SL(n) invariant notions of surface area

Monika Ludwig

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Felix Klein's Erlangen Program 1872

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- General linear group $\mathrm{GL}(n)$: $x \mapsto Ax$
  where A is an $n \times n$ matrix of determinant $\neq 0$
- Special affine group $\mathrm{SA}(n)$: $x \mapsto Ax + b$
  where A is an $n \times n$ matrix of determinant 1 and $b \in \mathbb{R}^n$

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- Theory of valuations:

Blaschke 1937, **Hadwiger** 1949, Sallee 1966, Schneider 1971, Groemer 1972, McMullen 1977, Goodey & Weil 1984, Betke & Kneser 1985, Klain 1995, Ludwig 1999, Reitzner 1999, Alesker 1999, Fu 2006, Bernig 2006, Haberl 2006, Schuster 2006, ...

Hadwiger's Classification Theorem 1952

Theorem

A functional $\Phi : \mathcal{K}^n \to \mathbb{R}$ *is a continuous, rigid motion invariant valuation*

$$\iff$$

$\exists \; c_0, c_1, \ldots, c_n \in \mathbb{R}$ *such that*

$$\Phi(K) = c_0 \, V_0(K) + \cdots + c_n \, V_n(K)$$

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$V_0(K), \ldots, V_n(K)$	intrinsic volumes of K
V_n	n-dimensional volume
$2\, V_{n-1}$	surface area
V_0	Euler characteristic ($V_0(K) = 1$ for $K \neq \emptyset$, $V_0(\emptyset) = 0$)

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New proof by Daniel Klain 1995

"Introduction to Geometric Probability" by Klain and Rota 1997

Equi-affine surface area of $K \in \mathcal{K}^n$

$$\Omega(K) = \int\limits_{\partial K} \kappa(K,x)^{\frac{1}{n+1}} \, dx$$

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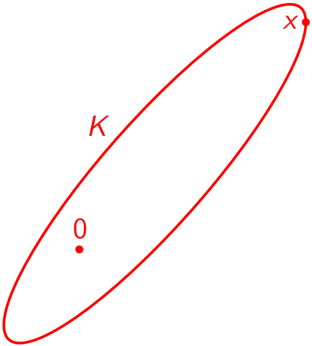
Definition for general convex bodies

Kurt Leichtweiß 1986, Carsten Schütt & Elisabeth Werner 1990, Erwin Lutwak 1991, Georg Dolzmann & Daniel Hug 1995

SL(n) invariance of Ω

$$\Omega(K) = \int_{\partial K} \kappa(K,x)^{\frac{1}{n+1}}\, dx = \int_{\partial K} \kappa_0(K,x)^{\frac{1}{n+1}}\, d\mu_K(x)$$



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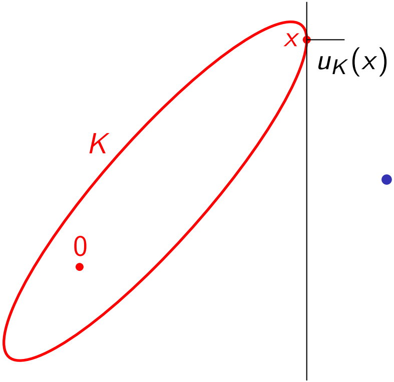

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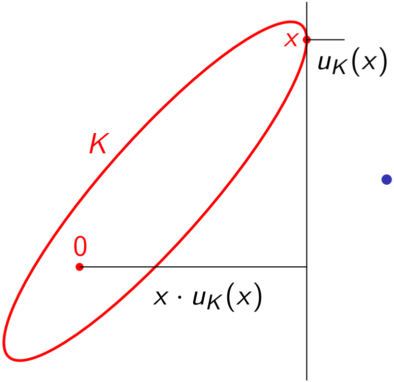

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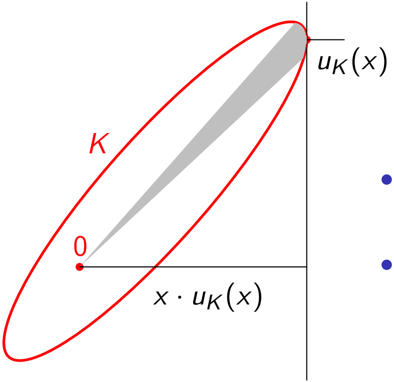

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- $d\mu_K(x) = x \cdot u_K(x)\,dx$ cone measure on ∂K

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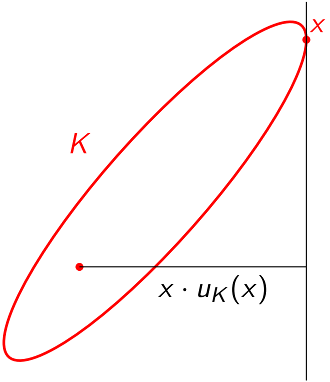

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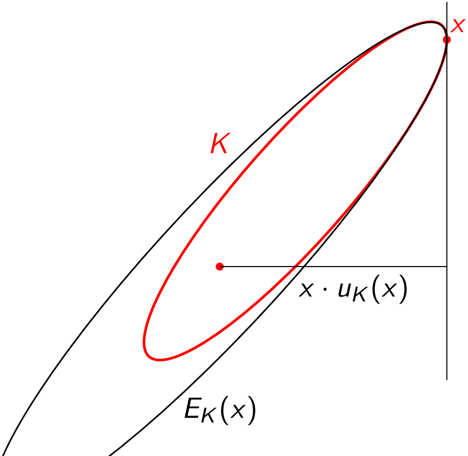

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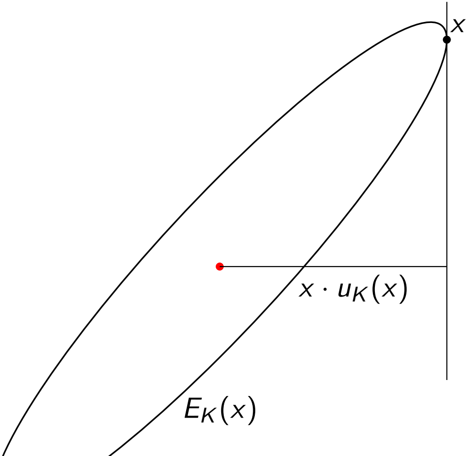

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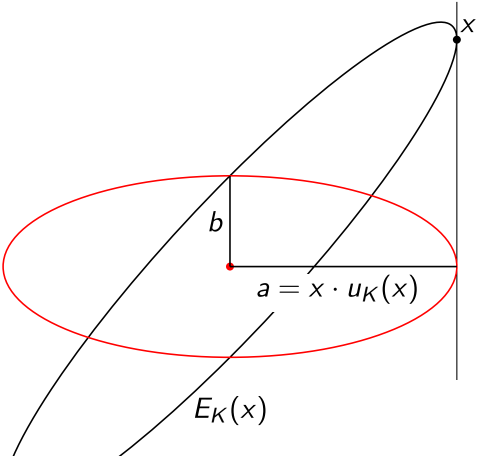

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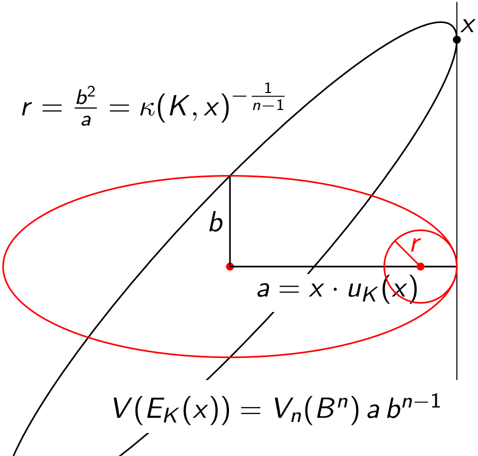

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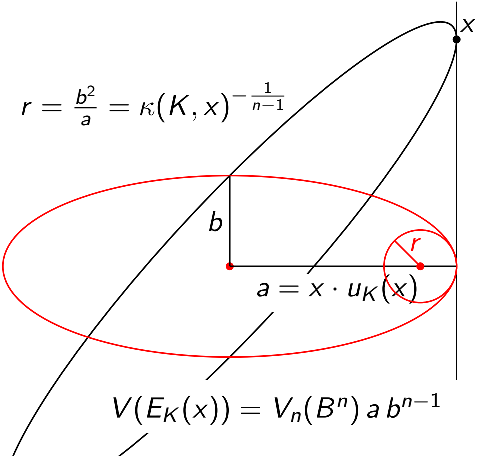

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Properties of $\Omega : \mathcal{K}^n \to \mathbb{R}$

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- Ω is *equi-affine invariant* (SA(n) invariant)

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- Ω is a valuation (Schütt 1993)

Theorem (Ludwig 1999, Ludwig & Reitzner 1999)

A functional $\Phi : \mathcal{K}^n \to \mathbb{R}$ *is an upper semicontinuous, equi-affine invariant valuation*

$$\Longleftrightarrow$$

$\exists \;\; c_0, c_1 \in \mathbb{R}, c_2 \geq 0$ *such that*

$$\Phi(K) = c_0 \, V_0(K) + c_1 \, V_n(K) + c_2 \, \Omega(K)$$

for every $K \in \mathcal{K}^n$.

Applications and recent results

- PDE
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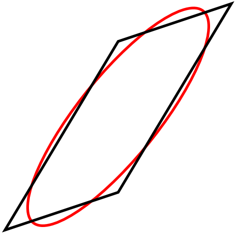
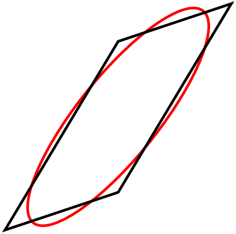
- Polytopal approximation

Polytopal Approximation



- $\mathcal{K}^n$ convex bodies in $\mathbb{R}^n$
- $\mathcal{P}_m$ convex polytopes with m vertices
- $K \triangle P = (K \cup P) \setminus (K \cap P)$
  symmetric difference of K and P
- $\delta(K, \mathcal{P}_m) = \min\{V_n(K \triangle P) : P \in \mathcal{P}_m\}$

Polytopal Approximation



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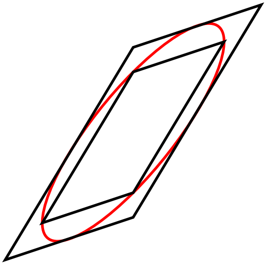
Fejes Tóth 1948, McClure & Vitale 1975, Schneider 1986, Gruber 1988, 1993 ($P \subset K$), Ludwig 1999: $\exists\, c_n > 0$

$$\lim_{m \to \infty} \delta(K, \mathcal{P}_m)\, m^{\frac{2}{n-1}} = c_n\, \Omega(K)^{\frac{n+1}{n-1}}$$

for $K \in \mathcal{K}^n$, $\partial K \in \mathcal{C}^2$

Polytopal Approximation



- $\mathcal{K}_0^n$ convex bodies in $\mathbb{R}^n$, $0 \in \operatorname{int} K$
- Banach-Mazur distance
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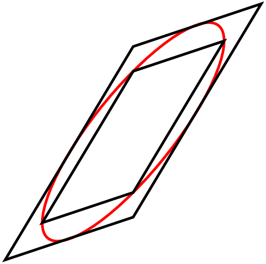

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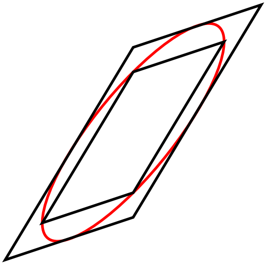
Gruber 1993: $\exists\, c_n > 0$

$$\lim_{m\to\infty} \left(\delta_{BM}(K, \mathcal{P}_m) - 1\right) m^{\frac{2}{n-1}} = c_n\, \Omega_n(K)^{\frac{2}{n-1}}$$

for $K \in \mathcal{K}^n$, $\partial K \in \mathcal{C}^2$

Polytopal Approximation



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- $\Omega_n(K)$ centro-affine surface area of K

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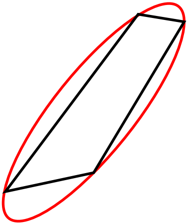
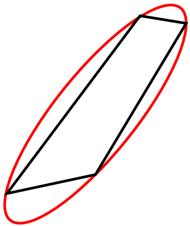


- Choose m points i.i.d. according to the cone measure μ_K on ∂K
- P_m convex hull of these m points

Polytopal Approximation



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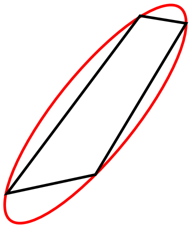
Reitzner 2002, Schütt & Werner 2003: $\exists\, c_n > 0$

$$\lim_{m\to\infty} \big(V_n(K) - \mathbb{E}\, V_n(P_m)\big)\, m^{\frac{2}{n+1}} = c_n\, \Omega_{\frac{n}{n-2}}(K)$$

for $K \in \mathcal{K}_0^n$, $\partial K \in \mathcal{C}^2$, $n \geq 3$

Polytopal Approximation



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- P_m convex hull of these m points
- $\Omega_p(K)$ L_p affine surface area of K

Reitzner 2002, Schütt & Werner 2003: $\exists\, c_n > 0$

$$\lim_{m\to\infty} \big(V_n(K) - \mathbb{E} V_n(P_m)\big)\, m^{\frac{2}{n+1}} = c_n\, \Omega_{\frac{n}{n-2}}(K)$$

for $K \in \mathcal{K}_0^n$, $\partial K \in \mathcal{C}^2$, $n \geq 3$

L_p affine surface area

- Lutwak 1996, $p > 1$; Hug 1996, $p > 0$

$$\Omega_p(K) = \int_{\partial K} \kappa_0(K,x)^{\frac{p}{n+p}} \, d\mu_K(x)$$

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- L_p Brunn Minkowski Theory:
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- Bastero, Gardner, Jensen, Klain, Koldobsky, Lonke, Romance, Rubin, Ryabogin, Volčič, Yaskina, Yaskin, . . .

Properties of $\Omega_p : \mathcal{K}_0^n \to \mathbb{R}$ (Lutwak 1996)

- Ω_p is SL(n) invariant and homogeneous of degree $q = n(n-p)/(n+p)$

$$\begin{aligned} \Omega_p(A\,K) &= \Omega_p(K) & \forall\, A \in \mathrm{SL}(n), \in \mathbb{R}^n \\ \Omega_p(\lambda K) &= \lambda^q\, \Omega_p(K) & \forall\, \lambda > 0 \end{aligned}$$

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- Ω_n is GL(n) invariant (centro-affine surface area)

SL(n) invariant valuations on $\mathcal{K}_0^n$

- $V_0(K)$ Euler characteristic
- $V_n(K)$ volume
- $V_n(K^*)$ volume of K^*
  $K^* = \{x \in \mathbb{R}^n : x \cdot y \leq 1 \text{ for } y \in K\}$ polar body of $K \in \mathcal{K}_0^n$

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- $\Omega_p(K)$ L_p affine surface area

Theorem (Ludwig 2002)

A functional $\Phi : \mathcal{P}_0^n \to \mathbb{R}$ *is a measurable,* $\mathrm{SL}(n)$ *invariant valuation that is homogeneous of degree* $p \in \mathbb{R}$

$$\Longleftrightarrow$$

$\exists\, c \in \mathbb{R}:$

$$\Phi(P) = \begin{cases} c\, V_0(P) & p = 0 \\ c\, V_n(P) & p = n \\ c\, V_n(P^*) & p = -n \\ 0 & \textit{otherwise} \end{cases}$$

for every $P \in \mathcal{P}_0^n$.

Definition

$\mathcal{P}_0^n$ convex polytopes in $\mathbb{R}^n$, $0 \in \operatorname{int} P$

Theorem (Ludwig & Reitzner 2007)

A functional $\Phi : \mathcal{K}_0^n \to \mathbb{R}$ *is an upper semicontinuous,* $\mathrm{SL}(n)$ *invariant valuation that is homogeneous of degree* q

$$\Longleftrightarrow$$

$\exists\ c_0 \in \mathbb{R}, c_1 \geq 0$ *such that*

$$\Phi(K) = \begin{cases} c_0\, V_0(K) + c_1\, \Omega_n(K) & \text{for } q = 0 \\ c_1\, \Omega_p(K) & \text{for } -n < q < n \text{ and } q \neq 0 \\ c_0\, V_n(K) & \text{for } q = n \\ c_0\, V_n(K^*) & \text{for } q = -n \\ 0 & \text{for } q < -n \text{ or } q > n \end{cases}$$

for every $K \in \mathcal{K}_0^n$ *where* $p = n(n-q)/(n+q)$.

Theorem (Ludwig & Reitzner 2007)

A functional $\Phi : \mathcal{K}_0^n \to \mathbb{R}$ *is an upper semicontinuous,* $\mathrm{GL}(n)$ *invariant valuation*

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$$\Phi(K) = c_0\, V_0(K) + c_1\, \Omega_n(K)$$

for every $K \in \mathcal{K}_0^n$.

Theorem (Ludwig & Reitzner 2007)

A functional $\Phi : \mathcal{K}_0^n \to \mathbb{R}$ *is an upper semicontinuous,* $\mathrm{SL}(n)$ *invariant valuation that vanishes on* $\mathcal{P}_0^n$

$$\iff$$

$\exists \phi \in \mathrm{Conc}[0, \infty)$ *such that*

$$\Phi(K) = \int_{\partial K} \phi(\kappa_0(K, x))\, d\mu_K(x)$$

for every $K \in \mathcal{K}_0^n$.

Definition

$\phi \in \mathrm{Conc}[0, \infty) \iff$

$\phi : [0, \infty) \to [0, \infty)$ concave, $\lim_{t \to 0} \phi(t) = \lim_{t \to \infty} \frac{\phi(t)}{t} = 0$

Affine isoperimetric inequalities

- Classical equi-affine isoperimetric inequality:
  for $K \in \mathcal{K}^n$ and B_K the centered ball s.t. $V_n(B_K) = V_n(K)$

$$\Omega(K) \leq \Omega(B_K)$$

  Blaschke 1916, Santaló 1949, Deicke 1953, Petty 1985, Leichtweiß 1988, Lutwak 1991

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- L^p affine isoperimetric inequality:
  for $K \in \mathcal{K}_c^n$ and B_K the centered ball s.t. $V_n(B_K) = V_n(K)$

$$\Omega_p(K) \leq \Omega_p(B_K)$$

  Lutwak 1999

Definition

$\mathcal{K}_c^n$ convex bodies in $\mathbb{R}^n$ with centroid at the origin $0 \in \operatorname{int} K$

Theorem (Ludwig 2008)

Let $K \in \mathcal{K}_c^n$ and let B_K be the centered ball s.t. $V_n(B_K) = V_n(K)$. If $\phi \in \mathrm{Conc}[0,\infty)$, then

$$\Omega_\phi(K) \leq \Omega_\phi(B_K)$$

and there is equality if K is a centered ellipsoid.

Definition

$$\Omega_\phi(K) = \int_{\partial K} \phi(\kappa_0(K,x))\, d\mu_K(x)$$

Thank you !!!