# Asymptotic Approximation by Quadratic Spline Curves

Monika Ludwig

**Abstract**

For a planar curve $C$ with positive affine curvature, we derive an asymptotic formula for the area approximation by quadratic spline curves with $n$ knots lying on $C$. The order of approximation is $1/n^4$ and the formula depends on an integral over the affine curvature.

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## 1 Introduction and statement of result

Let $C$ be a convex curve in the Euclidean plane $\mathbb{E}^2$ and let $\mathcal{P}_n(C)$ be the set of convex polygons with at most $n$ vertices all lying on $C$ and with the same endpoints as $C$. If the distance $\delta(C, P_n)$ of $C$ and $P_n \in \mathcal{P}_n(C)$ is measured by the area of the region between $C$ and $P_n$, then the problem of asymptotic best approximation is to study the behavior of

$$\delta(C, \mathcal{P}_n) = \inf\{\delta(C, P_n) : P_n \in \mathcal{P}_n(C)\}$$

as $n \to \infty$. For a $C$ of differentiability class $\mathcal{C}^2$ with positive curvature function $\kappa(t)$, the following asymptotic formula was given by L. Fejes Tóth [4], [5]

$$\delta(C, \mathcal{P}_n) \sim \frac{1}{12} \left( \int_0^l \kappa^{1/3}(t)\, dt \right)^3 \frac{1}{n^2} \quad \text{as } n \to \infty, \tag{1}$$

where $t$ is the arc length and $l$ the length of $C$. A complete proof of this result is due to D.E. McClure and R.A. Vitale [12]. Formula (1) holds also for general convex curves without any smoothness assumption, if $\kappa$ is now the generalized curvature (see [10]) and these polytopal approximation problems are also solved for approximation of smooth convex surfaces in $d$-dimensional space (see P.M. Gruber [7]). For more information we refer to the surveys [6], [8].

Here we consider the problem of approximating a planar curve $C$ by quadratic spline curves and derive an asymptotic formula in this case. Let $\mathcal{Q}_n$ be the set of quadratic spline curves with $n$ knots, i.e. of curves consisting of $n$ pieces of parabolas with common tangents at their common points. These curves are tangent continuous, i.e. their tangents (but not necessarily their tangent vectors) vary continuously. For a given planar curve $C$, let $\mathcal{Q}_n(C)$ be the set of $Q_n \in \mathcal{Q}_n$ with the same endpoints as $C$, with their knots lying on $C$, and with common tangents with $C$ at their knots. As in the case of polygons, we measure the distance of $C$ and $Q_n$ by the area of the region between $C$ and $Q_n$ and study

$$\delta(C, \mathcal{Q}_n) = \inf\{\delta(C, Q_n) : Q_n \in \mathcal{Q}_n(C)\} \tag{2}$$

as $n \to \infty$.

Like the determination of the asymptotic behavior of $\delta(C, \mathcal{P}_n)$, this problem of spline approximation is equi-affine invariant, i.e. invariant with respect to area preserving affine transformations. Therefore the asymptotic formula can be described with the help of notions from affine differential geometry (see the next section for definitions). Denote by $\lambda = \lambda(C)$ the affine length of $C$ and by $k(s)$ the affine curvature of $C$ given as a function of the affine arc length $s$, $0 \leq s \leq \lambda$.

**Theorem.** *Let $C$ be a curve in $\mathbb{E}^2$ of differentiability class $\mathcal{C}^4$ with positive affine curvature (or with negative affine curvature). Then*

$$\delta(C, \mathcal{Q}_n) \sim \frac{1}{240} \left( \int_0^\lambda |k^{1/5}(s)| ds \right)^5 \frac{1}{n^4} \tag{3}$$

*as $n \to \infty$.*

The restriction to curves with positive (or negative) affine curvature makes it possible to use in the proof arguments similar to that of the proof of the asymptotic formula (1). To determine just the order of approximation of this problem of spline approximation, it is also possible to use arguments from related approximation schemes (see [14], [2]).

For the problem of approximation of functions of one variable by spline functions asymptotic formulae were derived by D.D. Pence and P.W. Smith [13]. For polygons the problem of area approximation of a convex curve is equivalent to the problem of $L^1$-approximation of a function by linear splines. But quadratic spline functions have to be a class $\mathcal{C}^1$, whereas quadratic spline curves only have to be tangent continuous. The order of approximation is also different for these problems. For quadratic spline functions with $n$ knots the order of approximation is $1/n^3$, whereas for quadratic spline curves with $n$ knots we have $1/n^4$.

# 2 Tools from affine differential geometry

For the following notions from affine differential geometry, see the monographs by W. Blaschke [1] or K. Leichtweiß [9]. A recent survey on planar affine differential geometry is also contained in [3].

Let $C$ a planar curve of class $\mathcal{C}^2$ with positive curvature. The affine arc length is given by

$$s(t) = \int_0^t \kappa^{1/3}(\tau) d\tau, \ 0 \le t \le l, \tag{4}$$

where $t$ is the ordinary arc length, $\kappa(t)$ the curvature and $l$ is the length of $C$. The affine length $\lambda$ of $C$ is

$$\lambda = \int\limits_0^l \kappa^{1/3}(\tau) d\tau.$$

In the following, let $x : [0, \lambda] \to \mathbb{E}^2$ be an affine arclength parametrization of $C$. The affine curvature of $C$ at the point $x(s)$ is then given by

$$k(s) = |x''(s), x'''(s)|, \tag{5}$$

where $'$ denotes differentiation with respect to affine arc length and $|\cdot, \cdot|$ stands for determinant. The affine curvature $k(s)$ for $0 \le s \le \lambda$ determines a curve up to an area-preserving affine transformation. The curves with $k(s) = 0$ are parabolas. For a curve $C$ with $k(s) > 0$ (or $k(s) < 0$) for $0 \le s \le \lambda$, the quadratic arc, i.e. a piece of a parabola, with endpoints on $C$ and common tangents with $C$ at these points lies completely on one side of $C$.

# 3 Proof of the Theorem

We only give the proof for $k > 0$, the proof for $k < 0$ is analogue. We need the following lemma.

**Lemma.** *For $0 \le s_1 \le s_2 \le \lambda$, let $G(s_1, s_2)$ be the area of the moon-shaped piece between $C$ and the quadratic curve with endpoints $x(s_1)$ and $x(s_2)$. Then*

$$G(s_1, s_2) = \frac{1}{2} \left( k(s_1) \frac{(s_2 - s_1)^5}{5!} + o((s_2 - s_1)^5) \right)$$

*uniformly for all* $0 \leq s_1 \leq s_2 \leq \lambda$ *as* $(s_2 - s_1) \to 0$.

*Proof.* We first calculate the area $F(s_1, s_2)$ of the piece between the curve $C$ and the line segment joining $x(s_1)$ and $x(s_2)$. We have

$$F(s_1, s_2) = \frac{1}{2} \int_{s_1}^{s_2} |x(s) - x(s_1), x'(s)| ds$$

and applying Taylor's formula yields

$$F(s_1, s_2) = \frac{1}{2} \left( \frac{(s_2 - s_1)^3}{3!} - k(s_1) \frac{(s_2 - s_1)^5}{5!} + o((s_2 - s_1)^5) \right) \tag{6}$$

uniformly for all $0 \leq s_1 \leq s_2 \leq \lambda$ as $(s_2 - s_1) \to 0$. (Cf. [11], Lemma 1)

Second, we calculate the area $H(s_1, s_2)$ of the triangle formed by the line segment joining $x(s_1)$ and $x(s_2)$ and the tangents at $x(s_1)$ and $x(s_2)$. We have

$$H(s_1, s_2) = \frac{1}{2} \frac{|x'(s_1), x(s_2) - x(s_1)||x(s_2) - x(s_1), x'(s_2)|}{|x'(s_1), x'(s_2)|}$$

and by Taylor's formula and some calculation we see that

$$H(s_1, s_2) = \frac{1}{2} \left( \frac{(s_2 - s_1)^3}{4} + o((s_2 - s_1)^3) \right) \tag{7}$$

uniformly for all $0 \leq s_1 \leq s_2 \leq \lambda$ as $(s_2 - s_1) \to 0$. Since $k(s) > 0$ implies that the quadratic arc with endpoints at $x(s_1)$ and $x(s_2)$ lies completely on one side of $C$, we have

$$G(s_1, s_2) = \frac{2}{3} H(s_1, s_2) - F(s_1, s_2),$$

which together with (6) and (7) implies the lemma. $\square$

First we introduce a new parameter $r$ by

$$r(s) = \int_0^s k^{1/5}(s) ds,$$

which is possible, since $k(s) > 0$. Define

$$\rho = \int_0^\lambda k^{1/5}(s) ds.$$

By the mean value theorem of integral calculus there is a $\sigma$, $s_1 < \sigma < s_2$, such that

$$r(s_2) - r(s_1) = k^{1/5}(\sigma)(s_2 - s_1)$$

holds. Setting $r_1 = r(s_1)$ and $r_2 = r(s_2)$, we therefore have

$$(r_2 - r_1)^5 = k(s_1)(s_2 - s_1)^5 + o((s_2 - s_1)^5) \tag{8}$$

uniformly for all $0 \leq s_1 < s_2 \leq \lambda$ as $(s_2 - s_1) \to 0$. Let $K(r_1, r_2)$ be the area of the moon-shaped piece between $C$ and the quadratic arc with knots at $x(r_1)$ and $x(r_2)$. Then by Lemma 3 and (8) we obtain that

$$K(r_1, r_2) = \frac{1}{240}(r_2 - r_1)^5 + o((r_2 - r_1)^5) \tag{9}$$

uniformly for all $0 \leq r_1 \leq r_2 \leq \rho$ as $(r_2 - r_1) \to 0$.

We take $Q_{n+1} \in \mathcal{Q}_{n+1}(C)$ with knots at $r_{ni} = \frac{i}{n}\rho$ for $0 \leq i \leq n$. Then by (9)

$$\delta(C, \mathcal{Q}_{n+1}) \leq \delta(C, Q_{n+1}) = \sum_{i=1}^{n} K(r_{n,i-1}, r_{ni}) = \frac{1}{240}\frac{\rho^5}{n^4} + o(\frac{1}{n^4})$$

as $n \to \infty$ and consequently

$$\limsup_{n\to\infty} (n+1)^4 \delta(C, \mathcal{Q}_{n+1}) \leq \frac{1}{240}\rho^5. \tag{10}$$

In order to show the opposite inequality, let $Q_{n+1} \in \mathcal{Q}_{n+1}(C)$ be a sequence of best approximating quadratic spline curves, i.e. for $Q_{n+1}$ the infimum in (2) is attained. Such sequences exist, since $\delta(C, \mathcal{Q}_{n+1})$ depends continuously on its knots. Let $x(r_{ni})$ for $i = 0, \ldots, n$ be the knots of $Q_{n+1}$ and set $\rho_{ni} = r_{ni} - r_{n,i-1}$. Since $\lim_{n\to\infty} \delta(C, \mathcal{Q}_{n+1}) \to 0$, $\max_{i=1,\ldots,n} \rho_{ni} \to 0$ as $n \to \infty$. Choose $\varepsilon > 0$. (9) shows that there is an integer $n_0$ such that for $n \geq n_0$

$$|K(r_{n,i-1}, r_{ni})^{1/5} - (\frac{1}{240})^{1/5}\rho_{ni}| < \varepsilon\rho_{ni} \tag{11}$$

for $i = 1, \ldots, n$. Rewriting the mean value inequality

$$\Big(\frac{1}{n}\sum_{i=1}^{n} K(r_{n,i-1}, r_{ni})^{1/5}\Big)^5 \leq \frac{1}{n}\sum_{i=1}^{n} K(r_{n,i-1}, r_{ni})$$

in the form

$$\sum_{i=1}^{n} K(r_{n,i-1}, r_{ni})^{1/5} \leq n^{4/5}\big(\sum_{i=1}^{n} K(r_{n,i-1}, r_{ni})\big)^{1/5}$$

and combining this with (11), we see that

$$\begin{aligned}(\frac{1}{240})^{1/5}\rho = (\frac{1}{240})^{1/5}\sum_{i=1}^{n}\rho_{ni} &< \sum_{i=1}^{n}(K(r_{n,i-1},r_{ni})^{1/5}+\varepsilon\rho_{ni}) \\ &\leq n^{4/5}\Big(\sum_{i=1}^{n}K(r_{n,i-1},r_{ni})\Big)^{1/5}+\varepsilon\rho = \\ &= n^{4/5}\delta(C,\mathcal{Q}_{n+1})^{1/5}+\varepsilon\rho.\end{aligned}$$

Therefore

$$(\frac{1}{240})^{1/5}\rho \leq \liminf_{n\to\infty} n^{4/5}\delta(C,\mathcal{Q}_{n+1})^{1/5}+\varepsilon\rho.$$

Since $\varepsilon > 0$ was arbitrary, this implies that

$$\frac{1}{240}\rho^5 \leq \liminf_{n\to\infty}(n+1)^4\delta(C,\mathcal{Q}_{n+1}).$$

Combined with (10) this concludes the proof of the theorem.

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Monika Ludwig
Abteilung für Analysis
Technische Universität Wien
Wiedner Hauptstraße 8-10/1142
A 1040 Vienna
monika.ludwig@tuwien.ac.at