Valuations on Sobolev Spaces

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Abstract

All affinely covariant convex-body-valued valuations on the Sobolev space $W^{1,1}(\mathbb{R}^n)$ are completely classified. It is shown that there is a unique such valuation for Blaschke addition. This valuation turns out to be the operator which associates with each function $f \in W^{1,1}(\mathbb{R}^n)$ the unit ball of its optimal Sobolev norm.

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Let $| \cdot |$ denote a norm on $\mathbb{R}^n$ that is normalized so that its unit ball has the same volume, $v_n$, as the $n$-dimensional Euclidean unit ball. For such a norm, the sharp Gagliardo-Nirenberg-Sobolev inequality states that

$$\int_{\mathbb{R}^n} |\nabla f(x)|_p \, dx \geq n v_n^{1/n} |f|_{\frac{p}{n-1}}$$

for every $f \in W^{1,1}(\mathbb{R}^n)$. Here for $p \geq 1$, $|f|_p$ denotes the $L^p$ norm of $f$ and $| \cdot |_*$ the dual norm of $| \cdot |$ (see Section 1 for precise definitions). The Sobolev space $W^{1,1}(\mathbb{R}^n)$ is the space of functions $f \in L^1(\mathbb{R}^n)$ such that their weak gradient $\nabla f$ is in $L^1(\mathbb{R}^n)$. If the unit ball $B$ of $| \cdot |$ is the Euclidean unit ball, then inequality (1) goes back to Federer and Fleming [15] and Maz'ya [46] and is known to be equivalent to the Euclidean isoperimetric inequality. For general norms, (1) was established by Gromov [49, Appendix]. Note that the right hand side of (1) does not depend on $| \cdot |$. Hence for a given $f \in W^{1,1}(\mathbb{R}^n)$, $n \geq 2$, we may ask for its optimal Sobolev norm, that is, for the norm that minimizes the left-hand side of (1) among all norms whose unit balls have volume $v_n$.

This natural and important question was first asked by Lutwak, Yang and Zhang in [45]. They showed that the unit ball $\langle f \rangle$ corresponding to the optimal Sobolev norm of $f \in W^{1,1}(\mathbb{R}^n)$ is (up to normalization) the unique origin-symmetric convex body (that is, compact, convex set) in $\mathbb{R}^n$ such that

$$\int_{S^{n-1}} g(u) \, dS(\langle f \rangle, u) = \int_{\mathbb{R}^n} g(-\nabla f(x)) \, dx$$

for every $g \in C(\mathbb{R}^n)$ that is positively homogeneous of degree 1. Here $S(K, \cdot)$ is the Aleksandrov-Fenchel-Jessen surface area measure of $K \in \mathcal{K}^n$ and $\mathcal{K}^n$ is the set of origin-symmetric convex bodies in $\mathbb{R}^n$ with non-empty interiors together with the convex body $\{0\}$. The equations (2) are a functional version of the classical even Minkowski problem and define

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an operator \( \langle \cdot \rangle : W^{1,1}(\mathbb{R}^n) \to \mathcal{K}_n^\ast \) which associates with each \( f \in W^{1,1}(\mathbb{R}^n) \) its \textit{optimal Sobolev body} \( \langle f \rangle \). Thus (2) provides a second description of the optimal Sobolev norm. Lutwak, Yang and Zhang [45] showed that the optimal Sobolev body corresponds also to the optimal norm in a family of sharp Gagliardo-Nirenberg inequalities recently established by Cordero, Nazaret, and Villani [14]. Moreover, the optimal Sobolev body has proved to be critical in recent results on affine isoperimetric inequalities (see [13, 24, 40, 44, 45, 59, 60]).

Using valuations on Sobolev spaces, we obtain a new and totally different description of the operator \( f \mapsto \langle f \rangle \). A function \( z \) defined on a lattice \((\mathcal{L}, \lor, \land)\) and taking values in an abelian semigroup is called a \textit{valuation} if

\[
z(f \lor g) + z(f \land g) = z(f) + z(g)
\]

for all \( f, g \in \mathcal{L} \). A function \( z \) defined on some subset \( \mathcal{M} \) of \( \mathcal{L} \) is called a valuation on \( \mathcal{M} \) if (3) holds whenever \( f, g, f \lor g, f \land g \in \mathcal{M} \).

Investigations of valuations on convex bodies \((\mathcal{K}^n, \cup, \cap)\) have been an active and prominent part of mathematics ever since Dehn’s solution of Hilbert’s Third Problem in 1900. Blaschke obtained the first classification of real-valued valuations on convex bodies that are \( \text{SL}(n) \) invariant in the 1930s. This was greatly extended by Hadwiger in his famous classification of continuous, rigid motion invariant valuations and characterization of elementary mixed volumes. See [25, 30, 47, 48] for information on the classical theory of valuations on convex bodies and [1–5, 9, 16, 20–23, 33–35, 38, 39, 51, 53, 54, 58] for some of the more recent results. Valuations were also investigated on star shaped sets [27, 28], on manifolds [6–8, 10, 11] and on Lebesgue spaces [37, 56, 57].

In this paper, we classify valuations on \((W^{1,1}(\mathbb{R}^n), \lor, \land)\), where for \( f, g \in W^{1,1}(\mathbb{R}^n) \), the function \( f \lor g \) denotes the pointwise maximum and the function \( f \land g \) the pointwise minimum of \( f \) and \( g \). As in the classical results for valuations on convex bodies we use invariance and covariance properties to obtain characterizations of important operators. An operator \( z : W^{1,1}(\mathbb{R}^n) \to \mathcal{K}_n^\ast \) is called \( \text{GL}(n) \) \textit{covariant} if for some \( p \in \mathbb{R} \),

\[
z(f \circ \phi^{-1}) = |\det \phi|^p \phi z(f)
\]

for all \( f \in W^{1,1}(\mathbb{R}^n) \) and \( \phi \in \text{GL}(n) \), where \( \det \phi \) is the determinant of \( \phi \). An operator \( z \) is called \textit{translation invariant} if \( z(f \circ \tau^{-1}) = z(f) \) for all \( f \in W^{1,1}(\mathbb{R}^n) \) and translations \( \tau \). It is called \textit{homogeneous} if for some \( q \in \mathbb{R} \), we have \( z(sf) = |s|^q z(f) \) for all \( f \in W^{1,1}(\mathbb{R}^n) \) and \( s \in \mathbb{R} \). An operator \( z : W^{1,1}(\mathbb{R}^n) \to \mathcal{K}_n^\ast \) is called \textit{affinely covariant} if \( z \) is homogeneous, translation invariant and \( \text{GL}(n) \) covariant.

**Theorem 1.** An operator \( z : W^{1,1}(\mathbb{R}^n) \to \langle \mathcal{K}_n^\ast, \# \rangle \), where \( n \geq 3 \), is a \textit{continuous, affinely covariant valuation} if and only if there is a constant \( c \geq 0 \) such that

\[
z(f) = c \langle f \rangle
\]

for every \( f \in W^{1,1}(\mathbb{R}^n) \).

Here \( \# \) denotes Blaschke addition on \( \mathcal{K}_n^\ast \), that is, for \( K, L \in \mathcal{K}_n^\ast \), the convex body \( K \# L \) is the (uniquely determined) origin-symmetric convex body such that \( S(K \# L, \cdot) = S(K, \cdot) + S(L, \cdot) \) (see Section 1 for precise definitions). See [12, 18, 26, 29, 41–43, 52] for some of the recent results involving Blaschke addition and, in particular, Haberl [21], where a classification of Blaschke valuations on convex bodies was obtained.
Theorem 1 is in a certain sense dual to the following classification result for valuations \( z : W^{1,1}(\mathbb{R}^n) \to \langle \mathcal{K}_c^n, + \rangle \). Here + denotes Minkowski addition on \( \mathcal{K}_c^n \), that is, for \( K, L \in \mathcal{K}_c^n \), we have \( K + L = \{ x + y : x \in K, y \in L \} \). We say that an operator \( z : W^{1,1}(\mathbb{R}^n) \to \mathcal{K}_c^n \) is \( \text{GL}(n) \)-contravariant if for some \( p \in \mathbb{R} \),

\[
z(f \circ \phi^{-1}) = |\det \phi|^p \phi^{-t} z(f)
\]

for all \( f \in W^{1,1}(\mathbb{R}^n) \) and \( \phi \in \text{GL}(n) \), where \( \phi^{-t} \) is the transpose of the inverse of \( \phi \). An operator \( z : W^{1,1}(\mathbb{R}^n) \to \mathcal{K}_c^n \) is called affinely contravariant if \( z \) is homogeneous, translation invariant and \( \text{GL}(n) \)-contravariant.

**Theorem 2.** An operator \( z : W^{1,1}(\mathbb{R}^n) \to \langle \mathcal{K}_c^n, + \rangle \), where \( n \geq 3 \), is a continuous, affinely contravariant valuation if and only if there is a constant \( c \geq 0 \) such that

\[
z(f) = c \langle f \rangle
\]

for every \( f \in W^{1,1}(\mathbb{R}^n) \).

Here \( \Pi K \) denotes the projection body of a convex body \( K \). Projection bodies were introduced by Minkowski at the turn of the last century and have proved to be very useful in many ways and subjects (cf. [17]). They can be defined in the following way. Every convex body \( K \) is uniquely determined by its support function \( h(K, \cdot) \), where \( h(K, v) = \max\{ v \cdot x : x \in K \} \) for \( v \in \mathbb{R}^n \) and \( v \cdot x \) is the standard inner product of \( v, x \in \mathbb{R}^n \). The projection body of \( K \) is the convex body whose support function is given by

\[
h(\Pi K, v) = \frac{1}{2} \int_{S^{n-1}} |u \cdot v| \, dS(K, u), \quad v \in \mathbb{R}^n.
\]

Combined with (2), this gives

\[
h(\Pi \langle f \rangle, v) = \frac{1}{2} \int_{\mathbb{R}^n} |v \cdot \nabla f(x)| \, dx.
\]

Also the convex body \( \Pi \langle f \rangle \) has proved to be critical for affine isoperimetric inequalities. In particular, the affine Zhang-Sobolev inequality [60] is a volume inequality for the polar body of \( \Pi \langle f \rangle \) which strengthens and implies the Euclidean case of the Sobolev inequality (1).

1 **Background material on convex bodies**

General references on convex bodies are the books by Gardner [17], Gruber [19], Schneider [50], and Thompson [55]. We work in Euclidean \( n \)-space, \( \mathbb{R}^n \), and write \( x = (x_1, \ldots, x_n) \) for \( x \in \mathbb{R}^n \). Throughout this paper, \( u \cdot x \) denotes the standard inner product of \( u, x \in \mathbb{R}^n \) and \( | \cdot | \) denotes the standard Euclidean norm on \( \mathbb{R}^n \). The vectors of the standard basis of \( \mathbb{R}^n \) are denoted by \( e_1, \ldots, e_n \) and the \( k \)-dimensional volume of a \( k \)-dimensional convex body \( F \) by \( V_k(F) \).

Let \( \mathcal{K}_c^n \) denote the space of convex bodies in \( \mathbb{R}^n \). The subspace of convex bodies with non-empty interiors which contain the origin is denoted by \( \mathcal{K}_0^n \) and the subspace of origin-symmetric
bodies with non-empty interiors by $\mathcal{K}_n^n$. These spaces are equipped with the Hausdorff metric $\delta$ defined by
\[
\delta(K, L) = \max\{ |h(K, u) - h(L, u)| : u \in S^{n-1} \}.
\]
Minkowski addition can also be described by support functions, since
\[
h(K + L, v) = h(K, v) + h(L, v)
\]
for all $K, L \in \mathcal{K}^n$ and $v \in \mathbb{R}^n$. Note that $\langle \mathcal{K}_e^n, + \rangle$ is an abelian semigroup.

Blaschke addition is defined using the surface area measure $S(K, \cdot)$ for $K \in \mathcal{K}_0^n$. For a Borel set $\omega \subset S^{n-1}$, the surface area measure $S(K, \omega)$ is the $(n - 1)$-dimensional Hausdorff measure of the set of all boundary points of $K$ at which there exists a unit normal vector of $K$ belonging to $\omega$. The solution to the Minkowski problem (see [50]) states that a non-negative Borel measure $\mu$ on $S^{n-1}$ is the surface area measure of a convex body if and only if $\mu$ is not concentrated on a great subsphere and has its centroid, $\frac{1}{m(S^{n-1})} \int_{S^{n-1}} u d\mu(u)$, at the origin. If such a measure $\mu$ is given, there is a unique convex body $K \in \mathcal{K}_0^n$ with surface area measure $S(K, \cdot) = \mu$ that has its centroid, $\frac{1}{m_n(K)} \int_K x \, dx$, at the origin. For $K, L \in \mathcal{K}_0^n$, their Blaschke sum, $K \# L$, is defined as the unique convex body with centroid at the origin such that
\[
S(K \# L, \cdot) = S(K, \cdot) + S(L, \cdot).
\]
Since the sum of two surface area measures satisfies the necessary conditions of the Minkowski problem, Blaschke addition is well defined by the solution of the Minkowski problem. For $t > 0$ and $K \in \mathcal{K}_0^n$, the Blaschke multiple, $t \cdot K$, is defined as the unique convex body with centroid at the origin such that
\[
S(t \cdot K, \cdot) = t S(K, \cdot).
\]
Hence $t \cdot K = t^{1/(n-1)} K$, if $K$ has its centroid at the origin. A convex body is origin-symmetric if and only if its surface area measure is an even measure and its centroid is at the origin. Note that for $K \in \mathcal{K}_0^n$, the Blaschke symmetral $\frac{1}{2} \cdot (K \# (-K))$ is an origin-symmetric convex body. Also note that $\langle \mathcal{K}_e^n, \# \rangle$ is an abelian semigroup.

For $K \in \mathcal{K}^n$ which contains the origin in its interior, the polar body, $K^*$, of $K$ is defined by
\[
K^* = \{ x \in \mathbb{R}^n : x \cdot y \leq 1 \text{ for every } y \in K \}.
\]
For a normed space $E = (\mathbb{R}^n, | \cdot |)$, the dual space is $E^* = (\mathbb{R}^n, | \cdot |_*)$, where $| \cdot |_*$ is given for $v \in \mathbb{R}^n$ by
\[
| v |_* = \sup \{ x \cdot v : | x | \leq 1 \}.
\]
If $B$ is the unit ball of $E$, that is, $B = \{ x \in \mathbb{R}^n : | x | \leq 1 \}$, then its polar body, $B^*$, is the unit ball of $E^*$.

We require some facts about the projection operator $\Pi : \mathcal{K}^n \to \mathcal{K}^n$, which can be found in [17]. It is a simple consequence of the definition of $\Pi$ that
\[
\Pi(K \# L) = \Pi K + \Pi L
\]
for $K, L \in \mathcal{K}_0^n$. Note that for $K \in \mathcal{K}_0^n$, we have
\[
\Pi \left( \frac{1}{2} \cdot (K \# (-K)) \right) = \Pi K.
\]
The projection operator has strong contravariance and invariance properties: for all \( \phi \in \text{GL}(n) \) and translations \( \tau \), we have
\[
\Pi(\phi K) = |\det \phi| \phi^{-t} \Pi K \quad \text{and} \quad \Pi(\tau K) = \Pi K
\]
for all \( K \in K_0^n \). Further, \( \Pi \) is continuous on \( K_0^n \) and injective on \( K_c^n \). If \( Z^n \) denotes the range of \( \Pi \), the inverse operator \( \Pi^{-1} : Z^n \to K_c^n \) is also continuous.

The proofs of Theorems 1 and 2 make essential use of a classification result of convex-body-valued valuations established in [36]. To state the result, we need the following definitions.

Let \( P^n_0 \) denote the set of convex polytopes in \( \mathbb{R}^n \) that contain the origin in their interiors. The moment body, \( M \), of \( P \) is defined by
\[
h(M P, v) = \int_{P^n} |v \cdot x| dx, \quad v \in \mathbb{R}^n.
\]
We say that an operator \( Z : P^n_0 \to K_c^n \) is \( \text{GL}(n) \) contravariant of weight \( p \in \mathbb{R} \), if
\[
Z(\phi P) = |\det \phi|^p \phi^{-t} Z P
\]
for all \( P \in P^n_0 \) and \( \phi \in \text{GL}(n) \).

**Theorem 3** ([36]). An operator \( Z : P^n_0 \to \langle K_c^n, + \rangle \), where \( n \geq 3 \), is a valuation which is \( \text{GL}(n) \) contravariant of weight \( p \) if and only if there is a constant \( c \geq 0 \) such that
\[
Z P = \begin{cases} 
  cM P^* & \text{for } p = -1 \\
  c(P^* + (-P)^*) & \text{for } p = 0 \\
  c\Pi P & \text{for } p = 1 \\
  \{0\} & \text{otherwise}
\end{cases}
\]
for every \( P \in P^n_0 \).

For \( n = 2 \), there are additional convex-body-valued valuations (see [36]). Also note that if we replace \( \text{GL}(n) \) contravariance by \( \text{SO}(n) \) covariance, there is a much larger class of valuations (see, for example, [52]).

## 2 Background material on Sobolev spaces

For \( p \geq 1 \) and a measurable function \( f : \mathbb{R}^n \to \mathbb{R} \), let
\[
|f|_p = \left( \int_{\mathbb{R}^n} |f(x)|^p dx \right)^{1/p}.
\]
A measurable function \( f \) is in \( L^p(\mathbb{R}^n) \) if \( |f|_p < \infty \). A function \( f \in L^1(\mathbb{R}^n) \) has \( L^1 \) weak derivative, if there exists a measurable function \( \nabla f : \mathbb{R}^n \to \mathbb{R}^n \) such that \( \nabla f \in L^1(\mathbb{R}^n) \) (that is, \( |\nabla f| \in L^1(\mathbb{R}^n) \)) and
\[
\int_{\mathbb{R}^n} \nu(x) \cdot \nabla f(x) dx = -\int_{\mathbb{R}^n} f(x) \nabla \cdot \nu(x) dx
\]
for every compactly supported smooth vector field \( \nu(x) : \mathbb{R}^n \to \mathbb{R}^n \), where we use the notation \( \nabla \cdot \nu = \frac{\partial \nu_1}{\partial x_1} + \cdots + \frac{\partial \nu_n}{\partial x_n} \). The function \( \nabla f \) is called the \textit{weak gradient} of \( f \) and the \( L^1 \) norm of \( |\nabla f| \) is denoted by \( |\nabla f|_1 \).

An operator \( z : W^{1,1}(\mathbb{R}^n) \to K^*_c \) is continuous, if for every sequence \( f_k \in W^{1,1}(\mathbb{R}^n) \) with \( f_k \to f \) as \( k \to \infty \) in \( W^{1,1}(\mathbb{R}^n) \), we have \( \delta(z(f_k), z(f)) \to 0 \) as \( k \to \infty \). Here \( f_k \to f \) as \( k \to \infty \) in \( W^{1,1}(\mathbb{R}^n) \) if \( |f_k - f|_1 \to 0 \) and \( |\nabla(f_k - f)|_1 \to 0 \) as \( k \to \infty \). An operator \( z : W^{1,1}(\mathbb{R}^n) \to K^*_c \) is called trivial, if \( z(f) = \{0\} \) for all \( f \in W^{1,1}(\mathbb{R}^n) \). It is called \( \text{GL}(n) \) covariant of weight \( p \in \mathbb{R} \), if

\[
z(f \circ \phi^{-1}) = |\det \phi|^p \phi z(f)
\]

for all \( f \in W^{1,1}(\mathbb{R}^n) \) and \( \phi \in \text{GL}(n) \). It is called \( \text{GL}(n) \) contravariant of weight \( p \in \mathbb{R} \), if

\[
z(f \circ \phi^{-1}) = |\det \phi|^p \phi^{-t} z(f)
\]

for all \( f \in W^{1,1}(\mathbb{R}^n) \) and \( \phi \in \text{GL}(n) \). It is called \textit{homogeneous of degree} \( q \in \mathbb{R} \), if

\[
z(s f) = |s|^q z(f)
\]

for all \( f \in W^{1,1}(\mathbb{R}^n) \) and \( s \in \mathbb{R} \). If an operator \( z : W^{1,1}(\mathbb{R}^n) \to K^*_c \) is homogeneous of degree \( q \) and non-trivial, then setting \( s = 0 \) in the definition of homogeneity gives \( q \geq 0 \).

If \( z : W^{1,1}(\mathbb{R}^n) \to K^*_c \) is continuous and homogeneous of degree \( 0 \), then \( z(f) = z(0) \) for all \( f \in W^{1,1}(\mathbb{R}^n) \). If \( z \) is in addition \( \text{GL}(n) \) co- or contravariant, then we obtain that \( z \) is trivial. In particular, we have

\[
z(0) = \{0\} \tag{9}
\]

for all continuous, homogeneous and \( \text{GL}(n) \) co- or contravariant \( z : W^{1,1}(\mathbb{R}^n) \to K^*_c \).

For \( f, g \in W^{1,1}(\mathbb{R}^n) \), \( f \lor g, f \land g \in W^{1,1}(\mathbb{R}^n) \) and for almost every \( x \in \mathbb{R}^n \),

\[
\nabla(f \lor g)(x) = \begin{cases} \nabla f(x) & \text{when } f(x) > g(x) \\ \nabla g(x) & \text{when } f(x) < g(x) \\ \nabla f(x) = \nabla g(x) & \text{when } f(x) = g(x) \end{cases} \tag{10}
\]

and

\[
\nabla(f \land g) = \begin{cases} \nabla f(x) & \text{when } f(x) < g(x) \\ \nabla g(x) & \text{when } f(x) > g(x) \\ \nabla f(x) = \nabla g(x) & \text{when } f(x) = g(x) \end{cases} \tag{11}
\]

(see, for example, [32]). Hence \( (W^{1,1}(\mathbb{R}^n), \lor, \land) \) is a lattice.

Let \( L^{1,1}(\mathbb{R}^n) \subset W^{1,1}(\mathbb{R}^n) \) denote the space of \textit{piecewise affine functions} on \( \mathbb{R}^n \), where a function \( \ell : \mathbb{R}^n \to \mathbb{R} \) is called piecewise affine, if \( \ell \) is continuous and there are finitely many \( n \)-dimensional simplices \( S_1, \ldots, S_m \subset \mathbb{R}^n \) with pairwise disjoint interiors such that the restriction of \( \ell \) to each \( S_i \) is affine and \( \ell = 0 \) outside \( S_1 \cup \cdots \cup S_m \). Note that the simplices \( S_1, \ldots, S_m \) are a triangulation of the support of \( \ell \). Further, note that if \( V \) is the set of vertices of this triangulation, then \( V \) and the values \( \ell(v) \) for \( v \in V \) completely determine \( \ell \). Piecewise affine functions lie dense in \( W^{1,1}(\mathbb{R}^n) \) (see, for example, [31]).

For \( P \in \mathbb{P}_0^n \), define the piecewise affine function \( \ell_P \) by requiring that \( \ell_P(0) = 1 \), that \( \ell_P(x) = 0 \) for \( x \notin P \), and that \( \ell_P \) is affine on each simplex with apex at the origin and base
equal to a facet of $P$. Define $P^{1,1}(\mathbb{R}^n) \subset L^{1,1}(\mathbb{R}^n)$ as the set of all $\ell_P$ for $P \in \mathcal{P}_0^n$. Note that for $\phi \in \text{GL}(n)$,

$$\ell_{\phi P} = \ell_P \circ \phi^{-1}. \quad (12)$$

We remark that multiples and translates of $\ell_P$ for $P \in P^{1,1}(\mathbb{R}^n)$ correspond to linear elements within the theory of finite elements.

### 3 The operators $f \mapsto \langle f \rangle$ and $f \mapsto \Pi \langle f \rangle$

The operator $f \mapsto \langle f \rangle$ has strong covariance and invariance properties (see [45] and also [40]). In particular,

$$\langle sf \rangle = |s| \cdot \langle f \rangle, \quad \langle f \circ \phi^{-1} \rangle = \phi \langle f \rangle, \quad \langle f \circ \tau^{-1} \rangle = \langle f \rangle \quad (13)$$

for all $s \in \mathbb{R}$, $\phi \in \text{GL}(n)$ and for all translations $\tau$.

**Lemma 1.** The operator $z : W^{1,1}(\mathbb{R}^n) \rightarrow \langle \mathcal{K}_c^n, + \rangle$, defined by $z(f) = c \Pi \langle f \rangle$ with $c \geq 0$, is a continuous affinely contravariant valuation.

**Proof.** Using (10) and (11), we obtain from (4) and (5) that $z$ is a valuation. By (13) and (8), we see that $z$ is affinely contravariant. Suppose that $f_k \rightarrow f$ as $k \rightarrow \infty$ in $W^{1,1}(\mathbb{R}^n)$. Then for $u \in S^{n-1}$ we have by (4), the reverse triangle inequality and the Cauchy-Schwarz inequality,

$$|h(z(f_k), u) - h(z(f), u)| \leq \frac{c}{2} \int_{\mathbb{R}^n} |u \cdot \nabla(f_k - f)(x)| \, dx \leq \frac{c}{2} \int_{\mathbb{R}^n} |\nabla(f_k - f)(x)| \, dx.$$

Therefore we obtain $\delta(z(f_k), z(f)) \rightarrow 0$ as $k \rightarrow 0$ and thus $z$ is continuous. \qed

**Lemma 2.** The operator $z : W^{1,1}(\mathbb{R}^n) \rightarrow \langle \mathcal{K}_c^n, \# \rangle$, defined by $z(f) = c \langle f \rangle$ with $c \geq 0$, is a continuous affinely covariant valuation.

**Proof.** Since the inverse projection operator $\Pi^{-1}$ is continuous, Lemma 1 implies that $z$ is continuous. By (13), $z$ is affinely covariant. Since by Lemma 1 for all $f, g \in W^{1,1}(\mathbb{R}^n)$,

$$\Pi z(f) + \Pi z(g) = \Pi z(f \vee g) + \Pi z(f \wedge g),$$

we obtain by (6) that

$$\Pi (z(f) \# z(g)) = \Pi (z(f \vee g) \# z(f \wedge g)).$$

Applying $\Pi^{-1}$ gives

$$z(f) \# z(g) = z(f \vee g) \# z(f \wedge g)$$

for all $f, g \in W^{1,1}(\mathbb{R}^n)$. Thus $z : W^{1,1}(\mathbb{R}^n) \rightarrow \langle \mathcal{K}_c^n, \# \rangle$ is a valuation. \qed
Lemma 3. For $P \in \mathcal{P}^n_0$, $\langle \ell_P \rangle = \frac{1}{2} \cdot (P \# (-P))$.

Proof. By definition, $\langle \ell_P \rangle = \frac{1}{2} \cdot (P \# (-P))$ if for every even $g \in C(\mathbb{R}^n)$ that is homogeneous of degree 1,

$$\frac{1}{n} \int_{S^{n-1}} g(u) \, dS(\frac{1}{2} \cdot (P \# (-P)), u) = \int_{\mathbb{R}^n} g(-\nabla \ell_P) \, dx.$$

Let $P$ have facets $F_1, \ldots, F_m$. For the facet $F_i$, let $u_i$ be its unit outer normal vector and $T_i$ the convex hull of $F_i$ and the origin. Since for $x \in T_i$

$$\ell_P(x) = -\frac{u_i}{h(P, u_i)} \cdot x + 1$$

and

$$\nabla \ell_P(x) = -\frac{u_i}{h(P, u_i)},$$

we obtain that

$$\int_{\mathbb{R}^n} g(-\nabla \ell_P) \, dx = \sum_{i=1}^m \int_{T_i} g(-\nabla \ell_P(x)) \, dx$$

$$= \sum_{i=1}^m g(u_i) \frac{V_n(T_i)}{h(P, u_i)}$$

$$= \frac{1}{n} \sum_{i=1}^m g(u_i) V_{n-1}(F_i)$$

$$= \frac{1}{n} \int_{S^{n-1}} g(u) \, dS(P, u)$$

$$= \frac{1}{n} \int_{S^{n-1}} g(u) \, dS(\frac{1}{2} \cdot (P \# (-P)), u).$$

Thus $\langle \ell_P \rangle = \frac{1}{2} \cdot (P \# (-P))$.

\[ \square \]

4 Proof of Theorem 2

In Lemma 1, it was shown that $f \mapsto c \Pi \langle f \rangle$ is for $c \geq 0$ a continuous affinely contravariant valuation. Suppose that $z$ is a continuous affinely contravariant valuation. The following lemmas establish that there is a constant $c \geq 0$ such that $z(f) = c \Pi \langle f \rangle$ for all $f \in W^{1,1}(\mathbb{R}^n)$.

Lemma 4. If $z : P^{1,1}(\mathbb{R}^n) \to \mathcal{K}_c^n$ is continuous, non-trivial, and $\text{GL}(n)$ contravariant of weight $p$, then $p \geq 1$.

Proof. For $a > 0$ and $0 < \varepsilon < 1$, let $\phi_a \in \text{GL}(n)$ map $e_1$ to $a e_1$ and $e_i$ to $a^\varepsilon e_i$ for $i = 2, \ldots, n$. Using (12), we obtain for $P \in \mathcal{P}^n_0$ that $|\ell_{\phi_a}P|_1 = |\det \phi_a| |\ell_P|_1$ and

$$|\nabla \ell_{\phi_a}P|_1 = |\det \phi_a| \int_{\mathbb{R}^n} |\phi_a^{-1} \nabla \ell_P(x)| \, dx \leq |\det \phi_a| |\nabla \ell_P|_1 \max\{ |\phi_a^{-1} u| : u \in S^{n-1} \}.$$
Since $z$ is continuous, it follows from (9) that $q$ ε $\ell$ for every $f \in P_{0}^{1,1}(\mathbb{R}^{n})$. Consequently, $\ell_{a}P \rightarrow 0$ in $W^{1,1}(\mathbb{R}^{n})$ as $a \rightarrow 0$. Since $z$ is GL$(n)$ contravariant of weight $p$, we obtain by (12) that

$$z(\ell_{a}P) = a^{(1+(n-1)\varepsilon)}|\phi_{a}^{-1}z(\ell_{P})|.$$ 

Thus the first coordinates of points from $z(\ell_{P})$ are multiplied by $a^{(1+(n-1)\varepsilon)p-1}$. Since this happens for all $P \in P_{0}^{n}$ and $z$ is continuous, we conclude that $p \geq 1/(1 + (n-1)\varepsilon)$. Since $\varepsilon > 0$ was arbitrary, we obtain $p \geq 1$.

**Lemma 5.** If $z : P_{0}^{1,1}(\mathbb{R}^{n}) \rightarrow \langle K_{n}^{\varepsilon}, + \rangle$, where $n \geq 3$, is a continuous affinely contravariant valuation, then there is a constant $c \geq 0$ such that

$$z(f) = c\Pi\langle f \rangle$$

for every $f \in P_{0}^{1,1}(\mathbb{R}^{n})$.

**Proof.** Define the operator $Z : P_{0}^{n} \rightarrow \langle K_{n}^{\varepsilon}, + \rangle$ by setting

$$Z(P) = z(\ell_{P}).$$

If $\ell_{P}, \ell_{Q} \in P_{0}^{1,1}(\mathbb{R}^{n})$ are such that $\ell_{P} \vee \ell_{Q} \in P_{0}^{1,1}(\mathbb{R}^{n})$, then $\ell_{P} \vee \ell_{Q} = \ell_{P \cup Q}$ and $\ell_{P} \wedge \ell_{Q} = \ell_{P \cap Q}$. Since $z$ is a valuation on $P_{0}^{1,1}(\mathbb{R}^{n})$, it follows for $P, Q, P \cup Q \in P_{0}^{n}$ that

$$Z(P) + Z(Q) = z(\ell_{P}) + z(\ell_{Q}) = z(\ell_{P} \vee \ell_{Q}) + z(\ell_{P} \wedge \ell_{Q}) = Z(P \cup Q) + Z(P \cap Q).$$

Thus $Z : P_{0}^{n} \rightarrow \langle K_{n}^{\varepsilon}, + \rangle$ is a valuation.

By Lemma 4, the operator $z$ is GL$(n)$ contravariant of weight $p \geq 1$. Since for $\phi \in$ GL$(n)$

$$Z(\phi P) = z(\ell_{\phi}P) = z(\ell_{P} \circ \phi^{-1}) = |\det \phi|^{p} \phi^{-1}z(\ell_{P}) = |\det \phi|^{p} \phi^{-1}Z(P),$$

also $Z$ is GL$(n)$ contravariant of weight $p \geq 1$. Thus we obtain from Theorem 3 that there exists a constant $c \geq 0$ such that

$$z(\ell_{P}) = c\Pi P$$

for all $\ell_{P} \in P_{0}^{1,1}(\mathbb{R}^{n})$. The statement now follows from Lemma 3 and (7).

**Lemma 6.** If $z : W_{0}^{1,1}(\mathbb{R}^{n}) \rightarrow \langle K_{n}^{\varepsilon}, + \rangle$ is a continuous, non-trivial, translation invariant valuation which is homogeneous of degree $q$, then $q \geq 1$.

**Proof.** Let $P \in P_{0}^{n}$ and $\varepsilon > 0$. Take translations $\tau_{1}, \ldots, \tau_{k}$ such that the polytopes $\tau_{1}P$ are pairwise disjoint. Define

$$f_{k} = \frac{1}{k^{1+\varepsilon}}(\ell_{\tau_{1}P} \vee \cdots \vee \ell_{\tau_{k}P}).$$

Then $|f_{k}| = |\nabla f_{k}| = O(k^{-\varepsilon})$ as $k \rightarrow \infty$. Hence $f_{k} \rightarrow 0$ as $k \rightarrow \infty$ in $W^{1,1}(\mathbb{R}^{n})$. Since $z$ is a translation invariant and homogeneous valuation, we obtain using (9) that

$$z(f_{k}) = k^{q-(1+\varepsilon)}z(\ell_{P}).$$

Since $z$ is continuous, it follows from (9) that $q \geq 1$. 

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Lemma 7. If \( z : W^{1,1}(\mathbb{R}^n) \to \langle K_n^*, + \rangle \) is a continuous, non-trivial, translation invariant valuation which is \( \text{GL}(n) \) contravariant of weight 1 and homogeneous of degree \( q \), then \( q \leq 1 \).

Proof. Let \( P \in \mathcal{P}_n^0 \) and \( \alpha, \beta > 0 \). Take translations \( \tau_1, \ldots, \tau_k \) such that the polytopes \( \tau_i P \) are pairwise disjoint. Define

\[
f_k = k^\alpha (\ell_{\tau_1(P/k^\alpha)} \lor \cdots \lor \ell_{\tau_k(P/k^\alpha)}).
\]

Then \( |f_k|_1 = O(k^{1+\alpha-n/\beta}) \) and \( |\nabla f_k|_1 = O(k^{1+\alpha+\beta-n/\beta}) \) as \( k \to \infty \). Hence for \( \alpha < (n-1)\beta - 1 \), we have \( f_k \to 0 \) as \( k \to \infty \) in \( W^{1,1}(\mathbb{R}^n) \). Since \( z \) is a translation invariant and homogeneous valuation, we obtain using (9) that

\[
z(f_k) = k^\alpha q_k^{-n/\beta + \beta} z(\ell_P).
\]

Since \( z \) is continuous, it follows from (9) that \( q \leq (-1 + (n-1)\beta)/\alpha \). Since this holds for all \( \alpha < (n-1)\beta - 1 \) and all \( \beta \), we conclude that \( q \leq 1 \).

Lemma 8. Let \( z_1, z_2 : L^{1,1}(\mathbb{R}^n) \to \langle K_n^*, + \rangle \) be continuous, translation invariant valuations, which are homogeneous of the same degree. If \( z_1(f) = z_2(f) \) for all \( f \in P^{1,1}(\mathbb{R}^n) \), then

\[
z_1(f) = z_2(f) \quad (14)
\]

for all \( f \in L^{1,1}(\mathbb{R}^n) \).

Proof. Let \( z_1 \) and \( z_2 \) be homogeneous of degree \( q \). As noted before, \( q \geq 0 \). If \( q = 0 \), then \( z_i(f) = z_i(0) \) for all \( f \in L^{1,1}(\mathbb{R}^n) \) and the statement of the lemma is true. Therefore we assume that \( z_1 \) and \( z_2 \) are homogeneous of degree \( q > 0 \) and have

\[
z_1(0) = z_2(0) = \{0\} . \quad (15)
\]

Since \( z_1 \) and \( z_2 \) are homogeneous valuations, we obtain using (15) that for \( i = 1, 2 \),

\[
z_i(f \lor 0) + z_i(f \land 0) = z_i(f) + z_i(0) = z_i(f)
\]

and

\[
z_i(f \land 0) = z_i((-f) \lor 0) = z_i((-f) \lor 0).
\]

Thus it suffices to show that (14) holds for all \( f \in L^{1,1}(\mathbb{R}^n) \) with \( f \geq 0 \).

Let such a function \( f \) be given and let \( f \) not vanish identically. Triangulate the support of \( f \) so that \( f \) is affine on each simplex of the triangulation. Let \( V \) be the (finite) set of vertices and \( S \) the set of \( n \)-dimensional simplices of this triangulation. Note that \( f \) is determined by the values \( f(v) \) for \( v \in V \) and that if \( f(\bar{v}) > 0 \) for some \( \bar{v} \in V \), then by changing the value \( f(\bar{v}) \) we obtain again a function in \( L^{1,1}(\mathbb{R}^n) \). Since \( z_1 \) and \( z_2 \) are continuous, it suffices to prove (14) for a function \( f \) where the values \( f(v) \) are distinct for \( v \in V \) with \( f(v) > 0 \).
First, we show that for such a function $f$ there are $f_1, \ldots, f_m \in L^{1,1}(\mathbb{R}^n)$ which are non-negative and concave on their supports such that

$$f = f_1 \lor \cdots \lor f_m. \quad (16)$$

Define the function $f_i$ by setting $f_i(v) = f(v)$ on the vertices $v$ of the simplex $S_i$ of $S$. Choose a polytope $P_i$ such that $S_i \subset P_i$ and set $f_i(v) = 0$ on the vertices $v$ of $P_i$. The function $f_i$ determined by these data is concave on its support and piecewise linear. Moreover, if the polytopes $P_i$ are chosen suitably small, (16) holds.

Using the inclusion-exclusion principle, we obtain from (16) that for $i = 1, 2$,

$$z_i(f) = z_i(f_1 \lor \cdots \lor f_m) = \sum_j (-1)^{|j|-1} z_i(f_j)$$

where we sum over all non-empty $J \subset \{1, \ldots, m\}$ and

$$f_J = f_{j_1} \land \cdots \land f_{j_k}$$

for $J = \{j_1, \ldots, j_k\}$. Thus it suffices to prove (14) for non-negative $f \in L^{1,1}(\mathbb{R}^n)$ that are concave on their support.

For a given function $f \in L^{1,1}(\mathbb{R}^n)$, let $F \subset \mathbb{R}^{n+1}$ be the compact polytope bounded by the graph of $f$ and the hyperplane $\{x_{n+1} = 0\}$. We call $F$ singular if $F$ has $n$ facet hyperplanes that intersect in a line $L$ parallel to $\{x_{n+1} = 0\}$ but not contained in $\{x_{n+1} = 0\}$. Since $z_1$ and $z_2$ are continuous, it suffices to show (14) for $f \in L^{1,1}(\mathbb{R}^n)$ such that $F$ is not singular. So we assume for the rest of the proof that $f$ has this property.

Let such a function $f$ be given. Let $\bar{p}$ be the vertex of $F$ with the largest $x_{n+1}$ coordinate. We use induction on the number $m$ of facet hyperplanes of $F$ that are not passing through $\bar{p}$. If $m = 1$, then a translate of $f$ is in $P^{1,1}(\mathbb{R}^n)$. Since $z_1$ and $z_2$ are translation invariant and homogeneous, (14) is true. Suppose (14) is true for all $f \in L^{1,1}(\mathbb{R}^n)$ such that $F$ has at most $(m-1)$ facet hyperplanes not containing $\bar{p}$. We show that (14) then also holds for all $f \in L^{1,1}(\mathbb{R}^n)$ with $m$ such hyperplanes.

So let $F$ have $m$ such hyperplanes. Let $p_0 = (x_0, f(x_0))$ be a vertex of $F$ with minimal non-negative $x_{n+1}$-coordinate. Let $H_1, \ldots, H_j$ be the facet hyperplanes of $F$ through $p_0$ which do not contain $\bar{p}$. There is at least one such hyperplane. Define $\bar{F}$ as the polytope bounded by the intersection of all facet hyperplanes of $F$ with the exception of $H_1, \ldots, H_j$. Since $F$ has no edges parallel to $\{x_{n+1} = 0\}$ but not contained in $\{x_{n+1} = 0\}$, the polytope $\bar{F}$ is bounded and the piecewise affine function $\bar{f}$ corresponding to $\bar{F}$ is in $L^{1,1}(\mathbb{R}^n)$. Note that $\bar{F}$ has at most $(m-1)$ facet hyperplanes not containing $\bar{p}$. Let $H_{i_1}, \ldots, H_{i_k}$ be the facet hyperplanes of $\bar{F}$ that contain $p_0$. Choose suitable hyperplanes $\bar{H}_{i_1}, \ldots, \bar{H}_{i_k}$ containing $p_0$ so that the hyperplanes $\bar{H}_{i_1}, \ldots, \bar{H}_{i_k}$ and $\{x_{n+1} = 0\}$ bound a pyramid with apex at $p_0$ that is contained in $\bar{F}$, has $x_0$ in its base and has $\bar{H}_{i_1}, \ldots, \bar{H}_{i_k}$ among its facet hyperplanes. Define $\ell$ as the piecewise affine function determined by this pyramid and note that a suitable translate of $\ell$ is in $P^{1,1}(\mathbb{R}^n)$. Set $\bar{\ell} = f \land \ell \in L^{1,1}(\mathbb{R}^n)$. The polytope determined by $\bar{\ell}$ is a pyramid since it is bounded by $\{x_{n+1} = 0\}$ and hyperplanes containing $p_0$. Therefore a suitable translate of $\bar{\ell}$ is in $P^{1,1}(\mathbb{R}^n)$. Hence translates of $\bar{\ell}$ and $\ell$ are in $P^{1,1}(\mathbb{R}^n)$, the polytope $\bar{F}$ has at most $(m-1)$ facet hyperplanes not containing $\bar{p}$, and

$$f \lor \ell = \bar{f} \quad \text{and} \quad f \land \ell = \bar{\ell}.$$
Since $z$ is a valuation, we obtain for $i = 1, 2$ that

$$z_i(f) + z_i(\ell) = z_i(\tilde{f}) + z_i(\tilde{\ell}).$$

Thus by induction (14) holds for all $f \in L^{1,1}(\mathbb{R}^n)$ with $m$ facet hyperplanes not containing $\bar{p}$. This completes the proof of the lemma.

5 Proof of Theorem 1

Suppose that $z : W^{1,1}(\mathbb{R}^n) \to \langle K^n_c, \# \rangle$ is a continuous affinely covariant valuation. Then for all $f, g \in W^{1,1}(\mathbb{R}^n)$,

$$z(f) \# z(g) = z(f \vee g) \# z(f \wedge g).$$

Hence, applying $\Pi$, we obtain by (6) that

$$\Pi z(f) + \Pi z(g) = \Pi z(f \vee g) + \Pi z(f \wedge g)$$

for all $f, g \in W^{1,1}(\mathbb{R}^n)$, that is, $\Pi \circ z : W^{1,1}(\mathbb{R}^n) \to \langle K^n_c, + \rangle$ is a valuation. Since $z$ is affinely covariant, (8) implies that $\Pi \circ z$ is affinely contravariant. Since $z$ and $\Pi$ are continuous, also $\Pi \circ z$ is continuous. Thus by Theorem 2, there is a constant $\tilde{c} \geq 0$ such that

$$\Pi z(f) = \tilde{c} \Pi \langle f \rangle$$

for every $f \in W^{1,1}(\mathbb{R}^n)$. Since $\Pi$ is injective on $K^n_c$, we obtain that $z(f) = c\langle f \rangle$ for all $f \in W^{1,1}(\mathbb{R}^n)$ for some $c \in \mathbb{R}$. Combined with Lemma 2, this completes the proof of the theorem.

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References


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