[i,j]*factorial(nij(i,j))/2^nij(i,j));
  end;
  printf("\n");
  end;
```

```
         0          0          0          0          0
1          0          0         63          0          0
         0          0          0          0          1
0          0         31          0          0       1567
         0          0          0          1          0
0         15          0          0        480          0
         0          0          1          0          0
7          0          0        139          0          0
         0          1          0          0          3
0          0         36          0          0       1036
         1          0          0          1          0
0          7          0          0        139          0
```

check

K=3: http://oeis.org/A082162

```
> seq(factorial(nij(i,0))/2^nij(i,0)*bb[i,0],i=0..min(NN,10*K),K);
```

1, 1, 7, 139, 5711, 408354, 45605881, 7390305396, 1647470410551, 485292763088275, **(1.8.4)**

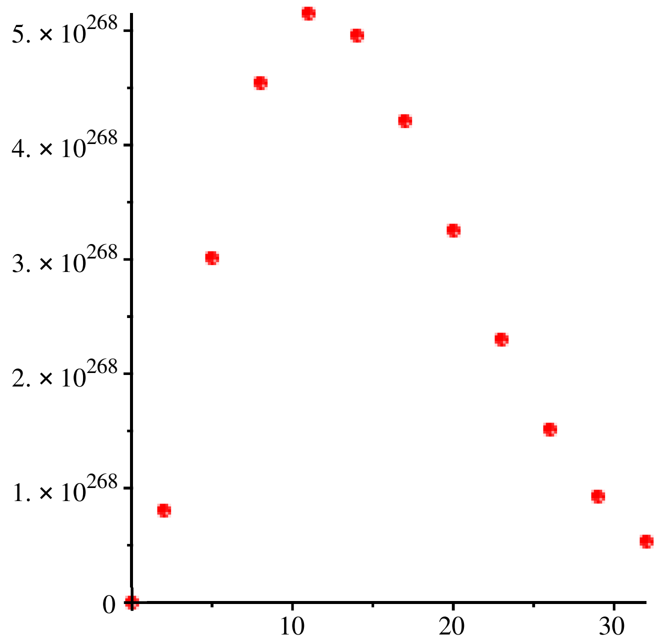
183049273155939442

```
> nij(x,y);
```

$$\frac{2x}{3} + \frac{y}{3} \tag{1.8.5}$$

```
> #N1 := K*floor(NN/K);
  N1 := 200*K;
  pointplot([[0,0],seq([j+K-1,bb[N1,j]*factorial(nij(N1,j))
  /factorial(ceil(nij(N1,j)/2))/factorial(ceil((nij(N1,j)-1)/2))],
  j=0..30,K)],color=red,symbol=solidcircle,symbolsize=15);
```

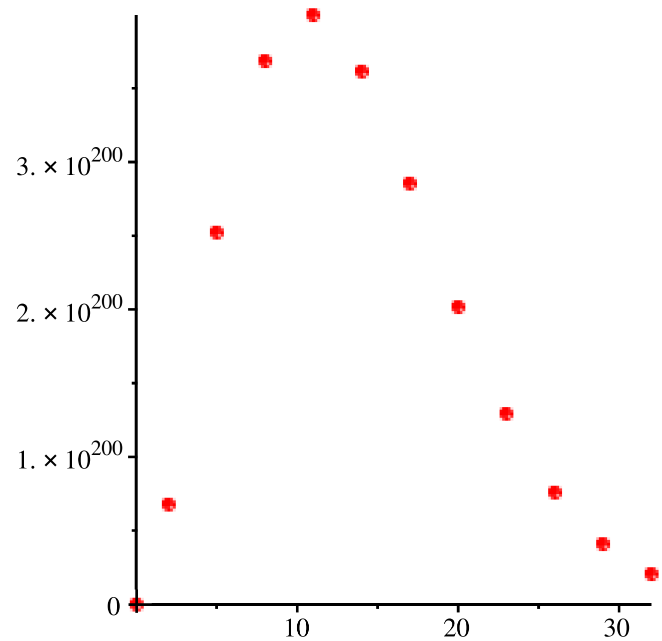
$N1 := 600$



new rescaling much better than old one

```
> #N1 := K*floor(NN/K);
  N1 := 150*K;
  pointplot([[0,0],seq([j+K-1,bb[N1,j]*2^nij(N1,j)],j=0..30,K)],
  color=red,symbol=solidcircle,symbolsize=15);
```

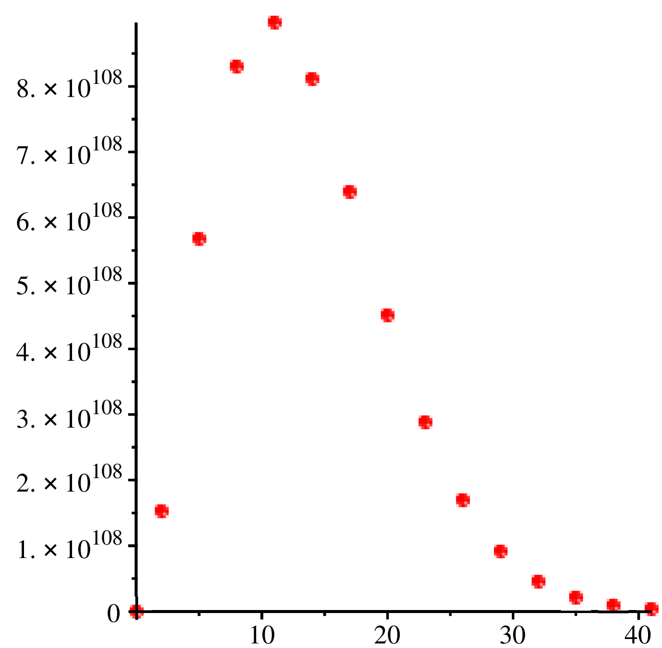
$N1 := 450$




```
> N1 := K*floor(NN/K);
  N1 := 150*K;
  pointplot([[0,0],seq([i+K-1,dd[N1,i]*(1/(K-1))^((K-1)*N1/K)*
  factorial(nij(N1,i))/factorial(ceil(nij(N1,i)/2))/factorial(ceil(
  (nij(N1,i)-1)/2))],i=0..40,K)],color=red,symbol=solidcircle,
  symbolsize=15);
```

$N1 := 999$

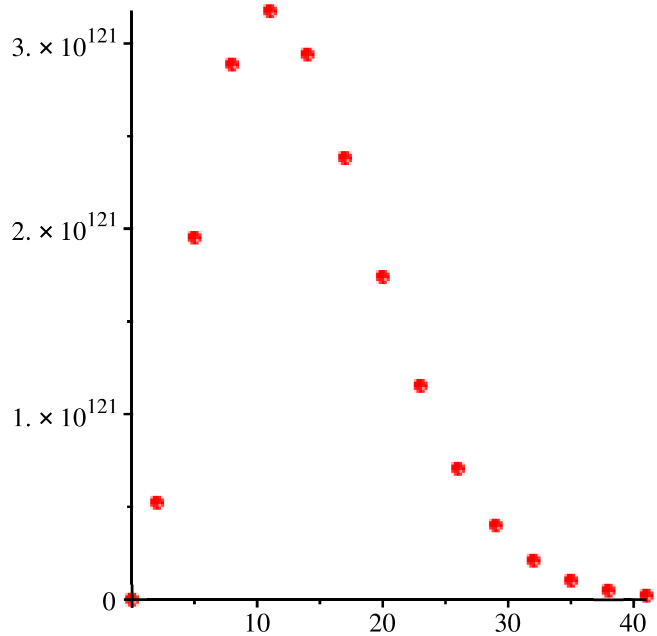
$N1 := 450$



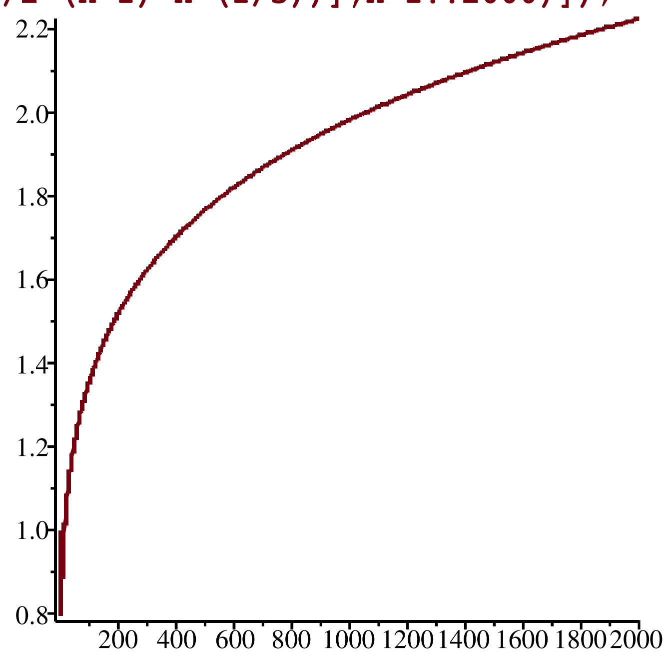

```
> N1 := K*floor(NN/K);
  #N1 := 50*K;
  pointplot([[0,0],seq([i+K-1,dd[N1,i]*(1/(K-1))^((K-1)*N1/K)*
  factorial((i+(K-1)*N1)/K)/factorial(ceil(((K-1)*N1+i)/2/K))
```

```
/factorial(ceil((((K-1)*N1+i)/K-1)/2))],i=0..40,K)],color=red,
symbol=solidcircle,symbolsize=15);
```

$N1 := 498$




```
> plot([seq([n,factorial(ceil(n/2))*factorial(ceil((n-1)/2))/
(factorial(n)/2^(n-1)*n^(1/3))],n=1..2000)]);
```




```
> 3
```

$$3 \tag{1.8.6}$$

Drift considerations

for large u, this recurrence has nearly 0 drift! (at least for w=o(u))
important to capture the Airy function

```
> newdrift := (K-1)^2*(K + u - w)/(K*u - u + w) - (K-1);
  subs(w = q*u,%);     # q=w/u
  map(simplify,series(%,u=infinity,3));
  series(simplify(subs(q=0,%)),u=infinity,2);  # for q very small,
  at least q=o(1)
```

$$newdrift := \frac{(K-1)^2\,(K+u-w)}{K\,u-u+w} - K + 1$$

$$\frac{(K-1)^2\,(-q\,u+K+u)}{K\,u+q\,u-u} - K + 1$$

$$-\frac{(K-1)^2\,(q-1)}{K+q-1} - K + 1 + \frac{(K-1)^2\,K}{(K+q-1)\,u}$$

$$\frac{(K-1)\,K}{u} \qquad \textbf{(1.9.1)}$$

```
> olddrift := (K + u - w)/(K*u - u + w) - (K-1);
  subs(w = q*u,%);     # q=w/u
  map(simplify,series(%,u=infinity,3));
  series(simplify(subs(q=0,%)),u=infinity,2);   # for q very small,
  at least q=o(1)
```

$$olddrift := \frac{K+u-w}{K\,u-u+w} - K + 1$$

$$\frac{-q\,u+K+u}{K\,u+q\,u-u} - K + 1$$

$$\frac{-q+1}{K+q-1} - K + 1 + \frac{K}{(K+q-1)\,u}$$

$$-\frac{K\,(K-2)}{K-1} + \frac{K}{(K-1)\,u} \qquad \textbf{(1.9.2)}$$

Weighted Dyck meanders now with zero drift

k-ary

```
> K := 3;
```

$$K := 3 \qquad \textbf{(1.10.1)}$$

```
> NN:=100;
```

$$NN := 100 \qquad \textbf{(1.10.2)}$$

compute the weighted Dyck meanders of size up to NN

```
> for i from 0 to NN do
  bb[0,i]:=0:
  for j from -1 to NN do
  bb[i,j] := 0:
  end:end:
```

```
bb[0,0] := 1:

for i from 1 to NN do
   for j from 0 to i do
      bb[i,j] := (K-1)^2*(i-j+K)/((K-1)*i+j)*bb[i-1,j-1] + bb
[i-1,j+K-1];
   end do:
end do:
```

number of rigth steps when reading (i,j)

```
> nij := (i,j) -> ((K-1)*i+j)/K
```

$$nij := (i, j) \mapsto \frac{(K-1)\cdot i + j}{K} \quad \textbf{(1.10.3)}$$

print the array

```
> for j from 5 to 0 by -1 do
  for i from 0 to 10 do
  printf("%10.0f ",bb[i,j]*factorial(nij(i,j))/(K-1)^(2*nij(i,j)));
  end;
  printf("\n");
  end;
```

```
         0          0          0          0          0
1          0          0         63          0          0
         0          0          0          0          1
0          0         31          0          0       1567
         0          0          0          1          0
0         15          0          0        480          0
         0          0          1          0          0
7          0          0        139          0          0
         0          1          0          0          3
0          0         36          0          0       1036
         1          0          0          1          0
0          7          0          0        139          0
```

check
K=3: http://oeis.org/A082162

```
> seq(factorial(nij(i,0))/(K-1)^(2*nij(i,0))*bb[i,0],i=0..min(NN,
  10*K),K);
```

1, 1, 7, 139, 5711, 408354, 45605881, 7390305396, 1647470410551, 485292763088275, 183049273155939442 **(1.10.4)**

recurrence with balanced weights (zero drift)

```
> eerec := ee[_n,_m] = (K-1)^2*(K+_n-_m)/((K-1)*_n+_m)*ee[_n-1,
  _m-1]+ee[_n-1,_m+K-1];
```

(2)

$$eerec := ee_{\_n,\,\_m} = \frac{(K-1)^2\,(K+\_n-\_m)\,ee_{\_n-1,\,\_m-1}}{(K-1)\,\_n+\_m} + ee_{\_n-1,\,\_m+K-1} \quad (2)$$

```
> A1 := AiryAiZeros(1);
```

$$A1 := \mathrm{AiryAiZeros}(1) \quad (3)$$

Newton polygons

Programs from [Elvey Price, Fang, Wallner 2021]

Get the maximal power of n^b for each m^a for a=0..M

```
> getMaxNewt := proc(M::posint, newt)
      local i, el, mnmax;

      for i from 0 to M do mnmax[i]:=-infinity end:
      for el in newt do
         if el[1]<=M then
         if mnmax[el[1]]<el[2] then mnmax[el[1]]:=el[2] end:
      end;
      end;

      return mnmax;
  end:
```

Compute the slopes and corners of the convex hull

```
> maxslope := proc(ll,M,i)
      local j,sl,tmp,sj;
      sl := ll[i+1]-ll[i]:   #initial slope between first 2 points
      #is this the one of the hull? find max starting from 0
      sj:=M;
      for j from i+1 to M do
         tmp := (ll[j]-ll[i])/(j-i);
         if tmp >= sl then sl:=tmp;sj:=j; end;
      end:
      return sl,sj;
  end:
```

Compute the slopes and corners of the convex hull

```
> getslopes := proc(ll,M)
      local sl,sj,li,ls;
      sj:=0;
      li:=sj:
      ls:=0:
      #go on and find other slopes of convex hull
      while sj<M do
         sl,sj := maxslope(ll,M,sj):
         #save it
         li:=li,sj;
         ls:=ls,sl;
      end:

      li:=[li];
      ls:=[seq(ls[i],i=2..nops(li))];
      return ls,li;
```

```
end:
```

List programs - Arrays

expansion to array (better performance than list)

```
> ex2Array := proc(ex,M,N,p)
    local tt,mdeg,ndeg,ar,mm,nn:
    ar := Array(0..M,0..p*N):

    for tt in expand(ex) do
       mdeg := m*diff(tt,m)/tt;
       ndeg := n*diff(tt,n)/tt;

       if mdeg>=0 and mdeg<=M and ndeg>=-N and ndeg<=0 then
          mm := mdeg;
          nn := -ndeg*p;
          ar[mm,nn] := ar[mm,nn] + subs(m=1,n=1,tt);
       end;
    end;

    return ar;
  end:
> mynewtArray := proc(ar,p)
    local newt,ad,mm,nn;
    newt := {}:

    ad := [ArrayDims(posCF)];

    for mm in `$`(ad[1]) do
    for nn in `$`(ad[2]) do
       if ar[mm,nn]<>0 then
          newt := {op(newt), [mm,-nn/p]};
       end;
    end:
    end:

    return newt;
  end:
> getArray := proc(ar,p,a,b)
    return ar[a,-b*p];
  end:
```

Shorthands

The largest root of the Airy funciton Ai(z)

```
> A1 := AiryAiZeros(1);
  evalf(%);
```

$$A1 := \mathrm{AiryAiZeros}(1)$$

$$-2.338107410 \tag{2.3.1}$$

This is the constant c in
s(n) = 2 + c*n^(-2/3) + ...

and will be responsible for the base of the stretched exponential

```
> csubs := c=a1*(K - 1)^(1/3)*K/2^(1/3);
```

$$csubs := c = \frac{a1\ (K-1)^{1/3}\ K\ 2^{2/3}}{2} \tag{2.3.2}$$

Labeling options to produce nice plots.

```
> myoptionsLo := labels=["i", "j"],symbolsize=25, symbol=diamond,
  axesfont = ["HELVETICA", "ROMAN", 15],labelfont = ["HELVETICA",
  18]:
  myoptionsUp := labels=["i", "j"],symbolsize=20, symbol=diamond,
  axesfont = ["HELVETICA", "ROMAN", 15],labelfont = ["HELVETICA",
  18]:
```

```
> (*
  fchoice3:
  wsubs:
  normal(isolate(%,lambda)):
  subs(%,%%%);
  *)
```

We introduce the following shorthands for the Airy function and its derivative
Keep the factor L undetermined; we will see it has to be 2/(K-1); it actually rescales n

```
> kaplam := normal(AiryAi(AiryAiZeros(1)+(L)^(1/3)*m/n^(1/3)))=
  kappa, normal(AiryAi(1, AiryAiZeros(1)+(L)^(1/3)*m/n^(1/3)))=
  lambda;
```

$$kaplam := \mathrm{AiryAi}\left(\frac{\mathrm{AiryAiZeros}(1)\ n^{1/3} + L^{1/3}\ m}{n^{1/3}}\right) = \kappa,\ \mathrm{AiryAi}\left(1, \frac{\mathrm{AiryAiZeros}(1)\ n^{1/3} + L^{1/3}\ m}{n^{1/3}}\right) = \lambda \tag{2.3.3}$$

```
> eerec;
```

$$ee_{\_n,\ \_m} = \frac{(K-1)^2\ (K + \_n - \_m)\ ee_{\_n-1,\ \_m-1}}{(K-1)\ \_n + \_m} + ee_{\_n-1,\ \_m+K-1} \tag{2.3.4}$$

The factor (coefficient) of ee[n-1,m-1]

```
> fac1 := (K - 1)^2*(K + n - m)/((K - 1)*n + m);
```

$$fac1 := \frac{(K-1)^2\ (K+n-m)}{(K-1)\ n + m} \tag{2.3.5}$$

The factor (coefficient) of ee[n-1,m+K-1]

```
> fac2 := 1;
```

$$fac2 := 1 \tag{2.3.6}$$

Expansions

```
> n:='n':m:='m':
```

We expand the Airy function around a1+(L)^(1/3)*m/n^(1/3) up to chosen order ordAi

```
> FFy := AiryAi(AiryAiZeros(1)+(L)^(1/3)*m/n^(1/3)+y);
```

$$FFy := \mathrm{AiryAi}\left(\mathrm{AiryAiZeros}(1) + \frac{L^{1/3} m}{n^{1/3}} + y\right) \quad \textbf{(2.4.1)}$$

We use two different expansion orders for the upper and lower bound
(to speed up the computations, and to produce the best pictures)
Here, we start with the **lower bound**, which needs less terms

```
> ordAiLo := 19;
  FFyserLo := map(expand,series(FFy,y,ordAiLo)):
```

$$ordAiLo := 19 \quad \textbf{(2.4.2)}$$

For y = x-(a1+L*m/n^(1/3)) we have that FF(x)=Ai(x),
i.e. an expansion of the Airy function

```
> FFxserLo := map(normal,subs(y=x-(AiryAiZeros(1)+(L)^(1/3)*m/n^
  (1/3)),FFyserLo)):
```

```
> indets(FFxserLo):
  map(x->subs(n=1,diff(x,n)),select(has,remove(has,%,L),n)):
  NordLo := min(%);
```

$$NordLo := -\frac{17}{3} \quad \textbf{(2.4.3)}$$

Replace the appearing Airy functions by our shorthands kappa and lambda

```
> indets(FFxserLo);
  FFxserLoKL := subs(kaplam,AiryAiZeros(1)=a1,FFxserLo):
  indets(%);
```

$$\left\{L, m, n, x, L^{1/3}, L^{2/3}, L^{4/3}, L^{5/3}, L^{7/3}, L^{8/3}, \frac{1}{n^{17/3}}, \frac{1}{n^{14/3}}, \frac{1}{n^{13/3}}, \frac{1}{n^{11/3}}, \frac{1}{n^{8/3}}, \frac{1}{n^{7/3}}, \frac{1}{n^{5/3}}, \frac{1}{n^{2/3}}, \frac{1}{n^{1/3}}, n^{1/3}, n^{2/3}, n^{4/3}, n^{5/3}, n^{7/3}, n^{8/3}, n^{10/3}, n^{11/3}, n^{13/3}, n^{14/3}, n^{16/3}, n^{17/3}, \mathrm{AiryAi}\left(\frac{\mathrm{AiryAiZeros}(1)\, n^{1/3} + L^{1/3} m}{n^{1/3}}\right), \mathrm{AiryAi}\left(1, \frac{\mathrm{AiryAiZeros}(1)\, n^{1/3} + L^{1/3} m}{n^{1/3}}\right)\right\}$$

$$\left\{L, a1, \kappa, \lambda, m, n, x, L^{1/3}, L^{2/3}, L^{4/3}, L^{5/3}, L^{7/3}, L^{8/3}, \frac{1}{n^{17/3}}, \frac{1}{n^{14/3}}, \frac{1}{n^{13/3}}, \frac{1}{n^{11/3}}, \frac{1}{n^{8/3}}, \frac{1}{n^{7/3}}, \frac{1}{n^{5/3}}, \frac{1}{n^{2/3}}, \frac{1}{n^{1/3}}, n^{1/3}, n^{2/3}, n^{4/3}, n^{5/3}, n^{7/3}, n^{8/3}, n^{10/3}, n^{11/3}, n^{13/3}, n^{14/3}, n^{16/3}, n^{17/3}\right\} \quad \textbf{(2.4.4)}$$

```
> #indets(FFxserLos);
  #FFxserLosKL := subs(kaplam,AiryAiZeros(1)=a1,FFxserLos):
  #indets(%);
```

Then, we use the generic ansatz for factor of FFy, i.e. the Airy function

```
> ## general ansatz to determine the constants ##
  ## in order to get better bounds, one would need to kill more
  terms in the Newton polygons ##
  # facAiryLo := 1+(add(q[i]*m^i,i=0..2))/(n) + add(qq[i]*m^i,i=0.
  .1)/(n^(2/3))
  #              + add(ss[i]*m^i,i=0..2)/(n^(4/3)) + add(rr[i]*m^i,
```

```
i=0..4)/(n^(5/3)) + add(uu[i]*m^i,i=0..4)/(n^(6/3));

## we will only need this ##
## we only need ss[2] and rr[3] in the end ##
facAiryLo := 1+(add(q[i]*m^i,i=0..2))/(n) + add(qq[i]*m^i,i=0..1)
/(n^(2/3))
                + add(ss[i]*m^i,i=2..2)/(n^(4/3)) + add(rr[i]*m^i,
i=3..3)/(n^(5/3));
```

$$facAiryLo := 1 + \frac{m^2 q_2 + m q_1 + q_0}{n} + \frac{m\, qq_1 + qq_0}{n^{2/3}} + \frac{ss_2\, m^2}{n^{4/3}} + \frac{rr_3\, m^3}{n^{5/3}} \tag{2.4.5}$$

This is our ansatz
(only the substitution is influenced by the parameters, i.e. not the ms and ns that are already in FFxser; that is what we want, as all of them should be expanded at a1+(L)^(1/3)*m/n(1/3))
Note that due to the replacement with kappa and lambda, the expansions are fixed already at this point and the ms and ns in the arguments of Ai and Ai' are not influenced.
Note that we substitute now a1+(L)^(1/3)*(m+1)/n^(1/3), i.e. m+1 instead of m around which we expanded above.

```
> XFL := (n0,m0) -> subs(n=n0,m=m0,facAiryLo)*subs(x=a1+(L)^(1/3)*
  (m0+1)/n0^(1/3),FFxserLoKL);
```

$$XFL := (n0, m0) \mapsto subs(n = n0, m = m0, facAiryLo) \cdot subs\left(x = a1 + \frac{L^{1/3} \cdot (m0 + 1)}{n0^{1/3}},\ FFxserLoKL\right) \tag{2.4.6}$$

Then, we do the same for the **upper bound**, with a few more terms

```
> ordAiUp := 30;
  FFyserUp := map(expand,series(FFy,y,ordAiUp)):
```

$$ordAiUp := 30 \tag{2.4.7}$$

For y = x-(a1+(L)^(1/3)*m/n^(1/3)) we have that FF(x)=Ai(x),
i.e. an expansion of the Airy function

```
> FFxserUp := normal(subs(y=x-(AiryAiZeros(1)+(L)^(1/3)*m/n^(1/3)),
  FFyserUp)):
```

Compute the real expansion order in n

```
> indets(FFxserUp):
  map(x->subs(n=1,diff(x,n)),select(has,remove(has,%,m),n)):
  NordUp := min(%);
```

$$NordUp := -\frac{26}{3} \tag{2.4.8}$$

```
> #kaplam;
```

Replace the appearing Airy functions by our shorthands kappa and lambda

```
> indets(FFxserUp);
  FFxserUpKL := subs(kaplam,AiryAiZeros(1)=a1,FFxserUp):
  indets(%);
```

$$\left\{L, m, n, x, L^{1/3}, L^{2/3}, L^{4/3}, L^{5/3}, L^{7/3}, L^{8/3}, L^{10/3}, L^{11/3}, L^{13/3}, L^{14/3}, \frac{1}{n^{26/3}},\right.$$

$\frac{1}{n^{25/3}}, \frac{1}{n^{23/3}}, \frac{1}{n^{20/3}}, \frac{1}{n^{19/3}}, \frac{1}{n^{17/3}}, \frac{1}{n^{14/3}}, \frac{1}{n^{13/3}}, \frac{1}{n^{11/3}}, \frac{1}{n^{8/3}}, \frac{1}{n^{7/3}},$

$\frac{1}{n^{5/3}}, \frac{1}{n^{2/3}}, \frac{1}{n^{1/3}}, n^{1/3}, n^{2/3}, n^{4/3}, n^{5/3}, n^{7/3}, n^{8/3}, n^{10/3}, n^{11/3}, n^{13/3},$

$n^{14/3}, n^{16/3}, n^{17/3}, n^{19/3}, n^{20/3}, n^{22/3}, n^{23/3}, n^{25/3}, n^{26/3},$

$\mathrm{AiryAi}\left(\frac{\mathrm{AiryAiZeros}(1)\, n^{1/3} + L^{1/3} m}{n^{1/3}}\right), \mathrm{AiryAi}\left(1, \frac{\mathrm{AiryAiZeros}(1)\, n^{1/3} + L^{1/3} m}{n^{1/3}}\right)\Bigg\}$

$$\Big\{L, a1, \kappa, \lambda, m, n, x, L^{1/3}, L^{2/3}, L^{4/3}, L^{5/3}, L^{7/3}, L^{8/3}, L^{10/3}, L^{11/3}, L^{13/3}, L^{14/3}, \frac{1}{n^{26/3}}, \frac{1}{n^{25/3}}, \frac{1}{n^{23/3}}, \frac{1}{n^{20/3}}, \frac{1}{n^{19/3}}, \frac{1}{n^{17/3}}, \frac{1}{n^{14/3}}, \frac{1}{n^{13/3}}, \frac{1}{n^{11/3}}, \frac{1}{n^{8/3}}, \frac{1}{n^{7/3}}, \frac{1}{n^{5/3}}, \frac{1}{n^{2/3}}, \frac{1}{n^{1/3}}, n^{1/3}, n^{2/3}, n^{4/3}, n^{5/3}, n^{7/3}, n^{8/3}, n^{10/3}, n^{11/3}, n^{13/3}, n^{14/3}, n^{16/3}, n^{17/3}, n^{19/3}, n^{20/3}, n^{22/3}, n^{23/3}, n^{25/3}, n^{26/3}\Big\} \tag{2.4.9}$$

In the upper bound, we use a extend the lower bound ansatz by additional terms p[i] which are divided by n^2

```
> facAiryUp := facAiryLo + (add(p[i]*m^i,i=0..4))/(n^2);
```

$$facAiryUp := 1 + \frac{m^2 q_2 + m q_1 + q_0}{n} + \frac{m\, qq_1 + qq_0}{n^{2/3}} + \frac{ss_2 m^2}{n^{4/3}} + \frac{rr_3 m^3}{n^{5/3}} + \frac{p_4 m^4 + p_3 m^3 + p_2 m^2 + p_1 m + p_0}{n^2} \tag{2.4.10}$$

This is our ansatz for the upper bound (comparable to XFL)
(only the substitution is influenced by the parameters, i.e. not the ms and ns that are already in FFxser; that is what we want, as all of them should be expanded at a1+(L)^(1/3)*m/n^(1/3))

```
> XFU := (n0,m0) -> subs(n=n0,m=m0,facAiryUp)*subs(x=a1+(L)^(1/3)*
  (m0+1)/n0^(1/3),FFxserUpKL);
```

$$XFU := (n0, m0) \mapsto subs(n = n0, m = m0, facAiryUp) \cdot subs\left(x = a1 + \frac{L^{1/3} \cdot (m0 + 1)}{n0^{1/3}}, FFxserUpKL\right) \tag{2.4.11}$$

```
> A1;
```

$$\mathrm{AiryAiZeros}(1) \tag{2.4.12}$$

```
> series(AiryAi(A1+x),x=0,2);
  evalf(%);
```

$$\mathrm{AiryAi}(1, \mathrm{AiryAiZeros}(1))\, x + \mathrm{O}(x^2)$$

$$0.7012108227\, x + \mathrm{O}\left(x^2\right) \qquad \mathbf{(2.4.13)}$$

Finally, the ansatz for the **quotient** of h(n)/h(n-1).
Note that pterm is a mnemonic for "polynomial term", as this value influences the polynomial term n^{alpha};
The other values have similar interpretations:
a .... exponantial growth, i.e. a^n
b .... b will be zero
c .... stretched exponential
pterm .... related to the critical exponent in polynomial term in asymptotics
dup .... technical choice, to simplify the proofs

```
> SF := n -> a+b/n^(1/3)+c/n^(2/3)+pterm/n+d/n^(7/6);
```

$$SF := n \mapsto a + \frac{b}{n^{1/3}} + \frac{c}{n^{2/3}} + \frac{pterm}{n} + \frac{d}{n^{7/6}} \qquad \mathbf{(2.4.14)}$$

Gen TCs - Lower bound

lower bound:
We will prove that this is positive

```
> i:='i':
  posansatz := -XX(n,m)*SS(n) + Fac1*XX(n-1,m-1) + Fac2*XX(n-1,m+K-1);
  subs(Fac1=fac1,Fac2=fac2,%);
```

$$posansatz := -XX(n, m)\, SS(n) + Fac1\, XX(n-1, m-1) + Fac2\, XX(n-1, m+K-1)$$

$$-XX(n, m)\, SS(n) + \frac{(K-1)^2\, (K+n-m)\, XX(n-1, m-1)}{(K-1)\, n + m} + XX(n-1, m+K-1) \qquad \mathbf{(2.5.1)}$$

Xtilde : specific values from proof below for experiments

From summary after proof below

```
> varsLowerBound :=
   [q[0] = 0, # arbitrary; not needed
  q[1] = (7*K - 11)/(6*(K - 1)),
  q[2] = -(K+2)/6/(K-1),
  qq[0] = 0,    # arbitrary, not needed
  qq[1] = -a1*(K - 2)*2^(2/3)/(6*(K - 1)^(2/3)),
  L=2/(K-1),
  a=K,
  b=0,
  csubs,
  pterm = K*(7*K-6)/6,
  d=-1
  ];
```

$$varsLowerBound := \left[q_0 = 0,\, q_1 = \frac{7\,K - 11}{6\,K - 6},\, q_2 = -\frac{K+2}{6\,(K-1)},\, qq_0 = 0,\, qq_1 = \right. \qquad \mathbf{(2.5.1.1)}$$

$$-\frac{a1\ (K-2)\ 2^{2/3}}{6\ (K-1)^{2/3}}, L=\frac{2}{K-1}, a=K, b=0, c=\frac{a1\ (K-1)^{1/3}\ K\ 2^{2/3}}{2}, pterm =\frac{K\ (7\ K-6)}{6}, d=-1\Big]$$

Fix K for experiments

```
> KK := 3;
```

$$KK := 3 \tag{2.5.1.2}$$

This is X tilde (this is derived later)

```
> subs(varsLowerBound,a1=A1,K=KK,(1 + (m^2*q[2] + m*q[1] + q[0])/n
  + (m*qq[1] + qq[0])/n^(2/3))*AiryAi(a1 + L^(1/3)*(m + 1)/n^(1/3))
  ):
  Xansatz := (n,m) -> subs(K=KK,((1 + (-m^2*(K + 2)/(6*(K - 1)) +
  m*(7*K - 11)/(6*K - 6))/n - m*AiryAiZeros(1)*(K - 2)*2^(2/3)/(6*
  (K - 1)^(2/3)*n^(2/3)))*AiryAi(AiryAiZeros(1) + 2^(1/3)*(1/(K -
  1))^(1/3)*(m + 1)/n^(1/3))));
```

$$Xansatz := (n, m) \mapsto subs\left(K=KK, \left(1+\frac{-\frac{m^2\cdot(K+2)}{6\cdot K-6}+\frac{m\cdot(7\cdot K-11)}{6\cdot K-6}}{n} - \frac{m\cdot \mathrm{AiryAiZeros}(1)\cdot(K-2)\cdot 2^{2/3}}{6\cdot(K-1)^{2/3}\cdot n^{2/3}}\right)\cdot \mathrm{AiryAi}\left(\mathrm{AiryAiZeros}(1) + \frac{2^{1/3}\cdot\left(\frac{1}{K-1}\right)^{1/3}\cdot(m+1)}{n^{1/3}}\right)\right) \tag{2.5.1.3}$$

(this is derived later)

```
> subs(varsLowerBound,a1=A1,a + b/n^(1/3) + c/n^(2/3) + pterm/n +
  d/n^(7/6));
  Sansatz := n -> subs(K=KK,K + AiryAiZeros(1)*(K - 1)^(1/3)*K*2^
  (2/3)/(2*n^(2/3)) + K*(7*K - 6)/(6*n) - 1/n^(7/6));
```

$$K+\frac{\mathrm{AiryAiZeros}(1)\ (K-1)^{1/3}\ K\ 2^{2/3}}{2\ n^{2/3}}+\frac{K\ (7\ K-6)}{6\ n}-\frac{1}{n^{7/6}}$$

$$Sansatz := n \mapsto subs\left(K=KK, K+\frac{\mathrm{AiryAiZeros}(1)\cdot(K-1)^{1/3}\cdot K\cdot 2^{2/3}}{2\cdot n^{2/3}}+\frac{K\cdot(7\cdot K-6)}{6\cdot n}-\frac{1}{n^{7/6}}\right) \tag{2.5.1.4}$$

```
> posXS := map(simplify,subs(XX=Xansatz,SS=Sansatz,Fac1=fac1,Fac2=
  fac2,K=KK,posansatz)):indets(%);
```

$$\left\{m, n, \frac{1}{n^{7/6}}, \frac{1}{n^{2/3}}, \frac{1}{n^{1/3}}, \frac{1}{(n-1)^{2/3}}, \frac{1}{(n-1)^{1/3}}, \mathrm{AiryAi}\left(\mathrm{AiryAiZeros}(1) + \frac{m}{(n-1)^{1/3}}\right), \mathrm{AiryAi}\left(\mathrm{AiryAiZeros}(1)+\frac{m+1}{n^{1/3}}\right), \mathrm{AiryAi}\Big(\mathrm{AiryAiZeros}(1)\right. \tag{2.5.1.5}$$

$$+ \frac{m+3}{(n-1)^{1/3}}\Big)\Big\}$$

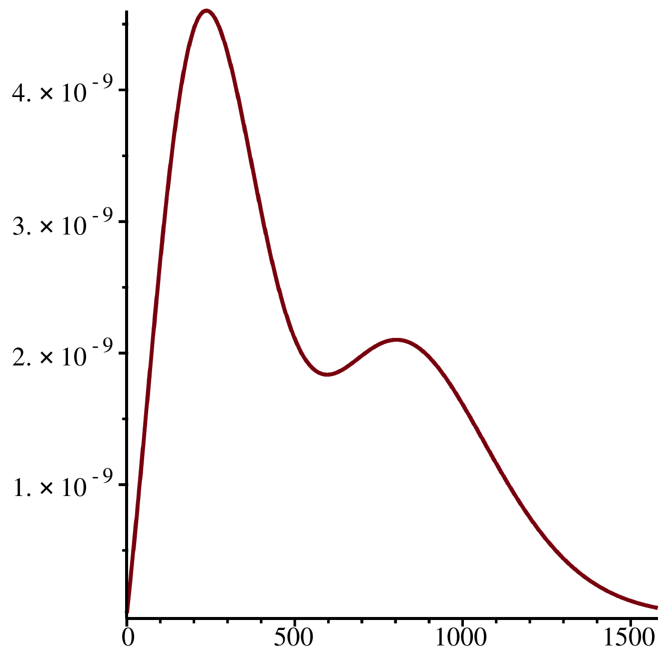
For a large n this function of m seems to be positive: looks good :)

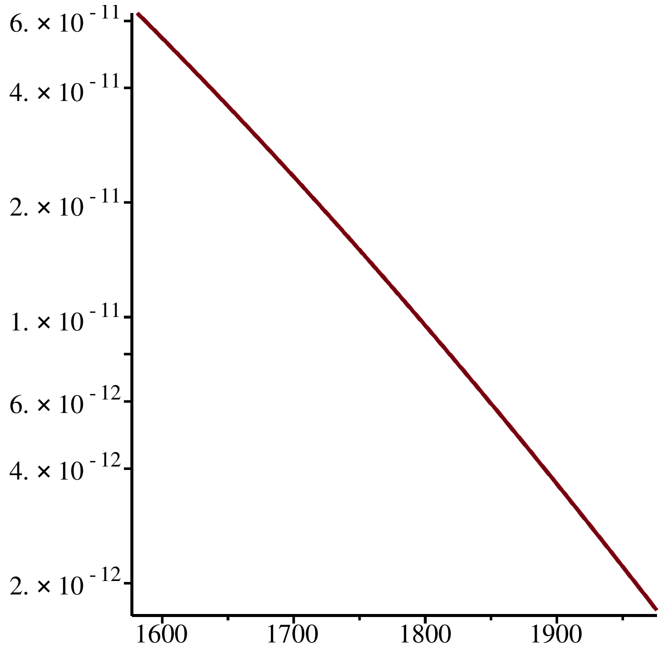
```
> Digits:=20:
  e1 := subs(csubs,a1=A1,pterm=1/6,posXS):indets(%);
  N := 10000000;
  M := floor(N^(1/2));
  P1 := plot([seq([mm,(subs(n=N,m=mm,e1))],mm=0..floor(M/2))])
  :display(P1);
  P2 := logplot([seq([mm,(subs(n=N,m=mm,e1))],mm=floor(M/2)..floor
  (5*M/8))]):display(P2);
```

$$\Big\{m, n, \frac{1}{n^{7/6}}, \frac{1}{n^{2/3}}, \frac{1}{n^{1/3}}, \frac{1}{(n-1)^{2/3}}, \frac{1}{(n-1)^{1/3}}, \mathrm{AiryAi}\Big(\mathrm{AiryAiZeros}(1) + \frac{m}{(n-1)^{1/3}}\Big), \mathrm{AiryAi}\Big(\mathrm{AiryAiZeros}(1) + \frac{m+1}{n^{1/3}}\Big), \mathrm{AiryAi}\Big(\mathrm{AiryAiZeros}(1) + \frac{m+3}{(n-1)^{1/3}}\Big)\Big\}$$

$$N := 10000000$$

$$M := 3162$$





Prove it

Recall the general ansatz

```
> facAiryLo*AiryAi(a1+(L)^(1/3)*(m+1)/n^(1/3));
  SF(n);
```

$$\left(1+\frac{q_2 m^2+q_1 m+q_0}{n}+\frac{m\,qq_1+qq_0}{n^{2/3}}+\frac{ss_2 m^2}{n^{4/3}}+\frac{rr_3 m^3}{n^{5/3}}\right)\mathrm{AiryAi}\left(a1+\frac{L^{1/3}(m+1)}{n^{1/3}}\right)$$

$$a+\frac{b}{n^{1/3}}+\frac{c}{n^{2/3}}+\frac{pterm}{n}+\frac{d}{n^{7/6}} \qquad \textbf{(2.5.2.1)}$$

```
> subs(Fac1=fac1,Fac2=fac2,K=2,posansatz);
```

$$-XX(n,m)\,SS(n)+\frac{(2+n-m)\,XX(n-1,m-1)}{n+m}+XX(n-1,m+1) \qquad \textbf{(2.5.2.2)}$$

Substitute ansatz into the sequence we want to be positive for large n and all m

```
> posF := map(expand, subs(XX=XFL,SS=SF,Fac1=fac1,Fac2=fac2,
  posansatz)):indets(%),nops(expand(%));
```

$$\left\{K, L, a, a1, b, c, d, \kappa, \lambda, m, n, pterm, q_0, q_1, q_2, qq_0, qq_1, rr_3, ss_2, L^{1/3}, L^{2/3}, L^{4/3}, L^{5/3}, L^{7/3}, L^{8/3}, \frac{1}{n^{17/3}}, \frac{1}{n^{14/3}}, \frac{1}{n^{13/3}}, \frac{1}{n^{11/3}}, \frac{1}{n^{8/3}}, \frac{1}{n^{7/3}}, \frac{1}{n^{7/6}}, \frac{1}{n^{5/3}}, \frac{1}{n^{4/3}}, \frac{1}{n^{2/3}}, \frac{1}{n^{1/3}}, n^{1/3}, n^{2/3}, n^{4/3}, n^{5/3}, n^{7/3}, n^{8/3}, n^{10/3}, n^{11/3}, n^{13/3}, n^{14/3}, n^{16/3}, n^{17/3}, \frac{1}{(n-1)^{5/3}}, \frac{1}{(n-1)^{4/3}}, \frac{1}{(n-1)^{2/3}}, \frac{1}{(n-1)^{1/3}}\right\}, 11162 \qquad \textbf{(2.5.2.3)}$$

```
> #subs(varsLowerBound, facAiryLo);
  #Sansatz := n -> K + AiryAiZeros(1)*(K - 1)^(1/3)*K*2^(2/3)/(2*n^
  (2/3)) + K*(7*K - 6)/(6*n) - 1/n^(7/6);
> #XFLtest := (n0,m0) -> subs(n=n0,m=m0,1 + (-m^2*(K + 2)/(6*(K -
  1)) + m*(7*K - 11)/(6*K - 6))/n - m*a1*(K - 2)*2^(2/3)/(6*(K - 1)
  ^(2/3)*n^(2/3)))*subs(x=a1+(L)^(1/3)*(m0+1)/n0^(1/3),FFxserLoKL);
> #posFXtest := map(expand, subs(XX=XFLtest,SS=Sansatz,Fac1=fac1,
  Fac2=fac2,posansatz)):
> #tmp := series(subs(K=K1+1,posFXtest),n=infinity,3) assuming
  K1>0,n::posint,m::posint:
> #tmp2 := collect(convert(tmp,polynom),[n,m],simplify) assuming
  K1>0,n::posint,m::posint:
> #series(tmp2,n=infinity,2);
```

The error terms are (to check, expand posF)

UPDATE

```
> simplify(O((2^(1/3)*(m+1)/n^(1/3)-2^(1/3)*m/n^(1/3))^ordAiLo));
  simplify(O((2^(1/3)*(m)/(n-1)^(1/3)-2^(1/3)*m/n^(1/3))^ordAiLo));
  simplify(O((2^(1/3)*(m+2)/(n-1)^(1/3)-2^(1/3)*m/n^(1/3))^ordAiLo)
  );
```

$$O\left(\frac{64\,2^{1/3}}{n^{19/3}}\right)$$

$$O\left(\frac{64\,2^{1/3}\,m^{19}\left(n^{1/3}-(n-1)^{1/3}\right)^{19}}{(n-1)^{19/3}\,n^{19/3}}\right)$$

$$O\left(\frac{64\,2^{1/3}\left(-m\,(n-1)^{1/3}+n^{1/3}\,(m+2)\right)^{19}}{n^{19/3}\,(n-1)^{19/3}}\right) \quad \textbf{(2.5.2.4)}$$

remove error terms

```
> posFd := convert(posF,polynom):indets(%);
```

$$\left\{K, L, a, a1, b, c, d, \kappa, \lambda, m, n, pterm, q_0, q_1, q_2, qq_0, qq_1, rr_3, ss_2, L^{1/3}, L^{2/3}, L^{4/3}, L^{5/3}, L^{7/3}, L^{8/3}, \frac{1}{n^{17/3}}, \frac{1}{n^{14/3}}, \frac{1}{n^{13/3}}, \frac{1}{n^{11/3}}, \frac{1}{n^{8/3}}, \frac{1}{n^{7/3}}, \frac{1}{n^{7/6}}, \frac{1}{n^{5/3}}, \frac{1}{n^{4/3}}, \frac{1}{n^{2/3}}, \frac{1}{n^{1/3}}, n^{1/3}, n^{2/3}, n^{4/3}, n^{5/3}, n^{7/3}, n^{8/3}, n^{10/3}, n^{11/3}, n^{13/3}, n^{14/3}, n^{16/3}, n^{17/3}, \frac{1}{(n-1)^{5/3}}, \frac{1}{(n-1)^{4/3}}, \frac{1}{(n-1)^{2/3}}, \frac{1}{(n-1)^{1/3}}\right\} \quad \textbf{(2.5.2.5)}$$

Next we fit the display size of the plots in the n (Nord) and m (Mord) direction.
(Note that everything up to ordAi is computed, but possibly not shown)

```
> Nord := -ordAiLo/3;
  Mord := floor(ordAiLo/3);
  myview := view=[0..Mord,Nord..0]:
```

$$Nord := -\frac{19}{3}$$

$$Mord := 6 \quad \textbf{(2.5.2.6)}$$

```
> Nord := -Mord;
```

(2.5.2.7)

$$Nord := -6 \qquad \textbf{(2.5.2.7)}$$

Expand again with respect to n,
these are then our unknowns

```
> posFe := series(posFd,n=infinity,ceil(-Nord)+1):indets(%);
  posFf := convert(%%,polynom):
  nops(expand(%));
```

$$\left\{K, L, a, a1, b, c, d, \kappa, \lambda, m, n, pterm, q_0, q_1, q_2, qq_0, qq_1, rr_3, ss_2, L^{1/3}, L^{2/3}, L^{4/3}, L^{5/3}, L^{7/3}, L^{8/3}, L^{10/3}, L^{11/3}, L^{13/3}, L^{14/3}, L^{16/3}, L^{17/3}, \left(\frac{1}{n}\right)^{1/3}, \left(\frac{1}{n}\right)^{2/3}, \left(\frac{1}{n}\right)^{3/2}, \left(\frac{1}{n}\right)^{4/3}, \left(\frac{1}{n}\right)^{5/2}, \left(\frac{1}{n}\right)^{5/3}, \left(\frac{1}{n}\right)^{7/2}, \left(\frac{1}{n}\right)^{7/3}, \left(\frac{1}{n}\right)^{7/6}, \left(\frac{1}{n}\right)^{8/3}, \left(\frac{1}{n}\right)^{9/2}, \left(\frac{1}{n}\right)^{10/3}, \left(\frac{1}{n}\right)^{11/2}, \left(\frac{1}{n}\right)^{11/3}, \left(\frac{1}{n}\right)^{11/6}, \left(\frac{1}{n}\right)^{13/3}, \left(\frac{1}{n}\right)^{13/6}, \left(\frac{1}{n}\right)^{14/3}, \left(\frac{1}{n}\right)^{16/3}, \left(\frac{1}{n}\right)^{17/3}, \left(\frac{1}{n}\right)^{17/6}, \left(\frac{1}{n}\right)^{19/6}, \left(\frac{1}{n}\right)^{23/6}, \left(\frac{1}{n}\right)^{25/6}, \left(\frac{1}{n}\right)^{29/6}, \left(\frac{1}{n}\right)^{31/6}, \left(\frac{1}{n}\right)^{35/6}, \mathrm{O}\left(\frac{1}{n^6}\right)\right\}$$

$$12252 \qquad \textbf{(2.5.2.8)}$$

```
> #map(simplify,series(convert(coeff(subs(varsLowerBound,K=K1+1,
  posFf),rr[1]),polynom),n=infinity,3)) assuming K1>0;
> #map(simplify,series(convert(coeff(subs(varsLowerBound,K=K1+1,
  posFf),ss[2]),polynom),n=infinity,3)) assuming K1>0;
```

lcm of denominators in exponents

```
> PP := 6;
```

$$PP := 6 \qquad \textbf{(2.5.2.9)}$$

convert the expansion into a list with respect to the degrees in m and n:
Format: [[mdeg,ndeg], coeff]

```
> posCF := ex2Array(posFf,Mord,-Nord,PP):ArrayNumElems(posCF);
```

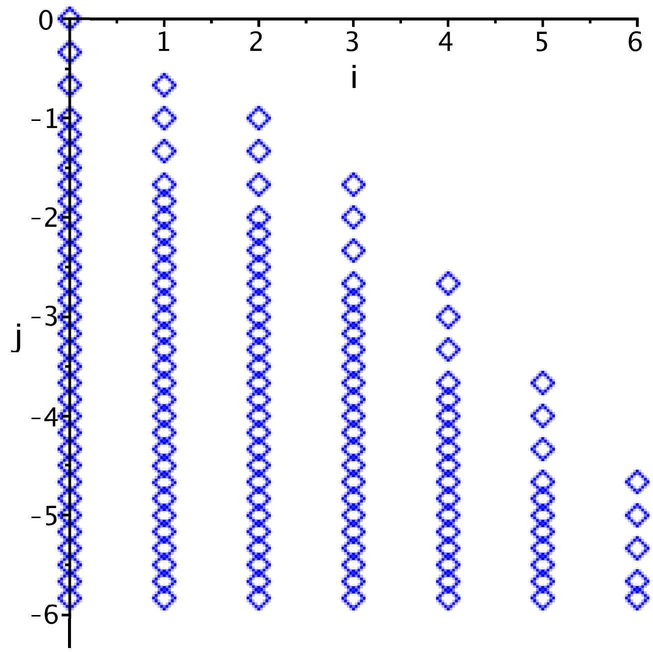
$$259 \qquad \textbf{(2.5.2.10)}$$

The mynewt function computes the Newton polygon of posFf

```
> newt1 := mynewtArray(posCF,PP):
```

First Newton polygon, where no unknowns have been fixed
Looks good, as usual!

```
> P1 := pointplot(newt1,myoptionsLo,color=blue):
  display(P1,myview);
```



Here, we want to kill the element (0,0)

```
> posCF[0,0];
  simplify(%);
```

$$\kappa - \frac{2\,\kappa\,K}{K-1} + \frac{\kappa\,K^2}{K-1} + \frac{\kappa}{K-1} - a\,\kappa$$

$$(K-a)\,\kappa \tag{2.5.2.11}$$

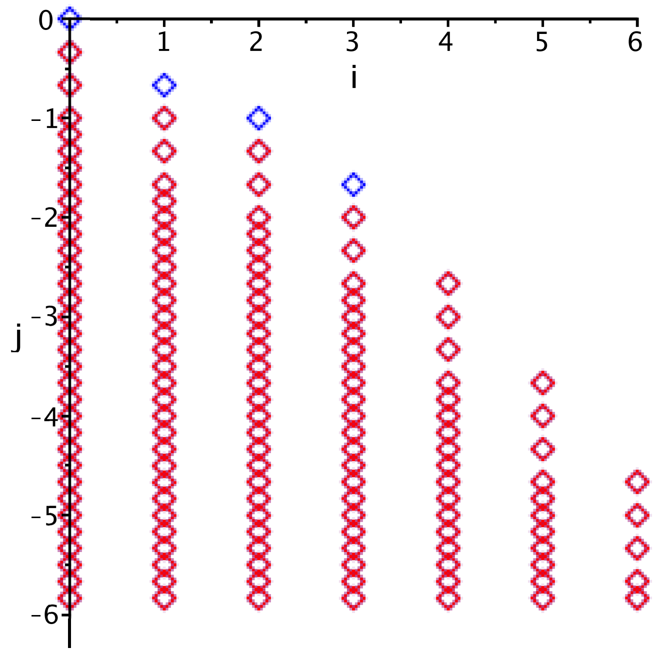
Set a=K

```
> posCFa := map(factor,subs(a=K,posCF)):
```

```
> newta := mynewtArray(posCFa,PP):
```

All blue points have been eliminated, and only the red ones remain:

```
> P1a := pointplot(newta,myoptionsLo,color=red):
  display(P1,P1a,myview);
```



These should all be zero:
we set b=0

```
> getArray(posCFa,PP,0,-1/3):
  collect(%,[kappa,lambda],factor);
```

$$-\kappa b \tag{2.5.2.12}$$

L = 2/(K-1)

```
> getArray(posCFa,PP,1,-1):
  collect(subs(b=0,qq[2]=0,qqs[2]=0,%),[kappa,lambda],
  factor@simplify) assuming K::posint;
  solve(%,L);
```

$$\frac{K\,(L\,K-L-2)\,\kappa}{2}$$

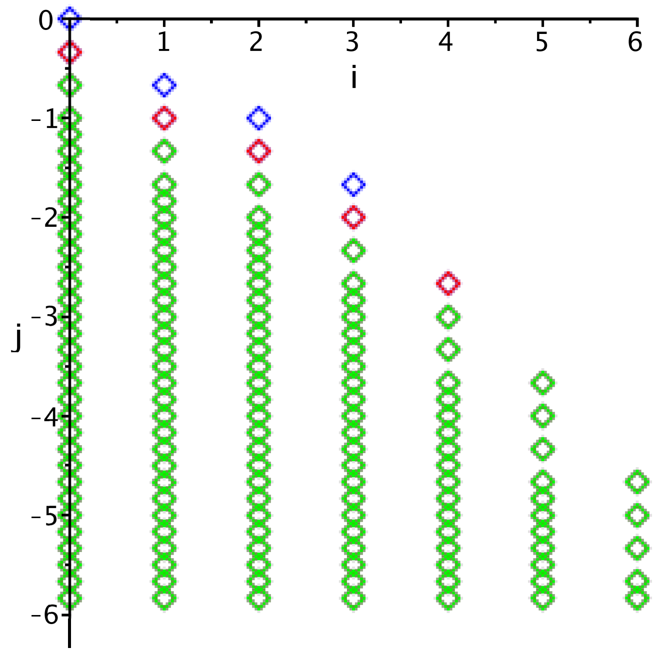
$$\frac{2}{K-1} \tag{2.5.2.13}$$

set a=K, b=0, L=2/(K-1)

```
> posCFab := map(factor,subs(b=0,L=2/(K-1),
  posCFa)):
> newtab := mynewtArray(posCFab,PP):
```

Now only the green points remain.

```
> P1ab := pointplot(newtab,myoptionsLo,color=green):
  display(P1,P1a,P1ab,myview);
```



at this point we find our choice for c, which we heuristically computed already before in Section 3
also possible here to set q[2]=-(K+2)/6/(K-1)

```
> csubs;
  simplify(getArray(posCFab,PP,0,-2/3));
  collect(%,[lambda,kappa],factor@simplify) assuming K::posint;
  simplify(subs(csubs,%)) assuming K::posint;
```

$$c = \frac{a1\ (K-1)^{1/3} K\, 2^{2/3}}{2}$$

$$\frac{\left(K\, a1\, 2^{2/3} (K-1) \left(\frac{1}{K-1}\right)^{2/3} - 2\, c\right) \kappa}{2}$$

$$\left(\frac{a1\ (K-1)^{1/3} K\, 2^{2/3}}{2} - c\right) \kappa$$

$$0 \qquad \textbf{(2.5.2.14)}$$

choose q[2]

```
> simplify(getArray(posCFab,PP,1,-4/3)):
  collect(subs(csubs,%),[lambda,kappa],factor@simplify) assuming
  K::posint;
  isolate(%,q[2]);
```

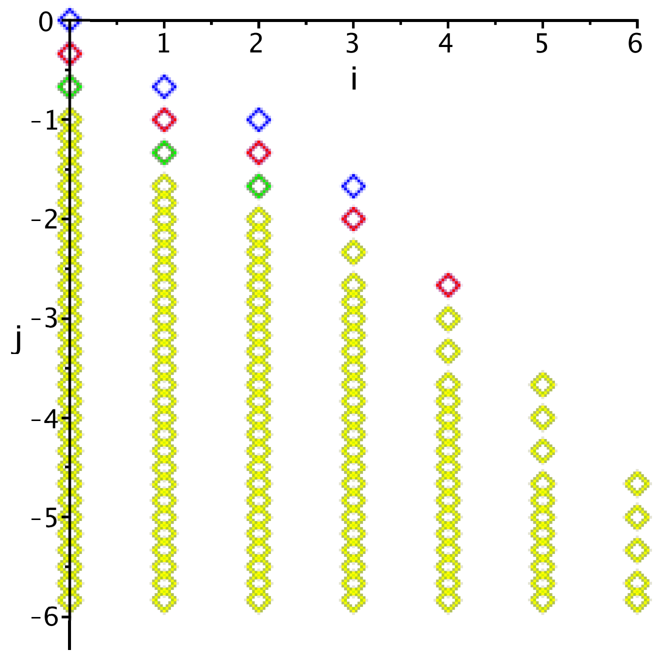
$$\frac{2^{1/3} K \left(6\, K\, q_2 + K - 6\, q_2 + 2\right) \lambda}{3\ (K-1)^{1/3}}$$

$$q_2 = \frac{-K-2}{6\, K - 6} \qquad \textbf{(2.5.2.15)}$$

set a=K, b=0, L=2/(K-1),c=a1*K*((K-1)/2)^(1/3),q[2]=-(K+2)/6/(K-1)

```
> posCFabc := map(simplify@factor,subs(csubs,q[2]=-(K+2)/6/(K-1),
  posCFab)) assuming K::posint:
> newtabc := mynewtArray(posCFabc,PP):
```

```
> P1abc := pointplot(newtabc,myoptionsLo,color=yellow):
  display(P1,P1a,P1ab,P1abc,myview);
```



Here we get pterm

Interesting: the qq[1] term is fixed here and is not visible in the binary case! (K=2)

```
> factor(getArray(posCFabc,PP,0,-1)):
  collect(%,[lambda,kappa],factor@simplify) assuming K::posint;
  coeff(%,kappa);
  factor(isolate(%,pterm));
  coeff(%%%,lambda);
  factor(isolate(%,qq[1]));
```

$$\frac{K\left(3\,K\,2^{1/3}\,qq_1+K\,(K-1)^{1/3}\,a1-3\,2^{1/3}\,qq_1-2\,(K-1)^{1/3}\,a1\right)\lambda}{3\,(K-1)^{1/3}}+\left(\frac{7}{6}\,K^2-K-pterm\right)\kappa$$

$$\frac{7}{6}\,K^2-K-pterm$$

$$pterm=\frac{K\,(7\,K-6)}{6}$$

$$\frac{K\left(3\,K\,2^{1/3}\,qq_1+K\,(K-1)^{1/3}\,a1-3\,2^{1/3}\,qq_1-2\,(K-1)^{1/3}\,a1\right)}{3\,(K-1)^{1/3}}$$

$$qq_1=-\frac{a1\,(K-2)\,2^{2/3}}{6\,(K-1)^{2/3}} \qquad \textbf{(2.5.2.16)}$$

For the polynomial term we need this correction by n^(-1/3)

```
> kaplam;
  subs(m=1,n=2*n,AiryAi(AiryAiZeros(1) + (2/(K-1))^(1/3)*m/n^(1/3))
  );
```

```
map(simplify,series(%,n=infinity,1));
```

$$\mathrm{AiryAi}\left(\frac{\mathrm{AiryAiZeros}(1)\, n^{1/3}+L^{1/3} m}{n^{1/3}}\right)=\kappa, \mathrm{AiryAi}\left(1, \frac{\mathrm{AiryAiZeros}(1)\, n^{1/3}+L^{1/3} m}{n^{1/3}}\right)=\lambda$$

$$\mathrm{AiryAi}\left(\mathrm{AiryAiZeros}(1)+\frac{\left(\frac{1}{K-1}\right)^{1/3}}{n^{1/3}}\right)$$

$$\mathrm{AiryAi}(1, \mathrm{AiryAiZeros}(1))\left(\frac{1}{K-1}\right)^{1/3}\left(\frac{1}{n}\right)^{1/3}+\mathrm{O}\left(\frac{1}{n}\right) \quad \textbf{(2.5.2.17)}$$

Conjectured polynomial term !!!
Divide by K and correct by -1/3
fits our conecture :)

```
> factor((K*(7*K - 6)/6)/K-1/3);
  subs(K=2,%);  #binary case ok
```

$$\frac{7K}{6}-\frac{4}{3}$$

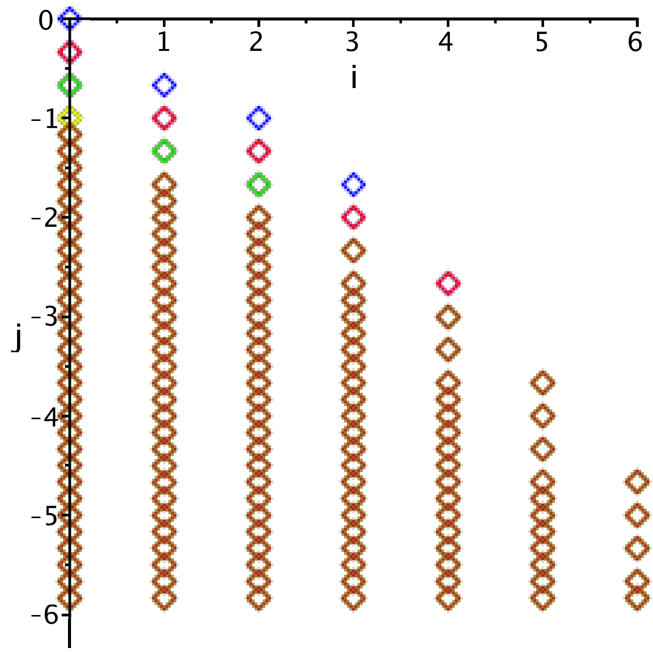
$$1 \quad \textbf{(2.5.2.18)}$$

set set a=K, b=0, L=2/(K-1),c=a1*K*((K-1)/2)^(1/3),q[2]=-(K+2)/6/(K-1), qq[1]=-a1*(K-2)*2^(2/3)/6/(K-1)^(2/3), pterm=K*(7*K-6)/6

```
> posCFabcd := map(simplify,subs(pterm=K*(7*K-6)/6,qq[1]=-a1*(K-2)*
  2^(2/3)/6/(K-1)^(2/3),posCFabc)) assuming K::posint:
> newtabcd := mynewtArray(posCFabcd,PP):
```

only the brown points remain
Now all points are strictly below n^{-1}

```
> P1abcd := pointplot(newtabcd,myoptionsLo,color=brown):
  display(P1,P1a,P1ab,P1abc,P1abcd,myview);
```



Here are the dominating corners and we see that we have to choose d=-1 to have a positive term; note that we will see that the second term should be negative, as lambda=Ai' is negative for large m

```
> getArray(posCFabcd,PP,0,-7/6);
  getArray(posCFabcd,PP,3,-14/6);
  (14/6-7/6)/3;   #slope
```

$$-\kappa d$$

$$-\frac{(K+2)^2 \lambda K 2^{1/3}}{18 (K-1)^{4/3}}$$

$$\frac{7}{18} \tag{2.5.2.19}$$

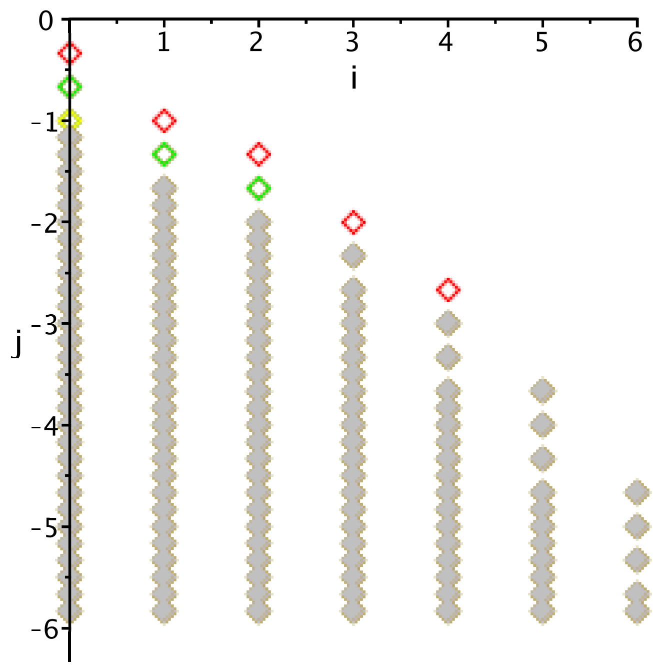
set a=K, b=0,c,pterm=..., and d=-1

```
> posCFabcde := subs(d=-1,rr[4]=0,uu[4]=0,posCFabcd):
> newtabcde := mynewtArray(posCFabcde,PP):
```

This is the final result, where only the solid diamonds are non-zero

```
> P1abcde := pointplot(newtabcde,myoptionsLo,symbol=soliddiamond,
  color=gray):
  display(P1,P1a,P1ab,P1abc,P1abcd,P1abcde,myview);
```

Warning, data could not be converted to float Matrix

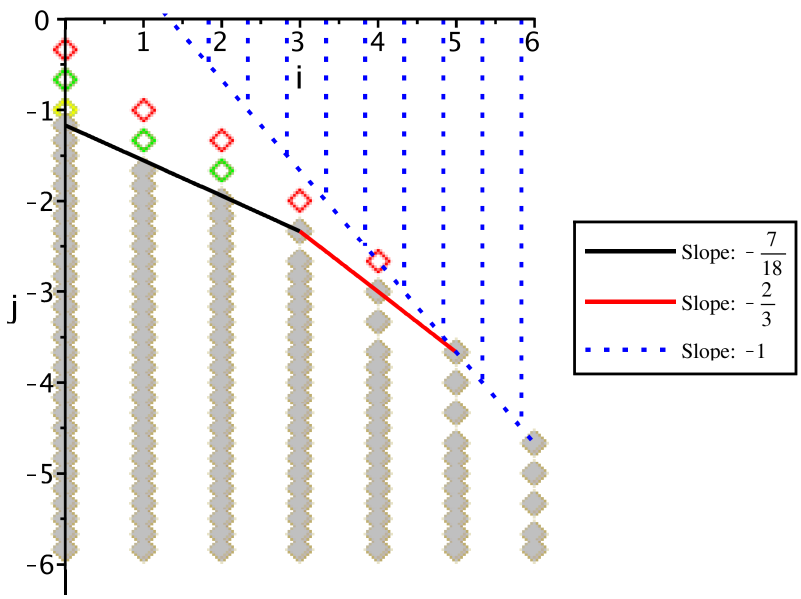


Plot the boundary and the slopes of the Newton polygon;
Note that we have already proved that there are now points above the blue dotted line

```
> LegendSize := size=[600,450]:
  P1dom1 := plot(-7/6-(7/18)*m,m=0..3,color=black,legend=[typeset
  ("Slope: ", -7/18)],legendstyle=[location=right]):
  P1dom2 := plot(-1/3-(2/3)*m,m=3..5,color=red,legend=[typeset
  ("Slope: ", -2/3)],legendstyle=[location=right]):
  P1dom3a := plot(4/3-m,m=0..6,color=blue,linestyle=dot,legend=
  [typeset("Slope: ", -1)],legendstyle=[location=right]):
  P1all := display(P1,P1a,P1ab,P1abc,P1abcd,P1abcde,P1dom1,P1dom2,
  P1dom3a,myview,LegendSize):

  for i from 1 to 10 do
     P1dom3[i] := plot([[4/3+i/2,0],[4/3+i/2,-i/2]],color=blue,
  linestyle=dot):
  end:
  display(P1all,seq(P1dom3[i],i=1..9));
```

Warning, data could not be converted to float Matrix
Warning, data could not be converted to float Matrix



This is the choice for SF

```
> SF(n);
  subs(a=K,b=0,csubs,pterm=K*(7*K-6)/6,d=-1,%);
```

$$a + \frac{b}{n^{1/3}} + \frac{c}{n^{2/3}} + \frac{pterm}{n} + \frac{d}{n^{7/6}}$$

$$K + \frac{a1\,(K-1)^{1/3}\,K\,2^{2/3}}{2\,n^{2/3}} + \frac{K\,(7\,K-6)}{6\,n} - \frac{1}{n^{7/6}} \tag{2.5.2.20}$$

recall

```
> subs(L=2/(K-1),[kaplam]);
```

$$\left[\mathrm{AiryAi}\left(\frac{\mathrm{AiryAiZeros}(1)\,n^{1/3} + 2^{1/3}\left(\frac{1}{K-1}\right)^{1/3} m}{n^{1/3}}\right) = \kappa, \mathrm{AiryAi}\left(1, \frac{\mathrm{AiryAiZeros}(1)\,n^{1/3} + 2^{1/3}\left(\frac{1}{K-1}\right)^{1/3} m}{n^{1/3}}\right) = \lambda\right] \tag{2.5.2.21}$$

Look at the corners
rr[3] will be fixed in lambda below (blue plot)

```
> getArray(posCFabcde,PP,0,-7/6);
  factor(getArray(posCFabcde,PP,3,-14/6));
  collect(getArray(posCFabcde,PP,4,-3),[lambda,kappa],factor)
  assuming K>1;
  collect(getArray(posCFabcde,PP,5,-11/3),[lambda,kappa],factor)
  assuming K>1;
```

$$\kappa$$

$$-\frac{(K+2)^2\,\lambda\,K\,2^{1/3}}{18\,(K-1)^{4/3}}$$

$$\frac{rr_3\, 2^{1/3}\,(K+2)\,K\,\lambda}{3\,(K-1)^{1/3}} - \frac{K\,\kappa\,(K+2)\,(K^2+K+11)}{36\,(K-1)^2}$$

$$\frac{\kappa\, rr_3\, K\,(K^2+K+11)}{6\,(K-1)} \qquad \textbf{(2.5.2.22)}$$

Now split the solid diamonds into the contributions from Ai and Ai'

```
> indets(posCFabcd);
```

$$\left\{K, a1, d, \kappa, \lambda, q_0, q_1, qq_0, rr_3, ss_2, \frac{1}{(K-1)^{14/3}}, \frac{1}{(K-1)^{13/3}}, \frac{1}{(K-1)^{11/3}}, \frac{1}{(K-1)^{10/3}}, \frac{1}{(K-1)^{8/3}}, \frac{1}{(K-1)^{7/3}}, \frac{1}{(K-1)^{5/3}}, \frac{1}{(K-1)^{4/3}}, \frac{1}{(K-1)^{2/3}}, \frac{1}{(K-1)^{1/3}}, (K-1)^{1/3}, (K-1)^{2/3}, (K-1)^{4/3}, (K-1)^{5/3}, (K-1)^{7/3}\right\} \qquad \textbf{(2.5.2.23)}$$

Sanity check that there are no other contributions

```
> subs(kappa=0,lambda=0,posCFabcd):
  ArrayTools[IsZero](%);
```

$$true \qquad \textbf{(2.5.2.24)}$$

Extract the coefficients of kappa=Ai and lambda=Ai'
and treat then seperately

```
> posFk := subs(kappa=1,lambda=0,rr[4]=0,uu[4]=0,posCFabcd):indets
  (%);
  posFl := subs(kappa=0,lambda=1,rr[4]=0,uu[4]=0,posCFabcd):indets
  (%);
```

$$\left\{K, a1, d, q_0, q_1, qq_0, rr_3, ss_2, \frac{1}{(K-1)^{14/3}}, \frac{1}{(K-1)^{13/3}}, \frac{1}{(K-1)^{11/3}}, \frac{1}{(K-1)^{10/3}}, \frac{1}{(K-1)^{8/3}}, \frac{1}{(K-1)^{7/3}}, \frac{1}{(K-1)^{5/3}}, \frac{1}{(K-1)^{4/3}}, \frac{1}{(K-1)^{2/3}}, \frac{1}{(K-1)^{1/3}}, (K-1)^{1/3}, (K-1)^{2/3}, (K-1)^{4/3}, (K-1)^{5/3}, (K-1)^{7/3}\right\}$$

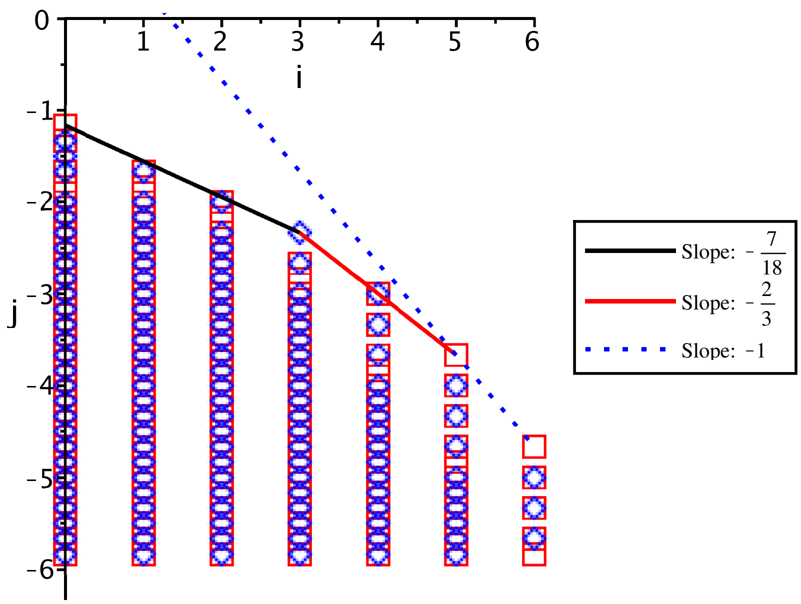
$$\left\{K, a1, d, q_0, q_1, qq_0, rr_3, ss_2, \frac{1}{(K-1)^{14/3}}, \frac{1}{(K-1)^{13/3}}, \frac{1}{(K-1)^{11/3}}, \frac{1}{(K-1)^{10/3}}, \frac{1}{(K-1)^{8/3}}, \frac{1}{(K-1)^{7/3}}, \frac{1}{(K-1)^{5/3}}, \frac{1}{(K-1)^{4/3}}, \frac{1}{(K-1)^{2/3}}, \frac{1}{(K-1)^{1/3}}, (K-1)^{1/3}, (K-1)^{2/3}, (K-1)^{4/3}, (K \qquad \textbf{(2.5.2.25)}$$

$$-1)^{5/3},\ (K-1)^{7/3}\Big\}$$

We color the non-zero nodes of the lastNewton polygon into
red squared .... coefficients of kappa=Ai
blue diamonds ...... coefficients of lambda=Ai'

```
> newt4a := mynewtArray(posFk,PP):
  newt4b := mynewtArray(posFl,PP):
```

```
> P4a := pointplot(newt4a,labels=["m deg", "n deg"], symbolsize=25,
  symbol=box,color=red):
  P4b := pointplot(newt4b,labels=["m deg", "n deg"], symbolsize=25,
  symbol=diamond, color=blue):
  P1dom3s := plot(1-m,m=4..5,color=black):
  display(P4a,P4b,P1dom1,P1dom2,P1dom3a,myview,myoptionsLo,
  LegendSize);
```



red extremes of Newton polygon

```
> mnmaxR := getMaxNewt(Mord,newt4a):
  seq([i,mnmaxR[i]],i=0..Mord);
```

$$\left[0, -\frac{7}{6}\right], \left[1, -\frac{5}{3}\right], [2, -2], \left[3, -\frac{8}{3}\right], [4, -3], \left[5, -\frac{11}{3}\right], \left[6, -\frac{14}{3}\right] \qquad \textbf{(2.5.2.26)}$$

These are the specific values at these points;
we see that we still have some degree of freedom: d and (p[4] if used)

```
> for i from 0 to Mord do
     i,factor(getArray(posFk,PP,i,mnmaxR[i]));
  end;
```

$$0,\ -d$$

$$1,\ -\frac{\left(-108\,(K-1)^{5/3}\,rr_3+7\,K^2\,2^{2/3}\,a1+4\,K\,2^{2/3}\,a1-24\,2^{2/3}\,a1\right)K}{36\,(K-1)^{2/3}}$$

$$2,\ -\frac{K\left(7\,K^2+40\,K-26\right)}{36\,(K-1)}$$

$$3,\ -\frac{1}{36\,(K-1)^{5/3}}\Big(K\,\big(-114\,rr_3\,(K-1)^{5/3}\,K+2\,K^3\,2^{2/3}\,a1-180\,(K-1)^{5/3}\,rr_3$$
$$+2\,K^2\,2^{2/3}\,a1+11\,K\,2^{2/3}\,a1-22\,2^{2/3}\,a1\big)\Big)$$

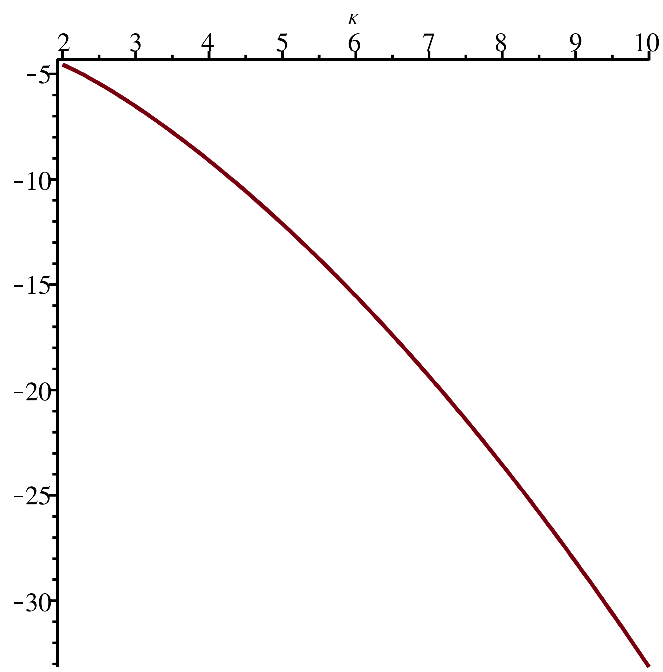
$$4,\ -\frac{K\,(K+2)\,(K^2+K+11)}{36\,(K-1)^2}$$

$$5,\ \frac{rr_3\,K\,(K^2+K+11)}{6\,(K-1)}$$

$$6,\ \frac{K\,rr_3\,(K^4+K^3+21\,K^2+11\,K-99)}{90\,(K-1)^2} \qquad \textbf{(2.5.2.27)}$$

```
> -K*(7*K^2 + 40*K - 26)/(36*(K - 1));
  plot(%,K=2..10);
```

$$-\frac{K\,(7\,K^2+40\,K-26)}{36\,K-36}$$



These are the slopes of the convex hull where the corners are given by the second sequence;
hence, in order to be positive when the slope > -1, we need to choose d>0, e.g. d=1;
note that p[4] is not important here, as the slope first slope -5/12 is less then -7/18;
and in the later regimes it will be dominated by the blue points.

```
> ls,li := getslopes(mnmaxR,Mord);
```

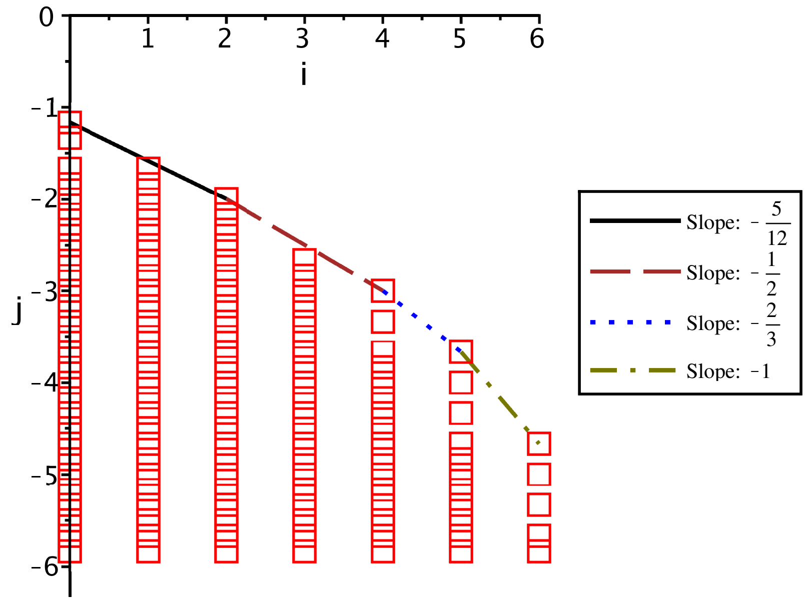
$$ls,\ li := \left[-\frac{5}{12},\ -\frac{1}{2},\ -\frac{2}{3},\ -1\right],\ [0, 2, 4, 5, 6] \qquad \textbf{(2.5.2.28)}$$

```
> colors := [green,black,brown,blue,olive,red]:
  styles := [spacedot,solid, dash, dot, dashdot, longdash,
  spacedash]:
```

Draw the conveux hull in red

```
> for i from 1 to nops(ls) do
```

```
    ls[i];li[i];
    tt[i] := plot((mnmaxR[li[i]]-ls[i]*li[i])+ls[i]*m,m=li[i]..li
[i+1],color=colors[i mod nops(colors)+1],linestyle=styles[i mod
nops(styles)+1],legend=[typeset("Slope: ", ls[i])],legendstyle=
[location=right]):
end:
Pconvred := seq(tt[i],i=1..nops(ls)):
#display(%);
> display(Pconvred,P4a,myview,myoptionsLo,LegendSize);
```




```
> #getArray(posFk,PP,6,-5);
```

We continue with the blue diamonds, i.e., the coefficients of Ai'

```
> mnmaxB := getMaxNewt(Mord,newt4b):
seq([i,mnmaxB[i]],i=0..Mord);
```

$$\left[0, -\frac{4}{3}\right], \left[1, -\frac{5}{3}\right], [2, -2], \left[3, -\frac{7}{3}\right], [4, -3], [5, -4], [6, -5] \qquad \textbf{(2.5.2.29)}$$

here we eliminate these corners: this is the reason to use q[1], ss[2], rr[3]
then the behavior will look up to n^(2/3-eps) like in the binary tree case
thus we can directly reuse the results/proofs from there

```
> for i from 0 to Mord-1 do
    i,factor(getArray(posFl,PP,i,mnmaxB[i]));
end;
```

$$0, \frac{\left(6 K q_1 - 7 K - 6 q_1 + 11\right) K 2^{1/3}}{6 (K-1)^{1/3}}$$

$$1, -\frac{2^{1/3} \left(K^2 2^{1/3} al^2 - 36 ss_2 (K-1)^{4/3} - 4 K 2^{1/3} al^2 + 4 2^{1/3} al^2\right) K}{18 (K-1)^{2/3}}$$

$$2, -\frac{K \left(-18 2^{1/3} (K-1)^{5/3} rr_3 + K^2 al - 4 al\right)}{6 (K-1)}$$

$$3, -\frac{(K+2)^2 K 2^{1/3}}{18 (K-1)^{4/3}}$$

$$4, \frac{K\, rr_3\, (K+2)\, 2^{1/3}}{3\, (K-1)^{1/3}}$$

$$5, \frac{2^{1/3}\, rr_3\, (K^3+K^2+K+11)\, K}{30\, (K-1)^{4/3}} \quad \textbf{(2.5.2.30)}$$

Differences to the binary case:

```
> factor(getArray(posFl,PP,0,mnmaxB[0]));
  factor(isolate(%,q[1]));
```

$$\frac{(6\, K\, q_1 - 7\, K - 6\, q_1 + 11)\, K\, 2^{1/3}}{6\, (K-1)^{1/3}}$$

$$q_1 = \frac{7\, K - 11}{6\, (K-1)} \quad \textbf{(2.5.2.31)}$$

this one is 0 in the binary case

```
> factor(getArray(posFl,PP,1,mnmaxB[1]));
  factor(isolate(%,ss[2]));
```

$$-\frac{2^{1/3}\, (K^2\, 2^{1/3}\, al^2 - 36\, ss_2\, (K-1)^{4/3} - 4\, K\, 2^{1/3}\, al^2 + 4\, 2^{1/3}\, al^2)\, K}{18\, (K-1)^{2/3}}$$

$$ss_2 = \frac{2^{1/3}\, al^2\, (K-2)^2}{36\, (K-1)^{4/3}} \quad \textbf{(2.5.2.32)}$$

this one is also 0 in the binary case

```
> factor(getArray(posFl,PP,2,mnmaxB[2]));
  factor(isolate(%,rr[3]));
```

$$-\frac{K\, (-18\, 2^{1/3}\, (K-1)^{5/3}\, rr_3 + K^2\, al - 4\, al)}{6\, (K-1)}$$

$$rr_3 = \frac{al\, (K-2)\, (K+2)\, 2^{2/3}}{36\, (K-1)^{5/3}} \quad \textbf{(2.5.2.33)}$$

Again we derive the slopes of the convex hull and its corners;
note that if we choose q[1] as shown above we eliminate the first term and decrease the slope.
This will be useful in the proof of Lemma 5.3, in the same way as it was used in the proof of Lemma 4.4.

```
> ls,li := getslopes(mnmaxB,Mord);
```

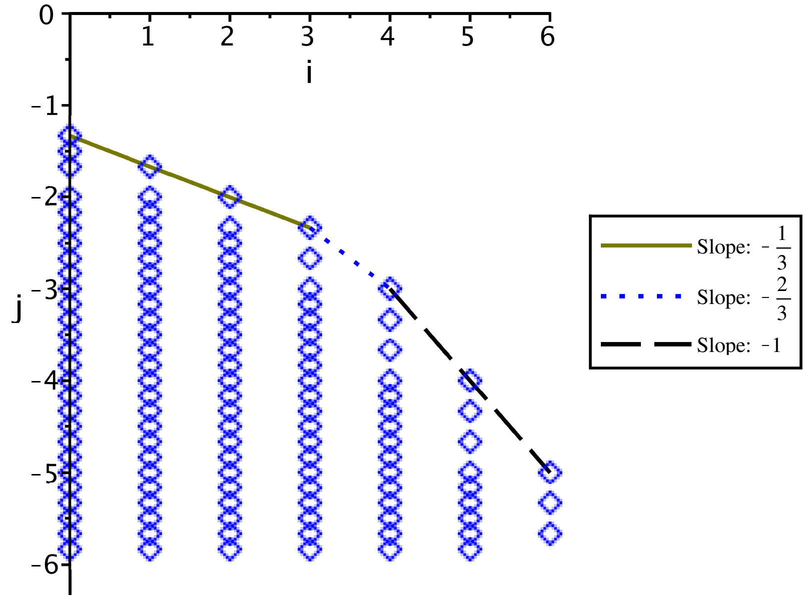
$$ls, li := \left[-\frac{1}{3}, -\frac{2}{3}, -1\right], [0, 3, 4, 6] \quad \textbf{(2.5.2.34)}$$

```
> colors := [green,olive,blue,black,brown,blue,red]:
  styles := [spacedot,solid, dot, dash, dashdot, longdash,
  spacedash]:
```

Draw the conveux hull in blue which still includes the term of order Theta(n^(-4/3))

```
> for i from 1 to nops(ls) do
     ls[i];li[i];
     tt[i] := plot((mnmaxB[li[i]]-ls[i]*li[i])+ls[i]*m,m=li[i]..li
  [i+1],color=colors[i mod nops(colors)+1],linestyle=styles[i mod
  nops(styles)+1],legend=[typeset("Slope: ", ls[i])],legendstyle=
```

```
[location=right]):
end:
Pconvblue := seq(tt[i],i=1..nops(ls)):
#display(%);
> display(Pconvblue,P4b,myview,myoptionsLo,LegendSize);
```



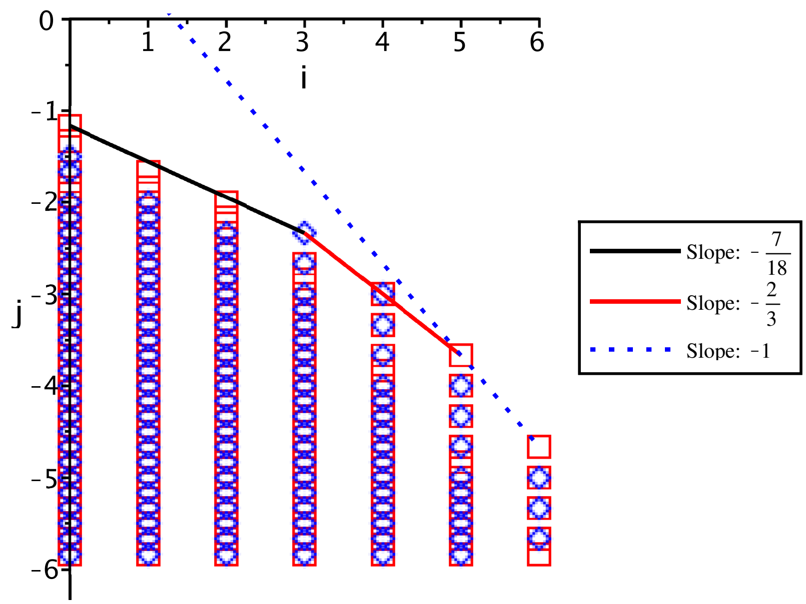
recall and fix these choices

```
> redLsubs := [q[1] = (7*K - 11)/(6*(K - 1)),
ss[2] = 2^(1/3)*a1^2*(K - 2)^2/(36*(K - 1)^(4/3)),
rr[3] = a1*(K - 2)*(K + 2)*2^(2/3)/(36*(K - 1)^(5/3))];
```

$$redLsubs := \left[q_1 = \frac{7\,K-11}{6\,K-6},\ ss_2 = \frac{2^{1/3}\,a1^2\,(K-2)^2}{36\,(K-1)^{4/3}},\ rr_3 = \frac{a1\,(K-2)\,(K+2)\,2^{2/3}}{36\,(K-1)^{5/3}}\right] \quad (2.5.2.35)$$

We kill this term by setting q[1] as above and recompute the Newton polygons.
In the next picture the left-top blue point disappeared.
(note that the coefficients of lower order terms of posFk change as well, which is why we recompute them; however the picture will not change as the convex hull is not influenced by this change.)

```
> posFk2 := map(factor,subs(redLsubs,posFk)):
newt4a2 := mynewtArray(posFk2,PP):
P4a2 := pointplot(newt4a2,labels=["m deg", "n deg"], symbolsize=
25, symbol=box, color=red):
posFl2 := map(factor,subs(redLsubs,posFl)):
newt4b2 := mynewtArray(posFl2,PP):
P4b2 := pointplot(newt4b2,labels=["m deg", "n deg"], symbolsize=
25, symbol=diamond, color=blue):
display(P4a2,P4b2,P1dom1,P1dom2,P1dom3a,myview,myoptionsLo,
LegendSize);
```



Recompute blue with q[1], ss[2], and rr[3] set as above

```
> mnmaxB := getMaxNewt(Mord,newt4b2):
  seq([i,mnmaxB[i]],i=0..Mord);
```

$$\left[0, -\frac{3}{2}\right], [1, -2], \left[2, -\frac{7}{3}\right], \left[3, -\frac{7}{3}\right], [4, -3], [5, -4], [6, -5] \quad \textbf{(2.5.2.36)}$$

here we see that the corners on the convex hull for i=4,5,6 are zero in the binary case
but we do not need to kill them, as we will only need the lower bound up to n^(2/3-eps)
maybe, we could kill them with right factors, but we don't do this here, and it is not clear, how many (infinitly many?) there are

```
> for i from 0 to Mord do
     i,factor(getArray(posFl2,PP,i,mnmaxB[i]));
  end;
```

$$0, -\frac{2^{1/3} d}{(K-1)^{1/3}}$$

$$1, -\frac{1}{45 (K-1)^2} \Big(K \big(-15 K 2^{1/3} (K-1)^{5/3} qq_0 - 30\, 2^{1/3} (K-1)^{5/3} qq_0 + 2 K^4 al$$
$$- 35 K^3 al + 170 K^2 al - 275 K al + 138 al\big)\Big)$$

$$2, \frac{\left(5 K^3 al^3 - 30 K^2 al^3 + 60 K al^3 - 57 K^3 - 40 al^3 + 168 K^2 + 63 K - 912\right) 2^{1/3} K}{540 (K-1)^{4/3}}$$

$$3, -\frac{(K+2)^2 K 2^{1/3}}{18 (K-1)^{4/3}}$$

$$4, \frac{K al (K-2) (K+2)^2}{54 (K-1)^2}$$

$$5, \frac{al (K-2) (K+2) \left(K^3 + K^2 + K + 11\right) K}{540 (K-1)^3}$$

$$6, \frac{al (K-2) (K+2) \left(K^5 + K^4 + 36 K^3 + K^2 + 281 K - 279\right) K}{11340 (K-1)^4} \quad \textbf{(2.5.2.37)}$$

And we get new slopes, yet at the same m powers given in the second sequence;
here we see that the term m^3 is negative, which will dominate in the regime when Ai' is negative.

```
> ls,li := getslopes(mnmaxB,Mord-1);
```

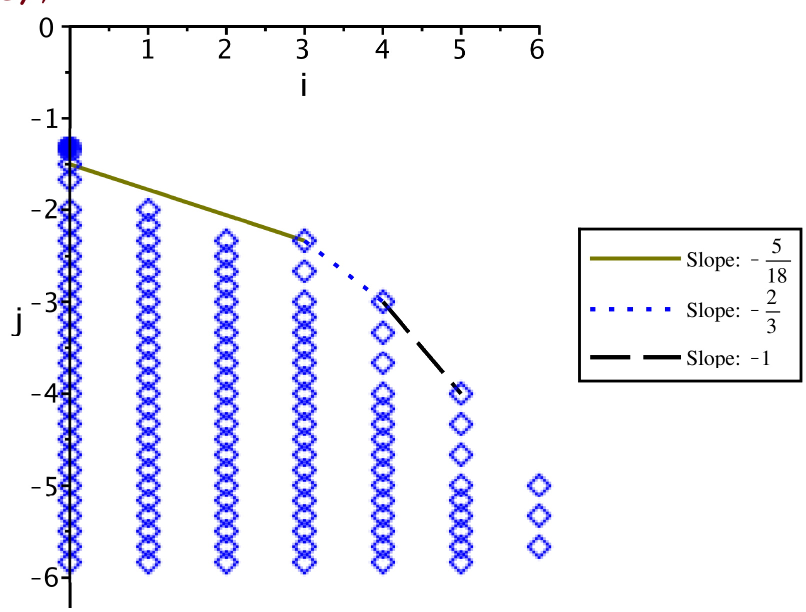
$$ls, li := \left[-\frac{5}{18}, -\frac{2}{3}, -1\right], [0, 3, 4, 5] \quad \textbf{(2.5.2.38)}$$

Draw the new conveux hull in blue

```
> for i from 1 to nops(ls) do
     ls[i];li[i];
     tt[i] := plot((mnmaxB[li[i]]-ls[i]*li[i])+ls[i]*m,m=li[i]..li
  [i+1],color=colors[i mod nops(colors)+1],linestyle=styles[i mod
  nops(styles)+1],legend=[typeset("Slope: ", ls[i])],legendstyle=
  [location=right]):
  end:
  Pconvblue := seq(tt[i],i=1..nops(ls)):
  #display(%);
```

Plot the difference to before:
The solid circle on the top-left and the corners fro i=1,2 disappeared
Like this, the convex hull looks up to i=j^(2/3-eps) as in the binary case
(note that in the binary case more terms vanish!!!)
We do not need these here!

```
> P4bdiffshort := pointplot([0,-4/3],labels=["m deg", "n deg"],
  symbolsize=25, symbol=solidcircle, color=blue):
  display(Pconvblue,P4bdiffshort,P4b2,myview,myoptionsLo,
  LegendSize);
```



Summary

The general ansatz for the lower bound

```
> facAiryLo*Airy(a1+(L)^(1/3)*(m+1)/n^(1/3));
  SF(n);
```

$$\left(1 + \frac{q_2\, m^2 + q_1\, m + q_0}{n} + \frac{m\, qq_1 + qq_0}{n^{2/3}} + \frac{ss_2\, m^2}{n^{4/3}} + \frac{rr_3\, m^3}{n^{5/3}}\right) Airy\left(a1 + \frac{L^{1/3}\,(m+1)}{n^{1/3}}\right)$$

$$a + \frac{b}{n^{1/3}} + \frac{c}{n^{2/3}} + \frac{pterm}{n} + \frac{d}{n^{7/6}} \quad \textbf{(2.5.3.1)}$$

Our choices

```
> varsLowerBound :=
  [q[0] = 0, # arbitrary; not needed
  q[1] = (7*K - 11)/(6*(K - 1)),
  q[2] = -(K+2)/6/(K-1),
  qq[0] = 0,     # arbitrary, not needed
  qq[1] = -a1*(K - 2)*2^(2/3)/(6*(K - 1)^(2/3)),
  ss[2] = 2^(1/3)*a1^2*(K - 2)^2/(36*(K - 1)^(4/3)),
  rr[3] = a1*(K - 2)*(K + 2)*2^(2/3)/(36*(K - 1)^(5/3)),
  L=2/(K-1),
  a=K,
  b=0,
  csubs,
  pterm = K*(7*K-6)/6,
  d=-1
  ];
```

$$varsLowerBound := \left[q_0 = 0, q_1 = \frac{7\,K - 11}{6\,K - 6}, q_2 = -\frac{K+2}{6\,(K-1)}, qq_0 = 0, qq_1 = -\frac{a1\,(K-2)\,2^{2/3}}{6\,(K-1)^{2/3}}, ss_2 = \frac{2^{1/3}\,a1^2\,(K-2)^2}{36\,(K-1)^{4/3}}, rr_3 = \frac{a1\,(K-2)\,(K+2)\,2^{2/3}}{36\,(K-1)^{5/3}}, L = \frac{2}{K-1}, a = K, b = 0, c = \frac{a1\,(K-1)^{1/3}\,K\,2^{2/3}}{2}, pterm = \frac{K\,(7\,K-6)}{6}, d = -1\right] \quad \textbf{(2.5.3.2)}$$

```
> facL2 := subs(varsLowerBound, facAiryLo*Airy(a1+L^(1/3)*(m+1)/n^
  (1/3)));
```

$$facL2 := \left(1 + \frac{-\frac{m^2\,(K+2)}{6\,(K-1)} + \frac{m\,(7\,K-11)}{6\,K-6}}{n} - \frac{m\,a1\,(K-2)\,2^{2/3}}{6\,(K-1)^{2/3}\,n^{2/3}} + \frac{2^{1/3}\,a1^2\,(K-2)^2\,m^2}{36\,(K-1)^{4/3}\,n^{4/3}} + \frac{a1\,(K-2)\,(K+2)\,2^{2/3}\,m^3}{36\,(K-1)^{5/3}\,n^{5/3}}\right) Airy\left(a1 + \frac{2^{1/3}\left(\frac{1}{K-1}\right)^{1/3}(m+1)}{n^{1/3}}\right) \quad \textbf{(2.5.3.3)}$$

```
> i:='i':
  facL1 := subs(varsLowerBound, SF(n));
  product(subs(n=i,%),i=1..K*n);
```

$$facL1 := K + \frac{a1\,(K-1)^{1/3} K\, 2^{2/3}}{2\, n^{2/3}} + \frac{K\,(7K-6)}{6\,n} - \frac{1}{n^{7/6}}$$

$$\prod_{i=1}^{K n} \left(K + \frac{a1\,(K-1)^{1/3} K\, 2^{2/3}}{2\, i^{2/3}} + \frac{K\,(7K-6)}{6\,i} - \frac{1}{i^{7/6}} \right) \qquad \textbf{(2.5.3.4)}$$

simplification of the product:
First factor by K^(K*n) and then perform an exp(log(...)) transform to get the stretched exponential

```
> K^(K*n)*product(subs(n=i,map(x->x/K,facL1)),i=1..K*n) assuming
  K>1,i>0;
```

$$K^{K n} \left(\prod_{i=1}^{K n} \left(1 + \frac{a1\,(K-1)^{1/3}\, 2^{2/3}}{2\, i^{2/3}} + \frac{7K-6}{6\,i} - \frac{1}{i^{7/6}\, K} \right) \right) \qquad \textbf{(2.5.3.5)}$$

looks at the structure using only constants

```
> product(1-c1/i^(2/3)-c2/i,i=1..K*n);
  sum(log(1-c1/i^(2/3)-c2/i),i=1..K*n);
  #series(%,n=infinity,1) assuming c1>0,c2>0,n>0,K>1;
  -int(c1/i^(2/3)+c2/i,i=1..K*n);
  map(simplify,series(%,n=infinity,1)) assuming c1>0,c2>0,n>0,K>1;
```

$$\prod_{i=1}^{K n} \left(1 - \frac{c1}{i^{2/3}} - \frac{c2}{i} \right)$$

$$\sum_{i=1}^{K n} \ln\left(1 - \frac{c1}{i^{2/3}} - \frac{c2}{i} \right)$$

Warning, unable to determine if 0 is between 1 and K*n; try to use assumptions or use the AllSolutions option

$$-\left(\int_{1}^{K n} \left(\frac{c1}{i^{2/3}} + \frac{c2}{i} \right) \mathrm{d}i \right)$$

$$-3\, c1\, K^{1/3}\, n^{1/3} - c2 \ln(K) - c2 \ln(n) + 3\, c1 \qquad \textbf{(2.5.3.6)}$$

the non-constant contributions

```
> c1val := -subs(varsLowerBound,c/K);
  c2val := -subs(varsLowerBound,pterm/K);
```

$$c1val := -\frac{a1\,(K-1)^{1/3}\, 2^{2/3}}{2}$$

$$c2val := -\frac{7K}{6} + 1 \qquad \textbf{(2.5.3.7)}$$

```
> facL1b := K^(K*n)*exp(-3*c1*K^(1/3)*n^(1/3))*n^(-c2);
  subs(c1=c1val,c2=c2val,%);
```

$$facL1b := K^{K n}\, \mathrm{e}^{-3\, c1\, K^{1/3} n^{1/3}}\, n^{-c2}$$

$$K^{K n}\, \mathrm{e}^{\frac{3\, a1\,(K-1)^{1/3}\, 2^{2/3} K^{1/3} n^{1/3}}{2}}\, n^{\frac{7K}{6} - 1} \qquad \textbf{(2.5.3.8)}$$

polynomial growth part 2:
correct by -1/3, coming from the Airy function
and by 1-k/2, coming from rewriting ((K-1)n)! into n!^(K-1)

```
> subs(m=0,n=K*n,a1=A1,Airy=AiryAi,facL2);
  map(normal,series(%,n=infinity,1));
  simplify(-c2val-1/3+1-K/2);
```

$$\mathrm{AiryAi}\left(\mathrm{AiryAiZeros}(1)+\frac{2^{1/3}\left(\frac{1}{K-1}\right)^{1/3}}{(K\,n)^{1/3}}\right)$$

$$\frac{\mathrm{AiryAi}(1,\mathrm{AiryAiZeros}(1))\,2^{1/3}\left(\frac{1}{K-1}\right)^{1/3}\left(\frac{1}{n}\right)^{1/3}}{K^{1/3}}+\mathrm{O}\left(\frac{1}{n}\right)$$

$$\frac{2\,K}{3}-\frac{1}{3} \qquad \textbf{(2.5.3.9)}$$

Gen TCs - Upper bound

upper bound: we want to prove positivity for large n and most m

```
> i:='i':
  posansatz := XX(n,m)*SS(n) - Fac1*XX(n-1,m-1) - Fac2*XX(n-1,m+
  K-1):
  subs(Fac1=fac1,Fac2=fac2,%);
```

$$XX(n,m)\,SS(n)-\frac{(K-1)^2\,(K+n-m)\,XX(n-1,m-1)}{(K-1)\,n+m}-XX(n-1,m+K-1) \qquad \textbf{(2.6.1)}$$

Update Xhat: specific values from proof below for experiments

From summary after proof below

```
> varsUpperBound := [q[0] = 0, q[1] = (7*K - 11)/(6*K - 6), q[2] =
  -(K + 2)/(6*(K - 1)), qq[0] = 0, qq[1] = -a1*(K - 2)*2^(2/3)/(6*
  (K - 1)^(2/3)), ss[2] = 2^(1/3)*a1^2*(K - 2)^2/(36*(K - 1)^(4/3)
  ), rr[3] = a1*(K - 2)*(K + 2)*2^(2/3)/(36*(K - 1)^(5/3)), L = 2/
  (K - 1), a = K, b = 0, c = a1*(K - 1)^(1/3)*K*2^(2/3)/2, pterm =
  K*(7*K - 6)/6, d = 1, p[0] = 0, p[1] = 0, p[2] = 0, p[3] = 0, p
  [4] = 1/3];
```

$$varsUpperBound := \left[q_0=0,\ q_1=\frac{7\,K-11}{6\,K-6},\ q_2=-\frac{K+2}{6\,K-6},\ qq_0=0,\ qq_1= \right. \qquad \textbf{(2.6.1.1)}$$

$$-\frac{a1\,(K-2)\,2^{2/3}}{6\,(K-1)^{2/3}},\ ss_2=\frac{2^{1/3}\,a1^2\,(K-2)^2}{36\,(K-1)^{4/3}},\ rr_3=\frac{a1\,(K-2)\,(K+2)\,2^{2/3}}{36\,(K-1)^{5/3}},\ L$$

$$= \frac{2}{K-1}, a = K, b = 0, c = \frac{a1\ (K-1)^{1/3}\ K\ 2^{2/3}}{2}, pterm = \frac{K\ (7\ K-6)}{6}, d = 1, p_0 = 0, p_1 = 0, p_2 = 0, p_3 = 0, p_4 = \frac{1}{3}\Big]$$

Fix K for experiments

```
> KK := 3;
```

$$KK := 3 \tag{2.6.1.2}$$

This is X tilde (this is derived later)

```
> subs(varsUpperBound,a1=A1,K=KK,(1 + (m^2*q[2] + m*q[1] + q[0])/n
  + (m*qq[1] + qq[0])/n^(2/3) + ss[2]*m^2/n^(4/3) + rr[3]*m^3/n^
  (5/3) + (m^4*p[4] + m^3*p[3] + m^2*p[2] + m*p[1] + p[0])/n^2)*
  AiryAi(a1 + L^(1/3)*(m + 1)/n^(1/3))):
  Xansatz := (n,m) -> subs(K=KK,((1 + (-5/12*m^2 + 5/6*m)/n - m*
  AiryAiZeros(1)/(6*n^(2/3)) + AiryAiZeros(1)^2*m^2/(72*n^(4/3)) +
  (5*AiryAiZeros(1)*m^3)/(72*n^(5/3)) + m^4/(3*n^2))*AiryAi
  (AiryAiZeros(1) + (m + 1)/n^(1/3))));
```

$$Xansatz := (n, m) \mapsto subs\left(K = KK, \left(1 + \frac{-\frac{5}{12}\cdot m^2 + \frac{5}{6}\cdot m}{n} - \frac{m\cdot \mathrm{AiryAiZeros}(1)}{6\cdot n^{2/3}} + \frac{\mathrm{AiryAiZeros}(1)^2\cdot m^2}{72\cdot n^{4/3}} + \frac{5\cdot \mathrm{AiryAiZeros}(1)\cdot m^3}{72\cdot n^{5/3}} + \frac{m^4}{3\cdot n^2}\right)\cdot \mathrm{AiryAi}\left(\mathrm{AiryAiZeros}(1) + \frac{m+1}{n^{1/3}}\right)\right) \tag{2.6.1.3}$$

(this is derived later)

```
> subs(varsUpperBound,a1=A1,a + b/n^(1/3) + c/n^(2/3) + pterm/n +
  d/n^(7/6));
  Sansatz := n -> subs(K=KK,K + A1*(K - 1)^(1/3)*K*2^(2/3)/(2*n^
  (2/3)) + K*(7*K - 6)/(6*n) + 1/n^(7/6));
```

$$K + \frac{\mathrm{AiryAiZeros}(1)\ (K-1)^{1/3}\ K\ 2^{2/3}}{2\ n^{2/3}} + \frac{K\ (7\ K-6)}{6\ n} + \frac{1}{n^{7/6}}$$

$$Sansatz := n \mapsto subs\left(K = KK, K + \frac{A1\cdot (K-1)^{1/3}\cdot K\cdot 2^{2/3}}{2\cdot n^{2/3}} + \frac{K\cdot (7\cdot K-6)}{6\cdot n} + \frac{1}{n^{7/6}}\right) \tag{2.6.1.4}$$

```
> posXS := map(simplify,subs(XX=Xansatz,SS=Sansatz,Fac1=fac1,Fac2=
  fac2,K=KK,posansatz)):indets(%);
```

$$\left\{m, n, \frac{1}{n^{7/6}}, \frac{1}{n^{5/3}}, \frac{1}{n^{4/3}}, \frac{1}{n^{2/3}}, \frac{1}{n^{1/3}}, \frac{1}{(n-1)^{5/3}}, \frac{1}{(n-1)^{4/3}}, \frac{1}{(n-1)^{2/3}}, \frac{1}{(n-1)^{1/3}}, \mathrm{AiryAi}\left(\mathrm{AiryAiZeros}(1) + \frac{m}{(n-1)^{1/3}}\right), \mathrm{AiryAi}\left(\mathrm{AiryAiZeros}(1)\right) \tag{2.6.1.5}$$

$$+\frac{m+1}{n^{1/3}}\Big), \mathrm{AiryAi}\left(\mathrm{AiryAiZeros}(1)+\frac{m+3}{(n-1)^{1/3}}\right)\Big\}$$

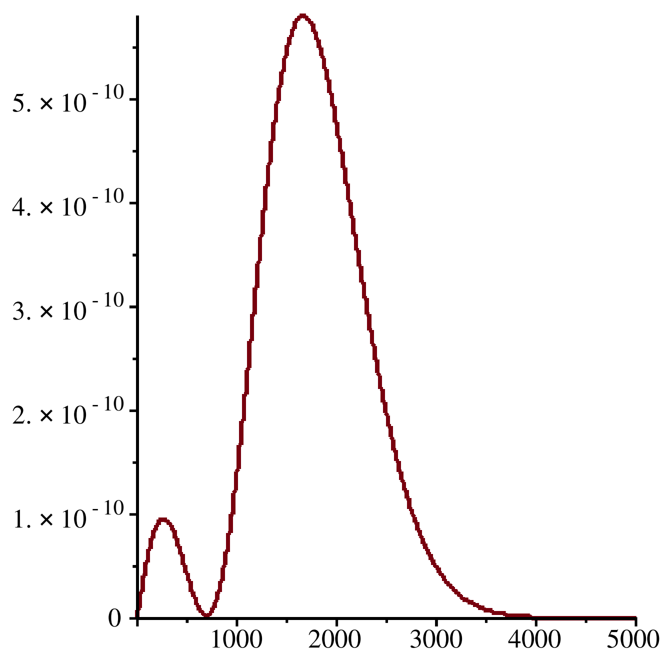
For a large n this function of m seems to be positive: looks good :)

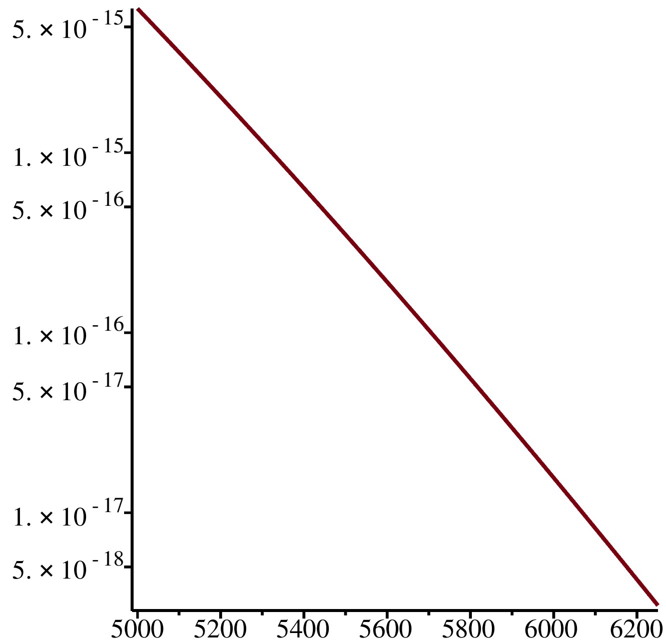
```
> 
Digits:=20:
e1 := subs(csubs,a1=A1,pterm=1/6,posXS):indets(%);
N := 100000000;
M := floor(N^(1/2));
P1 := plot([seq([mm,(subs(n=N,m=mm,e1))],mm=0..floor(M/2))])
:display(P1);
P2 := logplot([seq([mm,(subs(n=N,m=mm,e1))],mm=floor(M/2)..floor
(5*M/8))]):display(P2);
```

$$\Big\{m, n, \frac{1}{n^{7/6}}, \frac{1}{n^{5/3}}, \frac{1}{n^{4/3}}, \frac{1}{n^{2/3}}, \frac{1}{n^{1/3}}, \frac{1}{(n-1)^{5/3}}, \frac{1}{(n-1)^{4/3}}, \frac{1}{(n-1)^{2/3}},$$
$$\frac{1}{(n-1)^{1/3}}, \mathrm{AiryAi}\left(\mathrm{AiryAiZeros}(1)+\frac{m}{(n-1)^{1/3}}\right), \mathrm{AiryAi}\Big(\mathrm{AiryAiZeros}(1)$$
$$+\frac{m+1}{n^{1/3}}\Big), \mathrm{AiryAi}\left(\mathrm{AiryAiZeros}(1)+\frac{m+3}{(n-1)^{1/3}}\right)\Big\}$$

$$N := 100000000$$

$$M := 10000$$





Prove it

Recall the general ansatz

```
> facAiryUp*AiryAi(a1+(L)^(1/3)*(m+1)/n^(1/3));
  SF(n);
```

$$\left(1+\frac{m^2 q_2+m q_1+q_0}{n}+\frac{m\, qq_1+qq_0}{n^{2/3}}+\frac{ss_2\, m^2}{n^{4/3}}+\frac{rr_3\, m^3}{n^{5/3}}+\frac{p_4 m^4+p_3 m^3+m^2 p_2+m p_1+p_0}{n^2}\right)\mathrm{AiryAi}\left(a1+\frac{L^{1/3}(m+1)}{n^{1/3}}\right)$$

$$a+\frac{b}{n^{1/3}}+\frac{c}{n^{2/3}}+\frac{pterm}{n}+\frac{d}{n^{7/6}} \quad \textbf{(2.6.2.1)}$$

We will use the computed values and build on them;
note that only the term d=-1 will change to d=+1 and we will use addtional terms of order 1/n^2

```
> varsLowerBound;
```

$$\left[q_0=0,\ q_1=\frac{7K-11}{6K-6},\ q_2=-\frac{K+2}{6(K-1)},\ qq_0=0,\ qq_1=-\frac{a1\,(K-2)\,2^{2/3}}{6(K-1)^{2/3}},\ ss_2=\frac{2^{1/3}\,a1^2\,(K-2)^2}{36(K-1)^{4/3}},\ rr_3=\frac{a1\,(K-2)(K+2)\,2^{2/3}}{36(K-1)^{5/3}},\ L=\frac{2}{K-1},\ a=K,\ b=0,\ c=\frac{a1\,(K-1)^{1/3}K\,2^{2/3}}{2},\ pterm=\frac{K(7K-6)}{6},\ d=-1\right] \quad \textbf{(2.6.2.2)}$$

sub it into our ansatz

```
> subs(d=dup,varsLowerBound,facAiryUp*AiryAi(a1+(L)^(1/3)*(m+1)/n^
  (1/3)));
  subs(d=dup,varsLowerBound,SF(n));
```

$$\left(1+\frac{-\frac{m^2(K+2)}{6(K-1)}+\frac{m(7K-11)}{6K-6}}{n}-\frac{m\,a1\,(K-2)\,2^{2/3}}{6(K-1)^{2/3}n^{2/3}}+\frac{2^{1/3}a1^2(K-2)^2m^2}{36(K-1)^{4/3}n^{4/3}}+\frac{a1\,(K-2)(K+2)\,2^{2/3}m^3}{36(K-1)^{5/3}n^{5/3}}+\frac{p_4m^4+p_3m^3+m^2p_2+mp_1+p_0}{n^2}\right)\mathrm{AiryAi}\left(a1+\frac{2^{1/3}\left(\frac{1}{K-1}\right)^{1/3}(m+1)}{n^{1/3}}\right)$$

$$K+\frac{a1\,(K-1)^{1/3}K\,2^{2/3}}{2n^{2/3}}+\frac{K(7K-6)}{6n}+\frac{dup}{n^{7/6}} \qquad \textbf{(2.6.2.3)}$$

Substitute ansatz into the sequence we want to be positive for large n and all m
Note that we already fix the values we computed in the lower bound, as we want them to be the same.
only dup will change, it will flip signs

```
> map(expand, subs(XX=XFU,SS=SF,Fac1=fac1,Fac2=fac2,posansatz)):
  posF := subs(d=dup,varsLowerBound,%):indets(%),nops(expand(%));
```

$$\left\{K, a1, dup, \kappa, \lambda, m, n, p_0, p_1, p_2, p_3, p_4, \frac{1}{n^{26/3}}, \frac{1}{n^{25/3}}, \frac{1}{n^{23/3}}, \frac{1}{n^{20/3}}, \frac{1}{n^{19/3}}, \frac{1}{n^{17/3}}, \frac{1}{n^{14/3}}, \frac{1}{n^{13/3}}, \frac{1}{n^{11/3}}, \frac{1}{n^{8/3}}, \frac{1}{n^{7/3}}, \frac{1}{n^{7/6}}, \frac{1}{n^{5/3}}, \frac{1}{n^{4/3}}, \frac{1}{n^{2/3}}, \frac{1}{n^{1/3}}, n^{1/3}, n^{2/3}, n^{4/3}, n^{5/3}, n^{7/3}, n^{8/3}, n^{10/3}, n^{11/3}, n^{13/3}, n^{14/3}, n^{16/3}, n^{17/3}, n^{19/3}, n^{20/3}, n^{22/3}, n^{23/3}, n^{25/3}, n^{26/3}, \left(\frac{1}{K-1}\right)^{1/3}, \left(\frac{1}{K-1}\right)^{2/3}, \left(\frac{1}{K-1}\right)^{4/3}, \left(\frac{1}{K-1}\right)^{5/3}, \left(\frac{1}{K-1}\right)^{7/3}, \left(\frac{1}{K-1}\right)^{8/3}, \left(\frac{1}{K-1}\right)^{10/3}, \left(\frac{1}{K-1}\right)^{11/3}, \left(\frac{1}{K-1}\right)^{13/3}, \left(\frac{1}{K-1}\right)^{14/3}, \frac{1}{(K-1)^{5/3}}, \frac{1}{(K-1)^{4/3}}, \frac{1}{(K-1)^{2/3}}, (K-1)^{1/3}, \frac{1}{(n-1)^{5/3}}, \frac{1}{(n-1)^{4/3}}, \frac{1}{(n-1)^{2/3}}, \frac{1}{(n-1)^{1/3}}\right\}, \qquad \textbf{(2.6.2.4)}$$

$$69956$$

The error terms are (to check, expand posF)

UPDATE

```
> simplify(O((2^(1/3)*(m+1)/n^(1/3)-2^(1/3)*m/n^(1/3))^ordAiLo));
  simplify(O((2^(1/3)*(m)/(n-1)^(1/3)-2^(1/3)*m/n^(1/3))^ordAiLo));
  simplify(O((2^(1/3)*(m+2)/(n-1)^(1/3)-2^(1/3)*m/n^(1/3))^ordAiLo)
  );
```

$$O\left(\frac{64\,2^{1/3}}{n^{19/3}}\right)$$

$$O\left(\frac{64\,2^{1/3}\,m^{19}\left(n^{1/3}-(n-1)^{1/3}\right)^{19}}{(n-1)^{19/3}\,n^{19/3}}\right)$$

$$O\left(\frac{64\left(-m\,(n-1)^{1/3}+n^{1/3}\,(m+2)\right)^{19}\,2^{1/3}}{n^{19/3}\,(n-1)^{19/3}}\right) \quad (2.6.2.5)$$

remove error terms

```
> posFd := convert(posF,polynom):indets(%);
```

$$\left\{K, a1, dup, \kappa, \lambda, m, n, p_0, p_1, p_2, p_3, p_4, \frac{1}{n^{55/3}}, \frac{1}{n^{52/3}}, \frac{1}{n^{47/3}}, \frac{1}{n^{43/3}}, \frac{1}{n^{40/3}}, \frac{1}{n^{35/3}}, \frac{1}{n^{31/3}}, \frac{1}{n^{28/3}}, \frac{1}{n^{23/3}}, \frac{1}{n^{19/3}}, \frac{1}{n^{16/3}}, \frac{1}{n^{11/3}}, \frac{1}{n^{7/3}}, \frac{1}{n^{7/6}}, \frac{1}{n^{5/3}}, \frac{1}{n^{4/3}}, \frac{1}{n^{2/3}}, \frac{1}{n^{1/3}}, n^{1/3}, n^{2/3}, n^{4/3}, n^{5/3}, n^{7/3}, n^{8/3}, n^{10/3}, n^{11/3}, n^{13/3}, n^{14/3}, n^{16/3}, n^{17/3}, n^{19/3}, n^{20/3}, n^{22/3}, n^{23/3}, n^{25/3}, n^{26/3}, \left(\frac{1}{K-1}\right)^{1/3}, \left(\frac{1}{K-1}\right)^{2/3}, \left(\frac{1}{K-1}\right)^{4/3}, \left(\frac{1}{K-1}\right)^{5/3}, \left(\frac{1}{K-1}\right)^{7/3}, \left(\frac{1}{K-1}\right)^{8/3}, \left(\frac{1}{K-1}\right)^{10/3}, \left(\frac{1}{K-1}\right)^{11/3}, \left(\frac{1}{K-1}\right)^{13/3}, \left(\frac{1}{K-1}\right)^{14/3}, \frac{1}{(K-1)^{5/3}}, \frac{1}{(K-1)^{4/3}}, \frac{1}{(K-1)^{2/3}}, (K-1)^{1/3}, \frac{1}{(n-1)^{5/3}}, \frac{1}{(n-1)^{4/3}}, \frac{1}{(n-1)^{2/3}}, \frac{1}{(n-1)^{1/3}}\right\} \quad (2.6.2.6)$$

Next we fit the display size of the plots in the n (Nord) and m (Mord) direction.
(Note that everything up to ordAi is computed, but possibly not shown)

```
> Nord := -ordAiUp/3;
  Mord := floor(ordAiUp/3);
  myview := view=[0..Mord,Nord..0]:
```

$$Nord := -10$$

$$Mord := 10 \quad (2.6.2.7)$$

```
> Nord := -Mord;
```

$$Nord := -10 \quad (2.6.2.8)$$

Expand again with respect to n,
these are then our unknowns

```
> posFe := series(posFd,n=infinity,ceil(-Nord)+1):indets(%);
  posFf := convert(%%,polynom):
  nops(expand(%));
```

$$\left\{K, a1, dup, \kappa, \lambda, m, n, p_0, p_1, p_2, p_3, p_4, \left(\frac{1}{n}\right)^{1/3}, \left(\frac{1}{n}\right)^{2/3}, \left(\frac{1}{n}\right)^{3/2}, \left(\frac{1}{n}\right)^{4/3}, \left(\frac{1}{n}\right)^{5/2}, \left(\frac{1}{n}\right)^{5/3}, \left(\frac{1}{n}\right)^{7/2}, \left(\frac{1}{n}\right)^{7/3}, \left(\frac{1}{n}\right)^{7/6}, \left(\frac{1}{n}\right)^{8/3}, \left(\frac{1}{n}\right)^{9/2}, \left(\frac{1}{n}\right)^{10/3}, \left(\frac{1}{n}\right)^{11/2}, \left(\frac{1}{n}\right)^{11/3}, \left(\frac{1}{n}\right)^{11/6}, \left(\frac{1}{n}\right)^{13/3}, \left(\frac{1}{n}\right)^{13/6}, \left(\frac{1}{n}\right)^{14/3}, \left(\frac{1}{n}\right)^{16/3}, \left(\frac{1}{n}\right)^{17/3}, \left(\frac{1}{n}\right)^{17/6}, \left(\frac{1}{n}\right)^{19/6}, \left(\frac{1}{n}\right)^{23/6}, \left(\frac{1}{n}\right)^{25/6}, \left(\frac{1}{n}\right)^{29/6}, \left(\frac{1}{n}\right)^{31/6}, \left(\frac{1}{n}\right)^{35/6}, \left(\frac{1}{K-1}\right)^{1/3}, \left(\frac{1}{K-1}\right)^{2/3}, \left(\frac{1}{K-1}\right)^{4/3}, \left(\frac{1}{K-1}\right)^{5/3}, \left(\frac{1}{K-1}\right)^{7/3}, \left(\frac{1}{K-1}\right)^{8/3}, \left(\frac{1}{K-1}\right)^{10/3}, \left(\frac{1}{K-1}\right)^{11/3}, \left(\frac{1}{K-1}\right)^{13/3}, \left(\frac{1}{K-1}\right)^{14/3}, \left(\frac{1}{K-1}\right)^{16/3}, \left(\frac{1}{K-1}\right)^{17/3}, \frac{1}{(K-1)^{20/3}}, \frac{1}{(K-1)^{19/3}}, \frac{1}{(K-1)^{17/3}}, \frac{1}{(K-1)^{16/3}}, \frac{1}{(K-1)^{14/3}}, \frac{1}{(K-1)^{13/3}}, \frac{1}{(K-1)^{11/3}}, \frac{1}{(K-1)^{10/3}}, \frac{1}{(K-1)^{8/3}}, \frac{1}{(K-1)^{7/3}}, \frac{1}{(K-1)^{5/3}}, \frac{1}{(K-1)^{4/3}}, \frac{1}{(K-1)^{2/3}}, (K-1)^{1/3}, O\left(\frac{1}{n^6}\right)\right\}$$

$$18951 \qquad (2.6.2.9)$$

lcm of denominators in exponents

```
> PP := 6;
```

$$PP := 6 \qquad (2.6.2.10)$$

convert the expansion into a list with respect to the degrees in m and n:
Format: [[mdeg,ndeg], coeff]

```
> posCF := map(simplify,ex2Array(posFf,Mord+6,-Nord+6,PP)) assuming
  K>1:
  ArrayNumElems(posCF);
```

$$1649 \qquad (2.6.2.11)$$

```
> degree(posFf,m);
```

$$10 \qquad (2.6.2.12)$$

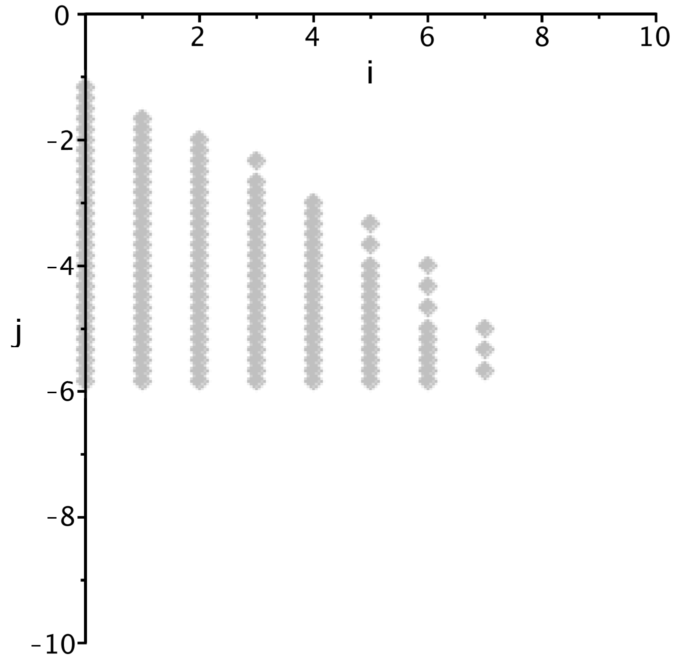
The mynewt function computes the Newton polygon of posFf

```
> newt1 := mynewtArray(posCF,PP):
```

Most terms are killed as expected from the choices of the lower bound

```
> P1 := pointplot(newt1,myoptionsUp,color=grey,symbol=soliddiamond)
```

```
:
display(P1,myview);
```



we want this term to be positive,
so we set dup=1

```
> getArray(posCF,PP,0,-7/6):
  collect(%,[kappa,lambda],factor);
```

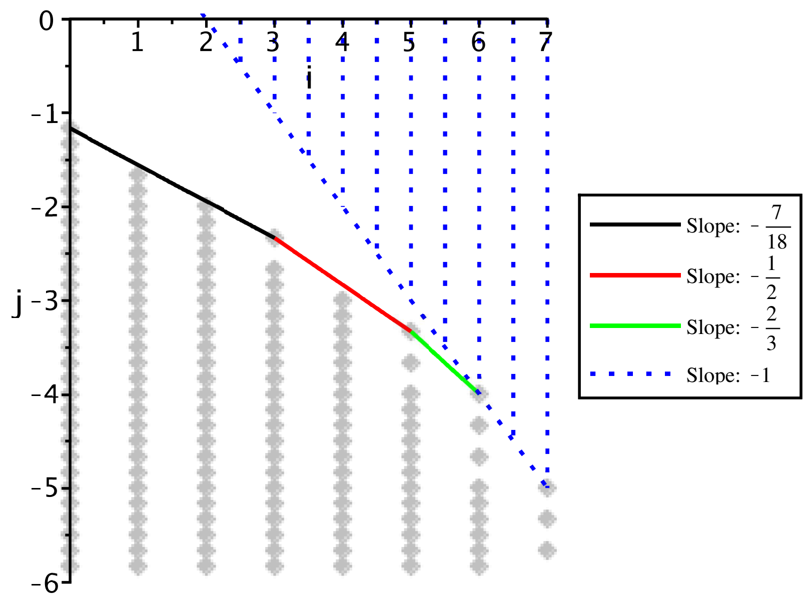
$$\kappa\, dup \tag{2.6.2.13}$$

Plot the boundary and the slopes of the Newton polygon;
Note that we have already proved that there are now points above the blue dotted line
This picture looks exactly the same as in the binary case

```
> LegendSize := size=[600,450]:
  P1dom1 := plot(-7/6-(7/18)*m,m=0..3,color=black,legend=[typeset
  ("Slope: ", -7/18)],legendstyle=[location=right]):
  P1dom2 := plot(-5/6-(1/2)*m,m=3..5,color=red,legend=[typeset
  ("Slope: ", -1/2)],legendstyle=[location=right]):
  P1dom2b := plot(0-(2/3)*m,m=5..6,color=green,legend=[typeset
  ("Slope: ", -2/3)],legendstyle=[location=right]):
  P1dom3a := plot(2-m,m=0..7,color=blue,linestyle=dot,legend=
  [typeset("Slope: ", -1)],legendstyle=[location=right]):
  P1all := display(P1,P1dom1,P1dom2,P1dom2b,P1dom3a,myview,
  LegendSize):

  for i from 1 to 10 do
     P1dom3[i] := plot([[2+i/2,0],[2+i/2,-i/2]],color=blue,
  linestyle=dot):
  end:

  myview := view=[0..7,0..-6]:
  display(P1all,seq(P1dom3[i],i=1..10),myview);
```



This is the choice for SF

```
> SF(n);
  subs(a=K,b=0,csubs,pterm=K*(7*K-6)/6,d=-1,%);
```

$$a+\frac{b}{n^{1/3}}+\frac{c}{n^{2/3}}+\frac{pterm}{n}+\frac{d}{n^{7/6}}$$

$$K+\frac{a1\,(K-1)^{1/3}\,K\,2^{2/3}}{2\,n^{2/3}}+\frac{K\,(7\,K-6)}{6\,n}-\frac{1}{n^{7/6}} \qquad \textbf{(2.6.2.14)}$$

recall

```
> subs(L=2/(K-1),[kaplam]);
```

$$\left[\mathrm{AiryAi}\left(\frac{\mathrm{AiryAiZeros}(1)\,n^{1/3}+2^{1/3}\left(\frac{1}{K-1}\right)^{1/3}m}{n^{1/3}}\right)=\kappa,\ \mathrm{AiryAi}\left(1,\right.\right. \qquad \textbf{(2.6.2.15)}$$

$$\left.\left.\frac{\mathrm{AiryAiZeros}(1)\,n^{1/3}+2^{1/3}\left(\frac{1}{K-1}\right)^{1/3}m}{n^{1/3}}\right)=\lambda\right]$$

Look at the corners

Here we see the necessary choice: p[4]>(K+2)^2/72/(K-1)^2 (note that for m>n^(1/3) lambda is negative)

```
> factor(getArray(posCF,PP,0,-7/6));
  factor(getArray(posCF,PP,3,-14/6));
  factor(getArray(posCF,PP,5,-20/6));
  factor(getArray(posCF,PP,6,-4));
  factor(getArray(posCF,PP,7,-5));
```

$$\kappa\,dup$$

$$-\frac{\left(72\,K^2\,p_4-K^2-144\,K\,p_4-4\,K+72\,p_4-4\right)K\,\lambda\,2^{1/3}}{18\,(K-1)^{4/3}}$$

$$-\frac{(K+2)\,K\,\lambda\,2^{1/3}\,p_4}{3\,(K-1)^{1/3}}$$

$$-\frac{\kappa\,K\,p_4\,(K^2+K+11)}{6\,(K-1)}$$

$$-\frac{\kappa\,K\,p_4\,(K^4+K^3+21\,K^2+11\,K-99)}{90\,(K-1)^2} \qquad \textbf{(2.6.2.16)}$$

```
> factor(getArray(posCF,PP,3,-14/6));
  collect(op(2,%),p[4],factor);
  solve(%,p[4]);
```

$$-\frac{(72\,K^2\,p_4-K^2-144\,K\,p_4-4\,K+72\,p_4-4)\,K\,\lambda\,2^{1/3}}{18\,(K-1)^{4/3}}$$

$$72\,(K-1)^2\,p_4-(K+2)^2$$

$$\frac{(K+2)^2}{72\,(K-1)^2} \qquad \textbf{(2.6.2.17)}$$

This is important!!!

```
> p[4] = (K + 2)^2/(72*(K - 1)^2);
  subs(K=2,%);     # what we got in the binary case
```

$$p_4=\frac{(K+2)^2}{72\,(K-1)^2}$$

$$p_4=\frac{2}{9} \qquad \textbf{(2.6.2.18)}$$

Now split the black dots into the contributions from Ai and Ai'

```
> indets(posCF);
```

$$\left\{K, a1, dup, \kappa, \lambda, p_0, p_1, p_2, p_3, p_4, \frac{1}{(K-1)^{14/3}}, \frac{1}{(K-1)^{13/3}}, \frac{1}{(K-1)^{11/3}}, \frac{1}{(K-1)^{10/3}}, \frac{1}{(K-1)^{8/3}}, \frac{1}{(K-1)^{7/3}}, \frac{1}{(K-1)^{5/3}}, \frac{1}{(K-1)^{4/3}}, \frac{1}{(K-1)^{2/3}}, \frac{1}{(K-1)^{1/3}}\right\} \qquad \textbf{(2.6.2.19)}$$

Sanity check that there are no other contributions

```
> map(factor,subs(kappa=0,lambda=0,posCF)):
  ArrayTools[IsZero](%);
```

$$true \qquad \textbf{(2.6.2.20)}$$

Extract the coefficients of kappa=Ai and lambda=Ai'
and treat then seperately

```
> posFk  := subs(kappa=1,lambda=0,posCF):indets(%);
  posFl  := subs(kappa=0,lambda=1,posCF):indets(%);
```

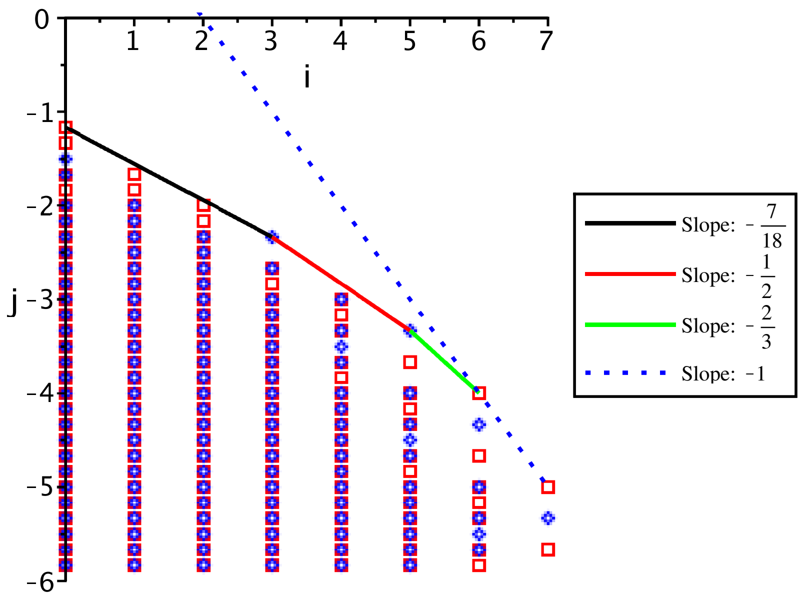
$$\left\{K, a1, dup, p_0, p_1, p_2, p_3, p_4, \frac{1}{(K-1)^{14/3}}, \frac{1}{(K-1)^{13/3}}, \frac{1}{(K-1)^{11/3}},\right.$$

$$\frac{1}{(K-1)^{10/3}}, \frac{1}{(K-1)^{8/3}}, \frac{1}{(K-1)^{7/3}}, \frac{1}{(K-1)^{5/3}}, \frac{1}{(K-1)^{4/3}}, \frac{1}{(K-1)^{2/3}}, \frac{1}{(K-1)^{1/3}}\Bigg\}$$

$$\Bigg\{K, a1, dup, p_0, p_1, p_2, p_3, p_4, \frac{1}{(K-1)^{14/3}}, \frac{1}{(K-1)^{13/3}}, \frac{1}{(K-1)^{11/3}}, \frac{1}{(K-1)^{10/3}}, \frac{1}{(K-1)^{8/3}}, \frac{1}{(K-1)^{7/3}}, \frac{1}{(K-1)^{5/3}}, \frac{1}{(K-1)^{4/3}}, \frac{1}{(K-1)^{2/3}}, \frac{1}{(K-1)^{1/3}}\Bigg\} \quad \textbf{(2.6.2.21)}$$

We color the non-zero nodes of the lastNewton polygon into
red squares............ coefficients of kappa=Ai
blue diamonds ...... coefficients of lambda=Ai'

```
> newt4a := mynewtArray(posFk,PP):
  newt4b := mynewtArray(posFl,PP):
> P4a := pointplot(newt4a,labels=["m deg", "n deg"], symbolsize=15,
  symbol=box,color=red):
  P4b := pointplot(newt4b,labels=["m deg", "n deg"], symbolsize=15,
  symbol=diamond, color=blue):
  P1dom3s := plot(1-m,m=4..5,color=black):
  display(P4a,P4b,P1dom1,P1dom2,P1dom2b,P1dom3a,myview,myoptionsUp,
  LegendSize,myview);
```



red extremes of Newton polygon

```
> mnmaxR := getMaxNewt(Mord,newt4a):
  seq([i,mnmaxR[i]],i=0..Mord);
```

$$\left[0, -\frac{7}{6}\right], \left[1, -\frac{5}{3}\right], [2, -2], \left[3, -\frac{8}{3}\right], [4, -3], \left[5, -\frac{11}{3}\right], [6, -4], [7, -5], [8, -\infty], [9, -\infty], [10, -\infty] \quad \textbf{(2.6.2.22)}$$

These are the specific values at these points;
we see that we still have some degree in freedom: d and p[4]

```
> for i from 0 to Mord do
     i,factor(getArray(posFk,PP,i,mnmaxR[i]));
  end;
```

$$0, dup$$

$$1, \frac{a1\, 2^{2/3} \left(K^2+K-3\right) K}{9\, (K-1)^{2/3}}$$

$$2, -\frac{K \left(216\, K^2 p_4 - 7\, K^2 - 432\, K\, p_4 - 40\, K + 216\, p_4 + 26\right)}{36\, (K-1)}$$

$$3, -\frac{\left(432\, K^3 p_4 + 7\, K^3 - 864\, K^2 p_4 + 18\, K^2 + 432\, K\, p_4 - 142\, K + 12\right) a1\, K\, 2^{2/3}}{216\, (K-1)^{5/3}}$$

$$4, -\frac{K \left(150\, K^3 p_4 - K^3 - 72\, K^2 p_4 - 3\, K^2 - 306\, K\, p_4 - 13\, K + 228\, p_4 - 22\right)}{36\, (K-1)^2}$$

$$5, -\frac{\left(36\, K^4 p_4 + K^4 - 36\, K^3 p_4 + K^3 - 36\, K^2 p_4 + 7\, K^2 + 36\, K\, p_4 - 4\, K - 44\right) a1\, 2^{2/3} K}{216\, (K-1)^{8/3}}$$

$$6, -\frac{K\, p_4 \left(K^2+K+11\right)}{6\, (K-1)}$$

$$7, -\frac{K\, p_4 \left(K^4+K^3+21\, K^2+11\, K-99\right)}{90\, (K-1)^2}$$

Error, (in getArray) bad index into Array

These are the slopes of the convex hull where the corners are given by the second sequence;
hence, in order to be positive when the slope > -1, we need to choose d>0, e.g. d=1;
note that p[4] is not important here, as the slope first slope -5/12 is less then -7/18;
and in the later regimes it will be dominated by the blue points.

```
> ls,li := getslopes(mnmaxR,Mord);
```

$$ls, li := \left[-\frac{5}{12}, -\frac{1}{2}, -1, -\infty\right], [0, 2, 6, 7, 10] \qquad \textbf{(2.6.2.23)}$$

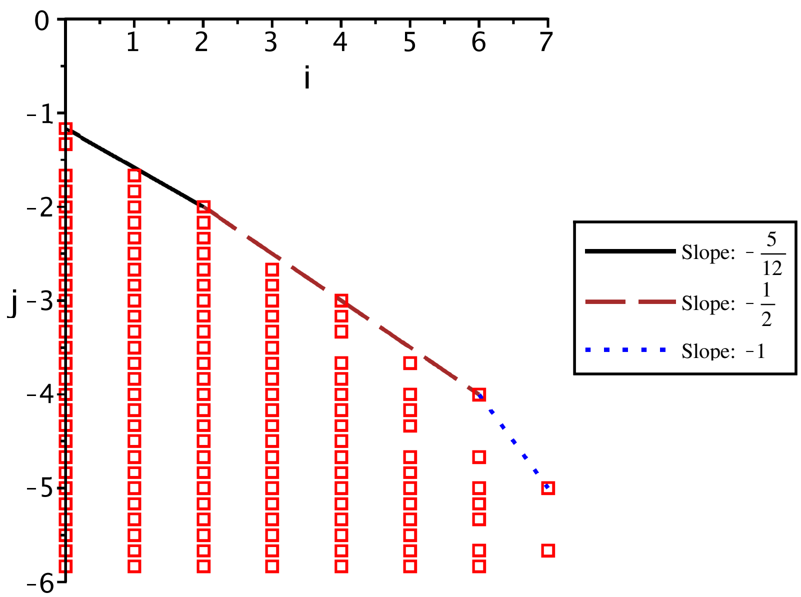
```
> colors := [green,black,brown,blue,olive,red]:
  styles := [spacedot,solid, dash, dot, dashdot, longdash,
  spacedash]:
```

Draw the conveux hull in red
picture is exactly the same as in the binary case!!!

```
> for i from 1 to nops(ls) do
     ls[i];li[i];
     tt[i] := plot((mnmaxR[li[i]]-ls[i]*li[i])+ls[i]*m,m=li[i]..li
  [i+1],color=colors[i mod nops(colors)+1],linestyle=styles[i mod
  nops(styles)+1],legend=[typeset("Slope: ", ls[i])],legendstyle=
  [location=right]):
  end:
  Pconvred := seq(tt[i],i=1..nops(ls)):
  #display(%);
```

Warning, unable to evaluate the function to numeric values in the region; see the plotting command's help page to ensure the calling sequence is correct

```
> display(Pconvred,P4a,myview,myoptionsUp,LegendSize);
```



We continue with the blue diamonds, i.e., the coefficients of Ai'

```
> mnmaxB := getMaxNewt(Mord,newt4b):
  seq([i,mnmaxB[i]],i=0..Mord-1);
```

$$\left[0, -\frac{3}{2}\right], [1, -2], \left[2, -\frac{7}{3}\right], \left[3, -\frac{7}{3}\right], [4, -3], \left[5, -\frac{10}{3}\right], \left[6, -\frac{13}{3}\right], \left[7, -\frac{16}{3}\right], [8, -\infty], [9, -\infty] \quad \textbf{(2.6.2.24)}$$

```
> for i from 0 to Mord-1 do
     i,factor(getArray(posFl,PP,i,mnmaxB[i]));
  end;
```

$$0, \frac{dup\, 2^{1/3}}{(K-1)^{1/3}}$$

$$1, \frac{a1\,(2K-3)\,(K^2-15K+46)\,K}{45\,(K-1)}$$

$$2, -\frac{1}{540\,(K-1)^{4/3}} \left(2^{1/3}\left(5K^3 a1^3 - 30K^2 a1^3 + 3240K^3 p_4 + 60K a1^3 - 57K^3 + 1620K^2 p_3 - 9720K^2 p_4 - 40 a1^3 + 168K^2 - 3240K p_3 + 9720K p_4 + 63K + 1620 p_3 - 3240 p_4 - 912\right) K\right)$$

$$3, -\frac{\left(72K^2 p_4 - K^2 - 144K p_4 - 4K + 72 p_4 - 4\right) K\, 2^{1/3}}{18\,(K-1)^{4/3}}$$

$$4, -\frac{K\, a1\,(K-2)\left(18K^2 p_4 + K^2 - 36K p_4 + 4K + 18 p_4 + 4\right)}{54\,(K-1)^2}$$

$$5, -\frac{(K+2)\,K\,2^{1/3}\, p_4}{3\,(K-1)^{1/3}}$$

$$6, -\frac{(K^3+K^2+K+11)\,K\,2^{1/3}\,p_4}{30\,(K-1)^{4/3}}$$

$$7, -\frac{(K^5+K^4+36\,K^3+K^2+281\,K-279)\,K\,2^{1/3}\,p_4}{630\,(K-1)^{7/3}}$$

Error, (in getArray) bad index into Array

Again we derive the slopes of the convex hull and its corners;

```
> ls,li := getslopes(mnmaxB,Mord-1);
```

$$ls, li := \left[-\frac{5}{18}, -\frac{1}{2}, -1, -\infty\right], [0, 3, 5, 7, 9] \qquad \textbf{(2.6.2.25)}$$

```
> colors := [brown,olive,purple,blue,olive,red,black]:
  styles := [spacedot,solid, dash, dot, dashdot, longdash,
  spacedash]:
```

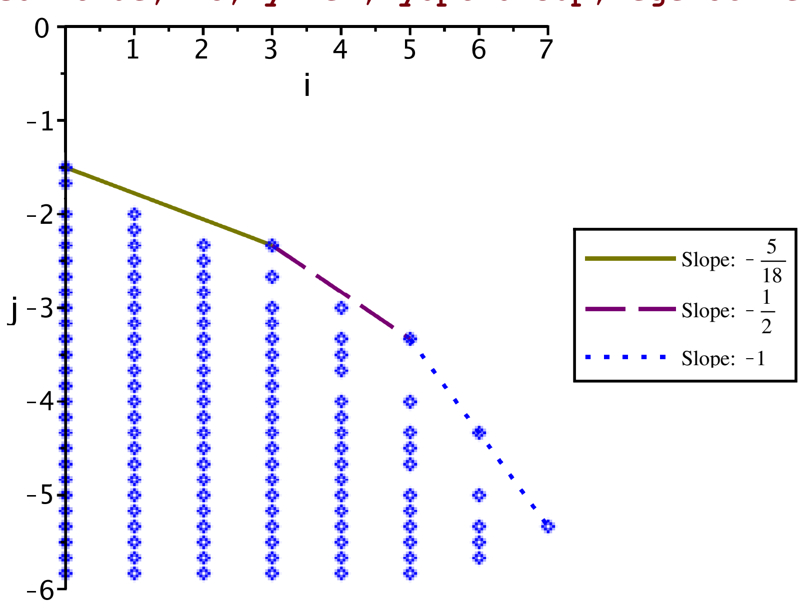
Draw the conveux hull in blue which still includes the term of order Theta(n^(-4/3))

```
> for i from 1 to nops(ls) do
     ls[i];li[i];
     tt[i] := plot((mnmaxB[li[i]]-ls[i]*li[i])+ls[i]*m,m=li[i]..li
  [i+1],color=colors[i mod nops(colors)+1],linestyle=styles[i mod
  nops(styles)+1],legend=[typeset("Slope: ", ls[i])],legendstyle=
  [location=right]):
  end:
  Pconvblue := seq(tt[i],i=1..nops(ls)):
  #display(%);
```

Warning, unable to evaluate the function to numeric values in the region; see the plotting command's help page to ensure the calling sequence is correct

picture is exactly the same as in the binary case!!!

```
> display(Pconvblue,P4b,myview,myoptionsUp,LegendSize);
```



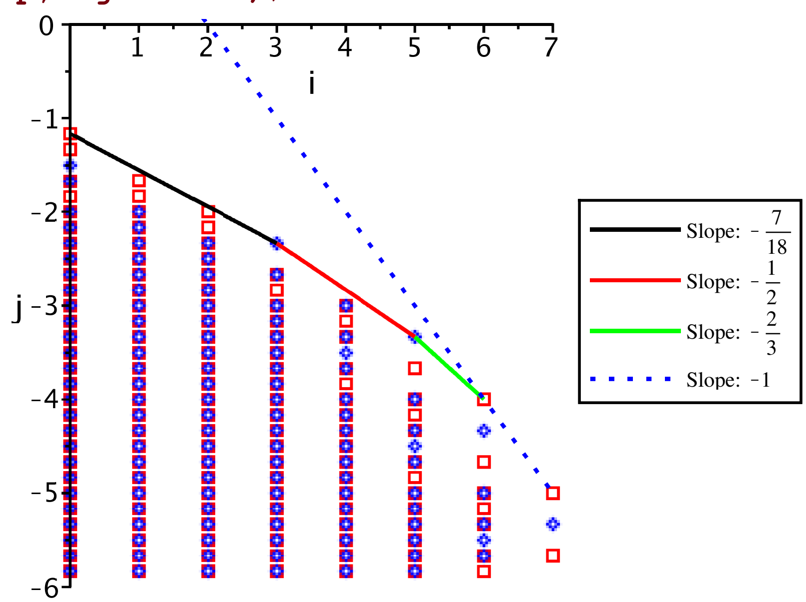

```
> varsLowerBound;
```

$$\left[q_0=0, q_1=\frac{7\,K-11}{6\,K-6}, q_2=-\frac{K+2}{6\,(K-1)}, qq_0=0, qq_1=-\frac{a1\,(K-2)\,2^{2/3}}{6\,(K-1)^{2/3}}, ss_2 \right. \qquad \textbf{(2.6.2.26)}$$

$$= \frac{2^{1/3}\, a1^2\, (K-2)^2}{36\,(K-1)^{4/3}},\ rr_3 = \frac{a1\,(K-2)\,(K+2)\,2^{2/3}}{36\,(K-1)^{5/3}},\ L = \frac{2}{K-1},\ a = K,\ b = 0,\ c$$
$$= \frac{a1\,(K-1)^{1/3}\,K\,2^{2/3}}{2},\ pterm = \frac{K\,(7\,K-6)}{6},\ d = -1 \Big]$$

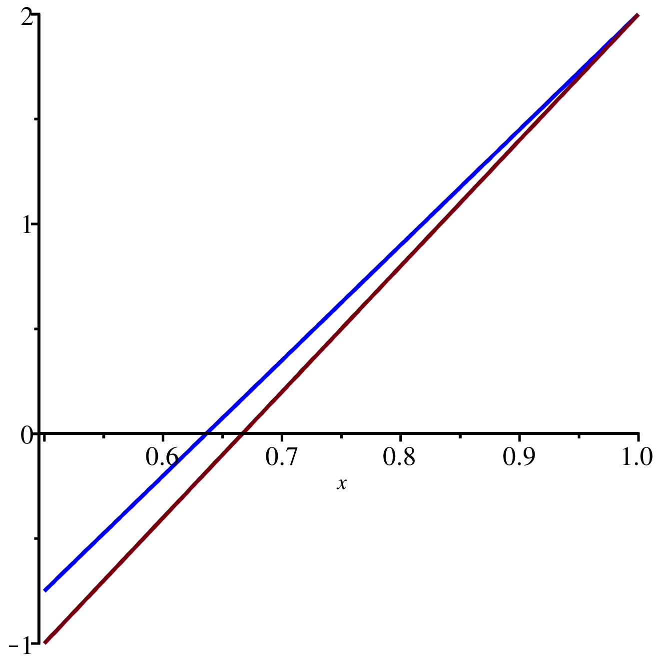
We remove the other p[0], p[1], p[2], and p[3] contributions and recompute the Newton polygons. (note that the coefficients of lower order terms of posFk change as well, which is why we recompute them; however the picture will not change as the convex hull is not influenced by this change.)

```
> posFk2 := map(factor,subs(p[3]=0,p[2]=0,p[1]=0,p[0]=0,q[0]=0,
  posFk)):
  newt4a2 := mynewtArray(posFk2,PP):
  P4a2 := pointplot(newt4a2,labels=["m deg", "n deg"], symbolsize=
  15, symbol=box, color=red):
  posFl2 := map(factor,subs(p[3]=0,p[2]=0,p[1]=0,p[0]=0,q[0]=0,
  posFl)):
  newt4b2 := mynewtArray(posFl2,PP):
  P4b2 := pointplot(newt4b2,labels=["m deg", "n deg"], symbolsize=
  15, symbol=diamond, color=blue):
  display(P4a2,P4b2,P1dom1,P1dom2,P1dom2b,P1dom3a,myview,
  myoptionsUp,LegendSize);
```



Proving the fourth regime 1/2<m<=n^(1-eps)
blue domiantes red

```
> P1 := plot(6*x-4,x=1/2..1):
  P2 := plot((5+1/2)*x-(10/3+1/6),x=1/2..1,color=blue):
  display(%,%%);
```



Summary

The general ansatz for the lower bound

```
> facAiryUp*AiryAi(a1+(L)^(1/3)*(m+1)/n^(1/3));
  SF(n);
```

$$\left(1 + \frac{m^2 q_2 + m q_1 + q_0}{n} + \frac{m\, qq_1 + qq_0}{n^{2/3}} + \frac{ss_2\, m^2}{n^{4/3}} + \frac{rr_3\, m^3}{n^{5/3}} + \frac{p_4 m^4 + m^3 p_3 + m^2 p_2 + m p_1 + p_0}{n^2}\right) \mathrm{AiryAi}\left(a1 + \frac{L^{1/3}(m+1)}{n^{1/3}}\right)$$

$$a + \frac{b}{n^{1/3}} + \frac{c}{n^{2/3}} + \frac{pterm}{n} + \frac{d}{n^{7/6}} \qquad \textbf{(2.6.3.1)}$$

Our choices additional to the lower case
Only d changes to d=+1 instead of d=-1 and p[4] is there additionally

```
> p[4]>(K + 2)^2/(72*(K - 1)^2);
  p[0]=0,p[1]=0,p[2]=0,p[3]=0,
  d=1;
```

$$\frac{(K+2)^2}{72\,(K-1)^2} < p_4$$

$$p_0 = 0, p_1 = 0, p_2 = 0, p_3 = 0, d = 1 \qquad \textbf{(2.6.3.2)}$$

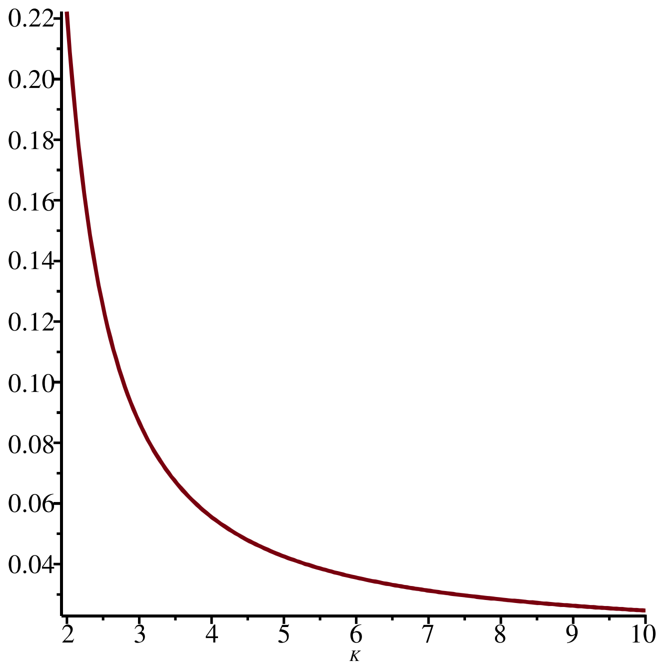
```
> p4lowbnd := (K + 2)^2/(72*(K - 1)^2);
  subs(K=2,%);
```

$$p4lowbnd := \frac{(K+2)^2}{72\ (K-1)^2}$$

$$\frac{2}{9} \tag{2.6.3.3}$$

this function lower bounding p[4] is monotonically decreasing,
so we can use the same as in the binary case: there we used p[4]=1/3

```
> plot(p4lowbnd,K=2..10);
  simplify(diff(p4lowbnd,K));
```



$$\frac{-K-2}{12\ (K-1)^3} \tag{2.6.3.4}$$

a possible choice

```
> varsUpperBound := [op(varsLowerBound[1..-2]),d=1,p[0]=0,p[1]=0,p
  [2]=0,p[3]=0,p[4]=1/3];
```

$$varsUpperBound := \left[q_0 = 0, q_1 = \frac{7\,K-11}{6\,K-6}, q_2 = -\frac{K+2}{6\ (K-1)}, qq_0 = 0, qq_1 = -\frac{a1\ (K-2)\ 2^{2/3}}{6\ (K-1)^{2/3}}, ss_2 = \frac{2^{1/3}\,a1^2\ (K-2)^2}{36\ (K-1)^{4/3}}, rr_3 = \frac{a1\ (K-2)\ (K+2)\ 2^{2/3}}{36\ (K-1)^{5/3}}, L = \frac{2}{K-1}, a = K, b = 0, c = \frac{a1\ (K-1)^{1/3}\ K\ 2^{2/3}}{2}, pterm = \frac{K\ (7\,K-6)}{6}, d = 1, p_0 = 0, p_1 = 0, p_2 = 0, p_3 = 0, p_4 = \frac{1}{3}\right] \tag{2.6.3.5}$$

Set p[4]=1/3 as in the relaxed case works

```
> subs(varsUpperBound, facAiryUp*AiryAi(a1+(L)^(1/3)*(m+1)/n^(1/3))
  );
  subs(varsUpperBound, SF(n));
```

$$\left(1+\frac{-\frac{m^2(K+2)}{6(K-1)}+\frac{m(7K-11)}{6K-6}}{n}-\frac{m\,a1\,(K-2)\,2^{2/3}}{6(K-1)^{2/3}n^{2/3}}+\frac{2^{1/3}a1^2(K-2)^2m^2}{36(K-1)^{4/3}n^{4/3}}\right.$$

$$\left.+\frac{a1\,(K-2)(K+2)\,2^{2/3}m^3}{36(K-1)^{5/3}n^{5/3}}+\frac{m^4}{3n^2}\right)\mathrm{AiryAi}\left(a1\right.$$

$$\left.+\frac{2^{1/3}\left(\frac{1}{K-1}\right)^{1/3}(m+1)}{n^{1/3}}\right)$$

$$K+\frac{a1\,(K-1)^{1/3}K\,2^{2/3}}{2n^{2/3}}+\frac{K(7K-6)}{6n}+\frac{1}{n^{7/6}} \tag{2.6.3.6}$$

Asymptotic expansion of directed recurrence

reset the (possibly) fixed values of these variables

```
> K:='K':n:='n':m:='m':
```

The generic type: Dyck-like

```
> rec1  := r(u,w)  = A(u,w)*r(u-1,w-1)  + B(u,w)*r(u-1,w+K-1);
  AA  := (K - 1)^2*(K + u -  w)/(K*u - u + w);
  BB  := 1;
```

$$rec1 := r(u,w) = A(u,w)\,r(u-1,w-1)+B(u,w)\,r(u-1,w+K-1)$$

$$AA := \frac{(K-1)^2(K+u-w)}{Ku-u+w}$$

$$BB := 1 \tag{3.1}$$

assume now

```
> rsubs := r(u,w)  = h(u)*(f(u^(-1/3)*(w+1))  + u^(-1/3)*g(u^(-1/3)*
  (w+1)));
```

$$rsubs := r(u,w) = h(u)\left(f\left(\frac{w+1}{u^{1/3}}\right)+\frac{g\left(\frac{w+1}{u^{1/3}}\right)}{u^{1/3}}\right) \tag{3.2}$$

and additionally this where lambda is a constant:
we want f(lambda)

```
> wsubs  := w = lambda*u^(1/3)-1;
```

$$wsubs := w = \lambda\,u^{1/3}-1 \tag{3.3}$$

plug everything into the recurrence

It seems to be a good idea to have a symmetric parameter in f.

```
> rec1a := subs(rsubs,subs(u=u-1,w=w-1,rsubs),subs(u=u-1,w=w+K-1,
  rsubs),rec1);
```

$$rec1a := h(u)\left(f\left(\frac{w+1}{u^{1/3}}\right)+\frac{g\left(\frac{w+1}{u^{1/3}}\right)}{u^{1/3}}\right)=A(u,w)\,h(u-1)\left(f\left(\frac{w}{(u-1)^{1/3}}\right)+\frac{g\left(\frac{w}{(u-1)^{1/3}}\right)}{(u-1)^{1/3}}\right)+B(u,w)\,h(u-1)\left(f\left(\frac{w+K}{(u-1)^{1/3}}\right)+\frac{g\left(\frac{w+K}{(u-1)^{1/3}}\right)}{(u-1)^{1/3}}\right) \quad (3.4)$$

additionally bind w to u

```
> wsubs;
  rec1b := subs(wsubs,rec1a);
```

$$w=\lambda u^{1/3}-1$$

$$rec1b := h(u)\left(f(\lambda)+\frac{g(\lambda)}{u^{1/3}}\right)=A\left(u,\lambda u^{1/3}-1\right)h(u-1)\left(f\left(\frac{\lambda u^{1/3}-1}{(u-1)^{1/3}}\right)+\frac{g\left(\frac{\lambda u^{1/3}-1}{(u-1)^{1/3}}\right)}{(u-1)^{1/3}}\right)+B\left(u,\lambda u^{1/3}-1\right)h(u-1)\left(f\left(\frac{\lambda u^{1/3}-1+K}{(u-1)^{1/3}}\right)+\frac{g\left(\frac{\lambda u^{1/3}-1+K}{(u-1)^{1/3}}\right)}{(u-1)^{1/3}}\right) \quad (3.5)$$

```
> wsubs;
  subs(wsubs,AA);
  #series(%,u=infinity,2);
  subs(wsubs,BB);
```

$$w=\lambda u^{1/3}-1$$

$$\frac{(K-1)^2\left(K+u-\lambda u^{1/3}+1\right)}{K u-u+\lambda u^{1/3}-1}$$

$$1 \quad (3.6)$$

replace also A and B by their closed forms

```
> rec1c := collect(normal(subs(op(subs(wsubs,[A(u,w)=AA,B(u,w)=BB])
  ), K=K, rec1b)/h(u-1)),f):
  indets(%);
```

$$\left\{K,\lambda,u,\frac{1}{u^{1/3}},u^{1/3},\frac{1}{(u-1)^{1/3}},(u-1)^{1/3},f(\lambda),f\left(\frac{\lambda u^{1/3}-1}{(u-1)^{1/3}}\right),f\left(\frac{\lambda u^{1/3}-1+K}{(u-1)^{1/3}}\right),g(\lambda),g\left(\frac{\lambda u^{1/3}-1}{(u-1)^{1/3}}\right),g\left(\frac{\lambda u^{1/3}-1+K}{(u-1)^{1/3}}\right),h(u),h(u-1)\right\} \quad (3.7)$$

expansion of the RHS for u large

```
> rec1rhs := map(simplify,series(rhs(rec1c),u=infinity,2)) assuming
  u::posint:
```

the LHS

```
> rec1lhs := lhs(rec1c);
```

$$rec1lhs := \frac{h(u)\,f(\lambda)}{h(u-1)} + \frac{h(u)\,g(\lambda)}{h(u-1)\,u^{1/3}} \quad \textbf{(3.8)}$$

asume such an expansion with constants independent of u

```
> huquotansatz := C[0] + C[1]*u^(-1/3) + C[2]*u^(-2/3) + C[3]*u^
  (-1);
```

$$huquotansatz := C_0 + \frac{C_1}{u^{1/3}} + \frac{C_2}{u^{2/3}} + \frac{C_3}{u} \quad \textbf{(3.9)}$$

use U3 instead of u^(1/3) in order to extract coefficients next
Here we want all of the coefficients should be zero; we match coefficients.

```
> collect(huquotansatz*subs(h(u)=1,h(u-1)=1,rec1lhs) - rec1rhs,u):
  convert(%,polynom):eqsys :=collect(simplify(subs(u=U3^3,%)),U3,
  factor) assuming U3>0:
```

constant coefficient, solve for C0

```
> cf0 := coeff(eqsys,U3,0);
  C0s := isolate(%,C[0]);
```

$$cf0 := -f(\lambda)\,(K - C_0)$$

$$C0s := C_0 = K \quad \textbf{(3.10)}$$

The coefficient of u^(-1/3) should be zero:
we Set C1 to to zero

```
> cf1 := factor(coeff(subs(C0s,eqsys),U3,-1));
  C1s := isolate(collect(numer(%),f,factor),C[1]);
```

$$cf1 := C_1\,f(\lambda)$$

$$C1s := C_1 = 0 \quad \textbf{(3.11)}$$

coefficient u^(-2/3):
The Airy function appears!!

```
> cf2 := coeff(subs(C0s,C1s,eqsys),U3,-2):
  collect(numer(%),f,factor);
  isolate(%,C[2]);
  fchoice1 := dsolve(%,f(lambda)) assuming K::posint;
```

$$(2\,K\,\lambda + 2\,C_2)\,f(\lambda) - K\,(K-1)\,\mathrm{D}^{(2)}(f)\,(\lambda)$$

$$C_2 = \frac{K\,(K-1)\,\mathrm{D}^{(2)}(f)\,(\lambda)}{2\,f(\lambda)} - K\,\lambda$$

$$fchoice1 := f(\lambda) = \_C1\ \mathrm{AiryAi}\left(\frac{2^{1/3}\,(K\,\lambda + C_2)}{(K-1)^{1/3}\,K}\right) + \_C2\ \mathrm{AiryBi}\left(\frac{2^{1/3}\,(K\,\lambda + C_2)}{(K-1)^{1/3}\,K}\right) \quad \textbf{(3.12)}$$

We choose the AiryAi function, as for lambda large it should decay and always stay above the x-axis (this is a density; to prove)

```
> fchoice2 := subs(_C1=1,_C2=0,fchoice1);
```

$$fchoice2 := f(\lambda) = \mathrm{AiryAi}\left(\frac{2^{1/3}\,(K\,\lambda + C_2)}{(K-1)^{1/3}\,K}\right) \quad \textbf{(3.13)}$$

now we have ee_{n,-1}=0,
which translates to f(0)=0.
Moreover, as noted above, this should be a density, so always be 0 afterwards, thus the parameter has to be equal to the largest Airy root a1

```
> (rhs(subs(_C1=1,lambda=0,AiryAi=ff,fchoice2))):ff:=x->x:%%;
  C2s := map(simplify,isolate(%=a1,C[2])) assuming K::posint;
```

$$\frac{2^{1/3}\,C_2}{(K-1)^{1/3}\,K}$$

$$C2s := C_2 = \frac{a1\,2^{2/3}\,(K-1)^{1/3}\,K}{2} \quad \textbf{(3.14)}$$

```
> fchoice3 := map(expand,subs(C2s,fchoice2)) assuming K::posint;
```

$$fchoice3 := f(\lambda) = \mathrm{AiryAi}\left(\frac{2^{1/3}\,\lambda}{(K-1)^{1/3}} + a1\right) \quad \textbf{(3.15)}$$

coefficient u^(-2/3):
should fix the polynomial term

```
> cf3 := coeff(subs(C0s,C1s,eqsys),U3,-3);
```

$$cf3 := C_3 f(\lambda) + C_2\,g(\lambda) - \frac{K^3\,\mathrm{D}^{(3)}(f)(\lambda)}{6} - \frac{K^2\,\mathrm{D}^{(2)}(g)(\lambda)}{2} + \frac{K^2\,\mathrm{D}^{(3)}(f)(\lambda)}{2} - K^2 f(\lambda) - \frac{4\,K\,\mathrm{D}(f)(\lambda)\,\lambda}{3} + K\,g(\lambda)\,\lambda + \frac{K\,\mathrm{D}^{(2)}(g)(\lambda)}{2} - \frac{K\,\mathrm{D}^{(3)}(f)(\lambda)}{3} \quad \textbf{(3.16)}$$

```
> fchoice2;
```

$$f(\lambda) = \mathrm{AiryAi}\left(\frac{2^{1/3}\,(K\lambda + C_2)}{(K-1)^{1/3}\,K}\right) \quad \textbf{(3.17)}$$

so we can make the ansatz for g:
polynomial multiples of AiryAi and AiryAiPrime
note that I want g(0)=0, so I start the polynomial at AiryAiPrime with lambda

```
> gansatz := add(g0[i]*lambda^i,i=0..2)*AiryAi(2^(1/3)*(K*lambda +
  C[2])/((K - 1)^(1/3)*K))
  + add(g1[i]*lambda^i,i=1..2)*AiryAi(1,2^(1/3)*(K*lambda + C[2])/(
  (K - 1)^(1/3)*K));
```

$$gansatz := \left(g0_2\,\lambda^2 + g0_1\,\lambda + g0_0\right)\mathrm{AiryAi}\left(\frac{2^{1/3}\,(K\lambda + C_2)}{(K-1)^{1/3}\,K}\right) + \left(g1_2\,\lambda^2 + g1_1\,\lambda\right)\mathrm{AiryAi}\left(1, \frac{2^{1/3}\,(K\lambda + C_2)}{(K-1)^{1/3}\,K}\right) \quad \textbf{(3.18)}$$

Works :)

```
> collect(convert(subs(fchoice2,g(lambda)=gansatz,convert(cf3,diff)
  ),D),[AiryAi,lambda],simplify@factor) assuming K::posint:
  tmp := subs(g1[2]=0,g1[1]=0,%);
```

$$tmp := \left(-\left(g0_2+\frac{4}{3}\right)K^2-\left(-g0_2-\frac{2}{3}\right)K+C_3\right)\mathrm{AiryAi}\left(\frac{2^{1/3}\left(K\lambda+C_2\right)}{(K-1)^{1/3}K}\right)+\left(-\frac{\left(6\,K\,g0_2+K-6\,g0_2+2\right)K\,2^{1/3}\lambda}{3\,(K-1)^{1/3}}-\frac{\left(K^2\,g0_1+\left(\frac{C_2}{3}-g0_1\right)K-\frac{2\,C_2}{3}\right)2^{1/3}}{(K-1)^{1/3}}\right)\mathrm{AiryAi}\left(1,\frac{2^{1/3}\left(K\lambda+C_2\right)}{(K-1)^{1/3}K}\right) \quad \textbf{(3.19)}$$

```
> 6*K*g0[2] + K - 6*g0[2] + 2;
  g02s := normal(isolate(%,g0[2]));
```

$$6\,K\,g0_2+K-6\,g0_2+2$$

$$g02s := g0_2=-\frac{K+2}{6\,(K-1)} \quad \textbf{(3.20)}$$

```
> K^2*g0[1] + (C[2]/3 - g0[1])*K - (2*C[2])/3;
  g01s := normal(isolate(%,g0[1]));
```

$$K^2\,g0_1+\left(\frac{C_2}{3}-g0_1\right)K-\frac{2\,C_2}{3}$$

$$g01s := g0_1=-\frac{C_2\,(K-2)}{3\,K\,(K-1)} \quad \textbf{(3.21)}$$

```
> -(g0[2] + 4/3)*K^2 - (-g0[2] - 2/3)*K + C[3];
  C3s := normal(isolate(subs(g0[2] = -(K + 2)/(6*(K - 1)), g0[1] =
  -C[2]*(K - 2)/(3*K*(K - 1)), %), C[3]));
```

$$-\left(g0_2+\frac{4}{3}\right)K^2-\left(-g0_2-\frac{2}{3}\right)K+C_3$$

$$C3s := C_3=\frac{(7\,K-6)\,K}{6} \quad \textbf{(3.22)}$$

additionally set g0[0]=0 and g1[1]=0 (not needed)
only The Airy function needed :)

```
> gchoice1 := g(lambda) = subs(g1[2]=0,g1[1]=0,g0[0]=0,g02s,g01s,
  gansatz);
  gchoice2 := subs(AiryAi(2^(1/3)*(K*lambda + C[2])/((K - 1)^(1/3)*
  K))=rhs(fchoice3),C2s,%);
```

$$gchoice1 := g(\lambda)=\left(-\frac{(K+2)\,\lambda^2}{6\,(K-1)}-\frac{C_2\,(K-2)\,\lambda}{3\,K\,(K-1)}\right)\mathrm{AiryAi}\left(\frac{2^{1/3}\left(K\lambda+C_2\right)}{(K-1)^{1/3}K}\right)$$

$$gchoice2 := g(\lambda)=\left(-\frac{(K+2)\,\lambda^2}{6\,(K-1)}-\frac{a1\,2^{2/3}\,(K-2)\,\lambda}{6\,(K-1)^{2/3}}\right)\mathrm{AiryAi}\left(\frac{2^{1/3}\lambda}{(K-1)^{1/3}}+a1\right) \quad \textbf{(3.23)}$$

```
> #expand(subs(C2s,2^(1/3)*(K*lambda + C[2])/((K - 1)^(1/3)*K)));
```

These are our constants

```
> hucf := [C0s,C1s,C2s,C3s];
  subs(%,huquotansatz);
```

$$hucf := \left[C_0 = K, C_1 = 0, C_2 = \frac{a1\, 2^{2/3}\, (K-1)^{1/3}\, K}{2}, C_3 = \frac{(7\,K-6)\,K}{6}\right]$$

$$K + \frac{a1\, 2^{2/3}\, (K-1)^{1/3}\, K}{2\, u^{2/3}} + \frac{(7\,K-6)\,K}{6\,u} \qquad \textbf{(3.24)}$$

polynomial term:

```
> simplify(rhs(C3s)/K-1/3);
  seq(subs(K=i,%),i=2..10);
```

$$\frac{7\,K}{6} - \frac{4}{3}$$

$$1, \frac{13}{6}, \frac{10}{3}, \frac{9}{2}, \frac{17}{3}, \frac{41}{6}, 8, \frac{55}{6}, \frac{31}{3} \qquad \textbf{(3.25)}$$

```
> i:='i':
```

Compute asymptotics

```
> huquotansatz;
  subs(hucf,%);
  K^n*product(expand(subs(u=i,%)/K),i=1..n);
```

$$C_0 + \frac{C_1}{u^{1/3}} + \frac{C_2}{u^{2/3}} + \frac{C_3}{u}$$

$$K + \frac{a1\, 2^{2/3}\, (K-1)^{1/3}\, K}{2\, u^{2/3}} + \frac{(7\,K-6)\,K}{6\,u}$$

$$K^n \left(\prod_{i=1}^{n} \left(1 + \frac{a1\, 2^{2/3}\, (K-1)^{1/3}}{2\, i^{2/3}} + \frac{7\,K}{6\,i} - \frac{1}{i}\right)\right) \qquad \textbf{(3.26)}$$

```
> aasubs := a = -a1*2^(2/3)*(K - 1)^(1/3)/(2);
  bbsubs := b = (7*K)/(6) - 1;
```

$$aasubs := a = -\frac{a1\, 2^{2/3}\, (K-1)^{1/3}}{2}$$

$$bbsubs := b = \frac{7\,K}{6} - 1 \qquad \textbf{(3.27)}$$

```
> int(log(1-a/x^(2/3))+b/x,x);
  map(simplify,series(subs(x=n,%),n=infinity,2)) assuming
  n::posint;
```

$$b \ln(x) + x \ln\left(1 - \frac{a}{x^{2/3}}\right) + 2\, a^{3/2} \operatorname{arctanh}\left(\frac{\sqrt{a}}{x^{1/3}}\right) - 2\, x^{1/3}\, a$$

$$-3\, n^{1/3}\, a + b \ln(n) + \frac{3\, a^2}{2\, n^{1/3}} + O\left(\frac{1}{n}\right) \qquad \textbf{(3.28)}$$

Hence (we are at the end interested in h(2n))
and we get the predicted stretched exponential
finally, also correct the polynomial term by 1/3

```
> h(n) = K^n * subs(aasubs,bbsubs,exp(-3*n^(1/3)*a)*n^(b-1/3))*(1+o
  (1));
  subs(n=K*n,%);
```

$$h(n) = K^n \mathrm{e}^{\frac{3 n^{1/3} a1\, 2^{2/3} (K-1)^{1/3}}{2}} n^{\frac{7K}{6} - \frac{4}{3}} (1 + o(1))$$

$$h(K n) = K^{K n} \mathrm{e}^{\frac{3 (K n)^{1/3} a1\, 2^{2/3} (K-1)^{1/3}}{2}} (K n)^{\frac{7K}{6} - \frac{4}{3}} (1 + o(1)) \quad (3.29)$$

```
> ((3*(K)^(1/3)*a1*2^(2/3)*(K - 1)^(1/3))/2)*n^(1/3);
  3*a1*(K*(K - 1)/2)^(1/3)*n^(1/3);
  simplify(%/%%) assuming K::posint;
```

$$\frac{3 K^{1/3} a1\, 2^{2/3} (K-1)^{1/3} n^{1/3}}{2}$$

$$\frac{3\, a1\, 2^{2/3} (K (K-1))^{1/3} n^{1/3}}{2}$$

$$1 \quad (3.30)$$

This should be the asymptotics

```
> factor((7*K)/6 - 4/3) assuming K::posint;
  factor((7*K-8)/6) assuming K::posint;
```

$$\frac{7K}{6} - \frac{4}{3}$$

$$\frac{7K}{6} - \frac{4}{3} \quad (3.31)$$

General Dyck case of Lemma 4.5 [Elvey Price, Fang, Wallner]

We want to prove
for integers 0 <= s1 < s2 <= r <= K*n,
k-j even, that

```
> p(r,s1,K*n)/(s1+1) >= p(r,s2,K*n)/(s2+1);
```

$$\frac{p(r, s2, K n)}{s2 + 1} \leq \frac{p(r, s1, K n)}{s1 + 1} \quad (4.1)$$

It suffices to prove that the following is non-negative

```
> LLp := p(r,s-1,K*n)/s-p(r,s+K-1,K*n)/(s+K);
```

$$LLp := \frac{p(r, s-1, K n)}{s} - \frac{p(r, s+K-1, K n)}{s+K} \quad (4.2)$$

this is the recurrence holding for m non-negative

```
> prec := p(r,s,K*n) = p(r+1,s-K+1,K*n) + (K-1)^2*(r-s+K)/((K-1)*
  (r+1)+s+1)*p(r+1,s+1,K*n);
```

$$prec := p(r, s, K n) = p(r+1, s-K+1, K n) + \frac{(K-1)^2 (r-s+K)\, p(r+1, s+1, K n)}{(K-1)(r+1) + s + 1} \quad (4.3)$$

These relations are used in the induction step

```
> pup := factor(subs(s=s+1,isolate(LLp,p(r,s-1,K*n))));
  pdown := factor(subs(s=s-K+1,isolate(LLp,p(r,s+K-1,K*n))));
```

$$pup := p(r,s,K\,n) = \frac{p(r,s+K,K\,n)\;(s+1)}{s+1+K}$$

$$pdown := p(r,s,K\,n) = -\frac{p(r,s-K,K\,n)\;(s+1)}{-s+K-1} \qquad \textbf{(4.4)}$$

Now, we perform an induction on s, with s=K*n being the base case.
We start by applying the recurrence to LLp.

```
> pe1 := collect(subs(subs(s=s-1,prec),subs(s=s+K-1,prec),LLp),p,
  factor);
```

$$pe1 := \frac{1}{(K\,r+K-r+s-1)\;s\;(s+K)}\Big(\big(K^4+K^3\,r+K^2\,r\,s-K^2\,s^2-K^3-2\,K^2\,r+K^2\,s-3\,K\,r\,s+2\,K\,s^2-K^2+K\,r-3\,K\,s+2\,r\,s-2\,s^2+K+2\,s\big)\;p(r+1,s,K\,n)\Big)+\frac{p(r+1,s-K,K\,n)}{s}-\frac{(K-1)^2\;(r-s+1)\;p(r+1,s+K,K\,n)}{(K\,r+2\,K-r+s-1)\;(s+K)} \qquad \textbf{(4.5)}$$

Now we can use the inductive hypothesis

Here we use the induction hypothesis and get the rational function which is we need to prove to be positive for r>=s>=1

```
> pe2 := collect(subs(
     subs(r=r+1,s=s-K,pup),
     subs(r=r+1,s=s+K,pdown),
  pe1), p,factor);
  pe2cf := subs(p(r+1,s,K*n)=1,%):
```

$$pe2 := \Big((K-1)\;\big(K^2\,r+K\,r\,s+2\,K^2-2\,K\,r+2\,K\,s-r\,s-3\,K+r-3\,s+1\big)\;K^2\,p(r+1,s,K\,n)\Big)\Big/\Big((K\,r+2\,K-r+s-1)\;(K\,r+K-r+s-1)\;s\;(s+K)\Big) \qquad \textbf{(4.6)}$$

binary case: clear

```
> subs(K=2,pe2cf);
```

$$\frac{4\;(r\,s+r+s+3)}{(r+s+3)\;(r+s+1)\;s\;(s+2)} \qquad \textbf{(4.7)}$$

ternary case: clear

```
> subs(K=3,pe2cf);
```

$$\frac{18\;(2\,r\,s+4\,r+3\,s+10)}{(2\,r+s+5)\;(2\,r+s+2)\;s\;(s+3)} \qquad \textbf{(4.8)}$$

4-ary case: clear

```
> subs(K=4,pe2cf);
```

$$\frac{48\;(3\,r\,s+9\,r+5\,s+21)}{(3\,r+s+7)\;(3\,r+s+3)\;s\;(s+4)} \qquad \textbf{(4.9)}$$

for K>=2 all coefficients in the nuemrator are positive

```
> collect(numer(pe2cf),[r,s],factor);
```

$$\big((K-1)^2\,K^2\,s+(K-1)^3\,K^2\big)\;r+(K-1)\;(2\,K-3)\;K^2\,s+(K-1)^2\;(2\,K-1)\;K^2 \qquad \textbf{(4.10)}$$

also the denominator is clearly positive for K>=2

```
> denom(pe2cf);
  %/s/(s+K);
  (K*r + K - r + s - 1)*(K*r + K - r + s - 1) + K*collect((K*r + K
  - r + s - 1),[r,s],factor);
  factor(%/%%);
```

$$(K\,r+2\,K-r+s-1)\;(K\,r+K-r+s-1)\;s\;(s+K)$$

$$
\left[\begin{matrix} (K\,r + K - r + s - 1)\,(K\,r + 2\,K - r + s - 1) \\ (K\,r + K - r + s - 1)^2 + K\,((K - 1)\,r + s + K - 1) \\ 1 \end{matrix}\right. \tag{4.11}
$$