Dyck paths and inversion tables

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Inversion table

Let $\pi = \pi_1\pi_2\dots\pi_n$ be a permutation. A pair (π_j, π_k) is called an *inversion* if $j < k$ and $\pi_j > \pi_k$. The *(right-)inversion table* $(r_1, r_2, \dots, r_n)$ of π is

$$r_i := |\{\pi_j : (i, \pi_j) \text{ is an inversion}\}|.$$

Observations:

- $0 \le r_i \le i - 1$
- $(r_i)_{i=1}^n$ uniquely characterizes π; see [4, Section 5.1.1]

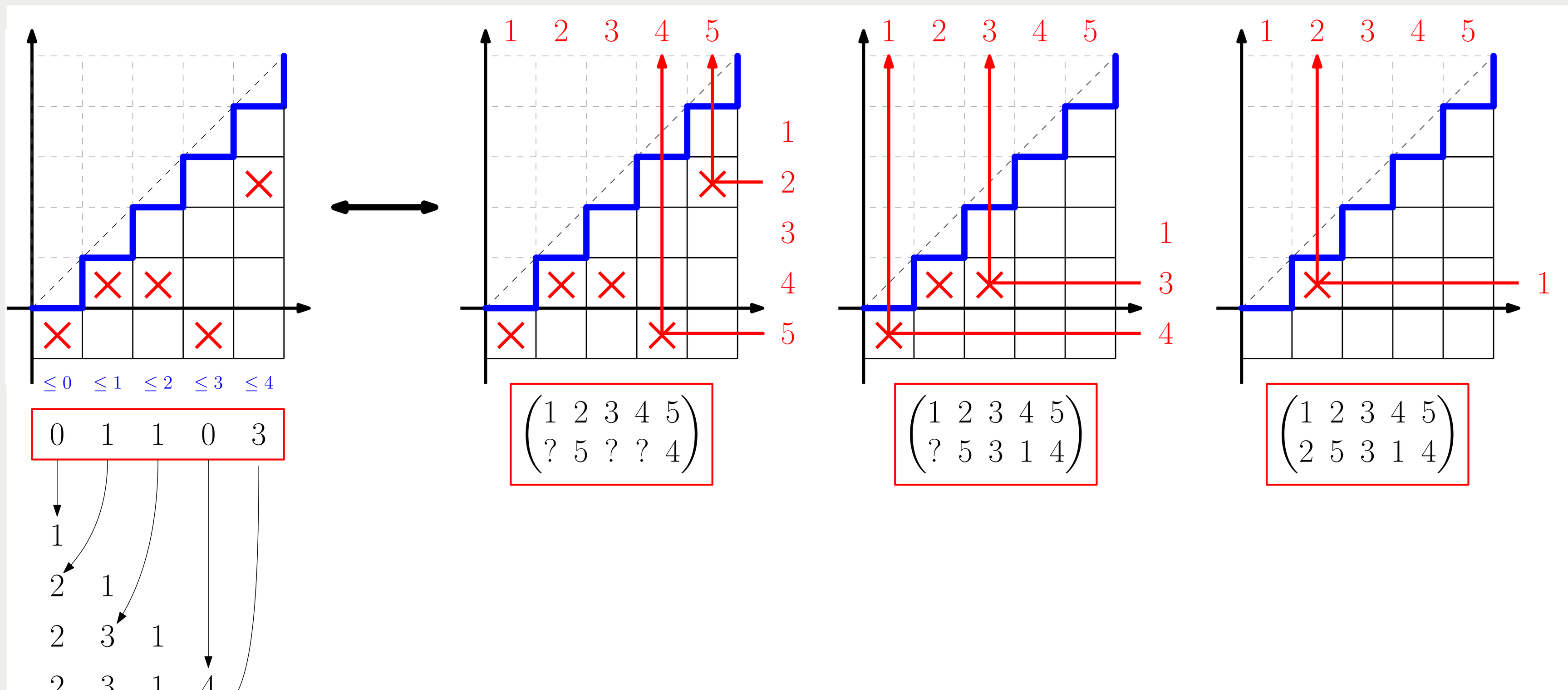
Permutation π	Inversion table $(r_i)_{i=1}^n$
12345	$(0,0,0,0,0)$
54321	$(0,1,2,3,4)$
25314	$(0,1,1,0,3)$

Key property: r_i mutually independent!

Corollary: There are $n!$ permutations of n.

$\Rightarrow$ Visualize inversion tables as *boxed staircase paths*:

- *Staircase path* $(EN)^n$
- One unit box between path and $y = -1$ is marked
- Bijection: Label lowest row 0, next 1, etc. If box k is marked in column i then $r_i = k$.



Top left: A visual representation of the inversion table $(0, 1, 1, 0, 3)$

Top right and bottom left: two ways to bijectively map it to the permutation $(2, 5, 3, 1, 4)$

Idea: Use other paths

- Path acts as an upper bound for the r_i's
- Staircase path gives $r_i \le i - 1$
- What happens for *other* Dyck paths?

A *Dyck path* D of length $2n$ is a path from $(0, 0)$ to (n, n) that takes steps $E = (1, 0)$ and $N = (0, 1)$, always staying weakly below the diagonal $y = x$. Let $y_i(D)$ be the ordinate of the ith E step in D.

Definition (Boxed Dyck paths and Dyck inversion tables)

- A *boxed Dyck path* B is a Dyck path D in which the ith E step is decorated by a number from $\{0, \dots, y_i(D)\}$.
- A *Dyck inversion table* $(r_1, r_2, \dots, r_n)$ for a Dyck path D is a sequence of nonnegative integers such that $0 \le r_i \le y_i(D)$.

Observations:

- Each boxed Dyck path is associated with a permutation
- The Dyck path imposes restrictions on the associated permutation

For example, in the path *EENEENNNEN* shown below we have:

$r_1 \le 0, \quad r_2 \le 0, \quad r_3 \le 1, \quad r_4 \le 1, \quad r_5 \le 4.$

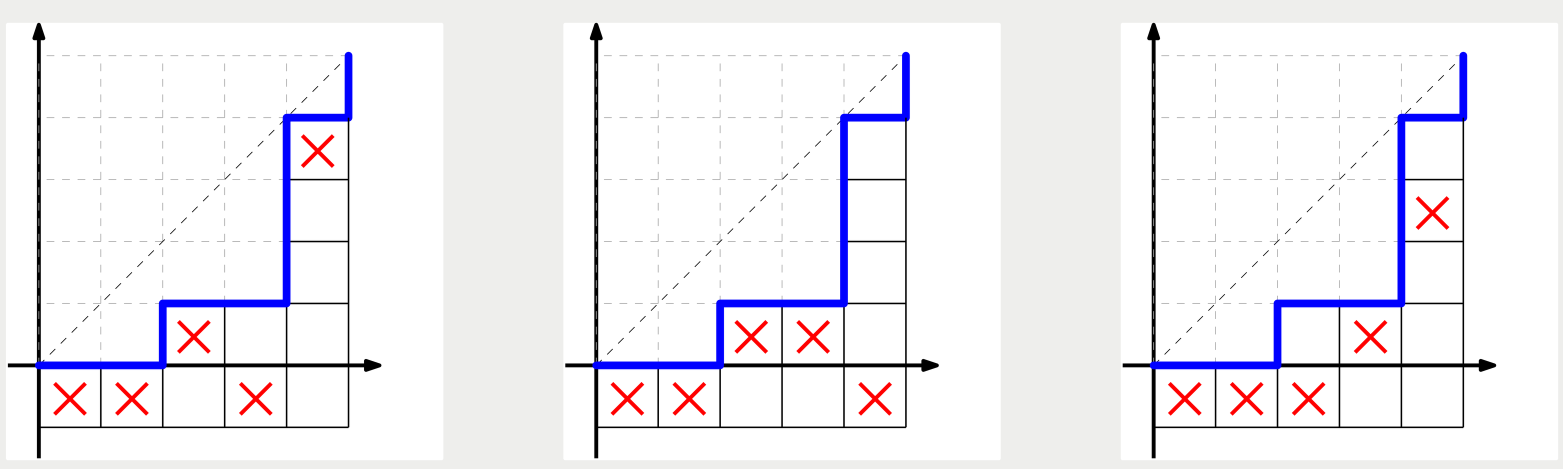


Figure: Three different boxed Dyck paths associated with the Dyck path *EENEENNNEN*.

Bijections

- Boxed Dyck paths of length $2n$ are in bijection with a class of directed acyclic graphs called relaxed binary trees [1].
- Inversion tables are in bijection with many objects [3]: regressive mappings, increasing Cayley trees, increasing plane binary trees, ...
- Dyck inversion tables allow *restricted classes*:
  - Fixed Dyck path (e.g., in a strip)
  - Restricted markers (e.g., weakly increasing; see Theorem below)
  - Avoiding marker patterns (connections with phylogenetic trees and automata [2]).

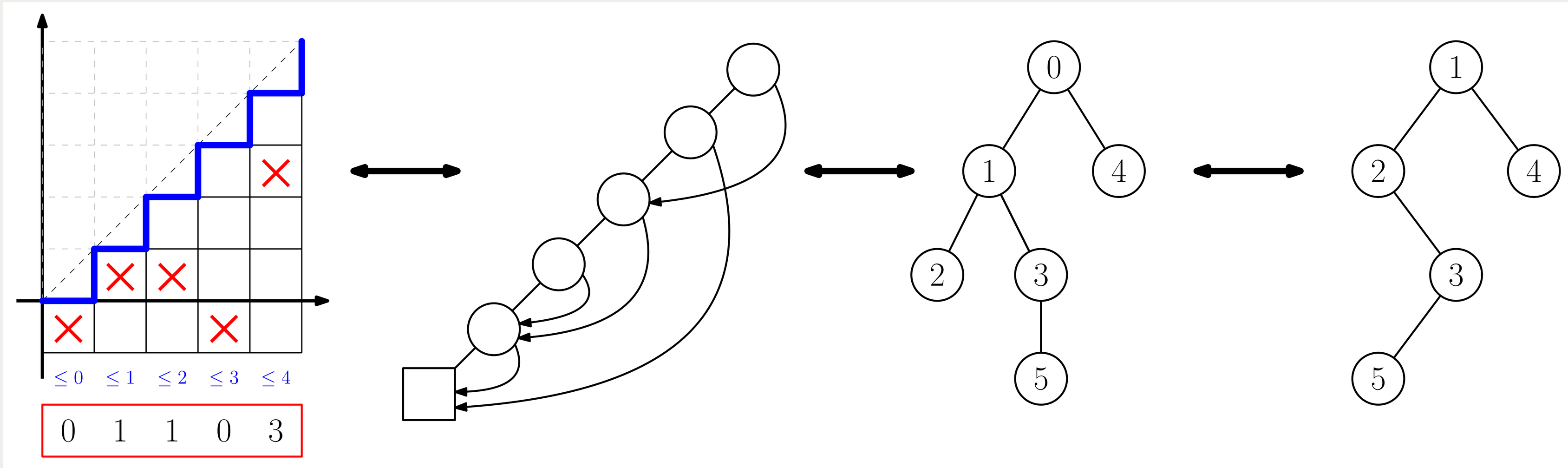


Figure: Four bijectively related combinatorial objects: (1) Boxed staircase paths, (2) relaxed (plane) binary chains, (3) increasing (non-plane) Cayley trees, (4) increasing (plane) binary trees.

Enumeration

The number b_n of boxed Dyck paths of length $2n$ satisfies

$$b_n = \Theta\left(n!\, 4^n e^{3a_1 n^{1/3}} n\right),$$

where $a_1 \approx -2.338$ is the largest root of the Airy function $\mathrm{Ai}(x)$ (solution of $\mathrm{Ai}''(x) = x\mathrm{Ai}(x)$ such that $\lim_{n\to\infty} \mathrm{Ai}(x) = 0$); see [1, Theorem 1.1].

Discussion:

- Catalan numbers $\mathrm{Cat}_n = \frac{1}{n+1}\binom{2n}{n} \sim \frac{4^n}{\sqrt{\pi n^3}}$ count Dyck paths of length $2n$
- Base of stretched exponential is quite small: $e^{3a_1 n^{1/3}} \approx 0.0008989^{n^{1/3}}$

Theorem

The probability that a random Dyck path of length $2n$ may be decorated by an independent random permutation of n, both drawn uniformly at random, is

$$\frac{b_n}{n!\,\mathrm{Cat}_n} = \Theta\left(e^{3a_1 n^{1/3}} n^{5/2}\right).$$

Theorem

The number of boxed Dyck paths with weakly increasing markers is equal to

$$\mathrm{Cat}_n \mathrm{Cat}_{n+2} - \mathrm{Cat}_{n+1}^2 = \frac{24}{\pi}\frac{16^n}{n^5}\left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right),$$

which is given by OEIS A005700. This sequence is D-finite but not algebraic.

Outlook

- Other variants of paths ending at (n, k):
  - No space constraints: Stirling numbers $S(n + k, k)$ of the second kind (set partitions of $n + k$ into k sets)
  - Markers below E *and* left of N: Eulerian numbers $E(n, k)$ (perm. of n with k ascents)
- Other bijections: OEIS A005700 enumerates many objects like Gouyou-Beauchamps excursions in $\mathbb{Z}^2_{\ge 0}$
- Other statistics: major index related to pos. of increasing markers $\sum_{r_i < r_{i+1}} i$

References

[1] Andrew Elvey Price, Wenjie Fang, and Michael Wallner. Compacted binary trees admit a stretched exponential. *J. Combin. Theory Ser. A*, 177:105306, 40, 2021.

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[4] Donald E. Knuth. *The Art of Computer Programming. Vol. 3*. Sorting and Searching. Addison-Wesley, Reading, MA, 1998.