On the semicontinuity of curvature integrals

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Abstract

Let $H_j(K,\cdot)$ be the j-th elementary symmetric function of the principal curvatures of a convex body K in Euclidean d-space. We show that the functionals $\int_{\mathrm{bd}\,K} f(H_j(K,x))\, d\mathcal{H}^{d-1}(x)$ depend upper semicontinuously on K, if $f:[0,\infty)\to[0,\infty)$ is concave, $\lim_{t\to 0} f(t)=0$, and $\lim_{t\to\infty} f(t)/t = 0$. An analogue statement holds for integrals of elementary symmetric functions of the radii of curvature.

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1 Introduction and statement of results

Let K be a convex body, i.e. a compact convex set, in Euclidean d-dimensional space $\mathbb{E}^d$. In many problems, especially in connection with asymptotic approximation by polytopes (see [5] and [6]), integrals of the following type are important:

$$\int\limits_{\mathrm{bd}\,K} H_j(K,x)^p \, d\mathcal{H}^{d-1}(x) \tag{1}$$

Here $\mathrm{bd}\,K$ denotes the boundary of K, $H_j(K,x)$ is the j-th elementary symmetric function of the principal curvatures at $x\in\mathrm{bd}\,K$, $\mathcal{H}^{d-1}$ is the $(d-1)$-dimensional Hausdorff measure, and $p\in\mathbb{R}$. These integrals are defined – but possibly infinite – for general convex bodies without smoothness assumption, since $H_j(K,\cdot)$ exists almost everywhere on $\mathrm{bd}\,K$ and is Lebesgue-measurable (see [1] or [3]).

The affine surface area for general convex bodies as defined by C. Schütt and E. Werner [19]

$$\Omega(K) = \int\limits_{\mathrm{bd}\,K} H_{d-1}(K,x)^{\frac{1}{d+1}} \, d\mathcal{H}^{d-1}(x) \tag{2}$$

is maybe the most important example of such integrals. Let $\mathcal{K}^d$ be the space of convex bodies in $\mathbb{E}^d$ equipped with the usual topology induced by the Hausdorff metric (cf. [17]). Then $\Omega(\cdot)$ is a functional defined on $\mathcal{K}^d$ and $\Omega(K)$ depends upper semicontinuously on K. This was first - even for smooth bodies – proved by E. Lutwak [14] using another definition for affine surface area. That Lutwak's definition is equivalent to (2) was shown by K. Leichtweiß [10], G. Dolzmann and D. Hug [4], and C. Schütt [18]. The proof of this equivalence can also be found in K. Leichtweiß' monograph [11]. Using the upper semicontinuity, it is possible to give a characterization of affine surface area (see [13]).

We give a direct proof for the upper semicontinuity of the functional defined in (2) and for the following larger class of functionals. Let $\mathcal{D}$ be the set of functions $f : [0,\infty) \to [0,\infty)$ such that f is concave, $\lim_{t\to 0} f(t) = 0$, and $\lim_{t\to\infty} f(t)/t = 0$. Then we have the following result.

Theorem 1 *Let $f \in \mathcal{D}$. Then for $j = 1,\ldots,d-1$*

$$\mu(K) = \int\limits_{\operatorname{bd} K} f(H_j(K,x))\, d\mathcal{H}^{d-1}(x) \tag{3}$$

is finite for every $K \in \mathcal{K}^d$ and depends upper semicontinuously on K.

The functions $f(t) = t^p$ for $0 < p < 1$ are in $\mathcal{D}$. Therefore Theorem 1 implies that in this case the functionals defined in (1) are upper semicontinuous.

For $d = 2$, the class $\mathcal{D}$ is the largest possible, i.e. for $K \in \mathcal{K}^2$ and $H_{d-1}(K,x) = \kappa(K,x)$, the curvature of $\operatorname{bd} K$ at x, the functional

$$\int\limits_{\operatorname{bd} K} f(\kappa(K,x))\, d\mathcal{H}^1(x)$$

is upper semicontinuous if and only if $f \in \mathcal{D}$ or f is constant (in the second case, the functional is proportional to the length of $\operatorname{bd} K$, which depends continuously on K). This follows from results of [12], where a characterization of upper semicontinuous and rigid motion invariant valuations on $\mathcal{K}^2$ is given.

The first definition of affine surface area for a general convex body K was given by K. Leichtweiß [9]. It is

$$\Omega(K) = \int\limits_{S^{d-1}} P_{d-1}(K,u)^{\frac{d}{d+1}}\, d\mathcal{H}^{d-1}(u) \tag{4}$$

where $P_j(K,u)$ is the j-th elementary symmetric function of the principal radii of curvature at $u \in S^{d-1}$ and S^{d-1} is the unit sphere in $\mathbb{E}^d$. This integral is well defined for a general convex body, since the principal radii of curvature exist almost everywhere on S^{d-1} and are Lebesgue measurable (see [9]). This definition was shown to be equivalent to (2) in [18]. Also for other integrals over the elementary symmetric functions of the principal radii of curvature, we obtain upper semicontinuous functionals.

Theorem 2 *Let $f \in \mathcal{D}$. Then for $j = 1, \ldots, d-1$*

$$\mu(K) = \int\limits_{S^{d-1}} f(P_j(K,u))\, d\mathcal{H}^{d-1}(u) \tag{5}$$

is finite for every $K \in \mathcal{K}^d$ and depends upper semicontinuously on K.

Since the proofs of these two theorems are similar, we only give the proof of Theorem 2.

Results related to Theorem 2 are contained in work [15] of E. Lutwak. He showed that the functionals

$$\int\limits_{S^{d-1}} P_j(K,x)^p\, d\mathcal{H}^{d-1}(x) \tag{6}$$

for $0 < p < 1$ defined for convex bodies with boundary of class $\mathcal{C}^2$ and positive curvatures, can be extended to functionals defined for every $K \in \mathcal{K}^d$, in such a way that the extension is upper semicontinuous. We remark that similarly to the case of affine surface area, i.e. $p = d/(d+1)$, it can be seen that this extension coincides with the functionals defined in (5) for $f(t) = t^p$. For the proof of this equivalence in the case of affine surface area see [10] and [4].

D. Hug gave in [7] a direct proof for the equivalence of the definition of affine surface area in (2) and in (4). We remark that with similar methods the connection between the functionals (3) and (5) in the case $j = d-1$ can be established, i.e., for $f \in \mathcal{D}$

$$\int\limits_{\operatorname{bd} K} f(H_{d-1}(K,x))\, d\mathcal{H}^{d-1}(x) = \int\limits_{S^{d-1}} g(P_{d-1}(K,u))\, d\mathcal{H}^{d-1}(u)$$

where $g(t) = t\, f(1/t)$. For $j = 1, \ldots, d-2$, the corresponding transformation formulae are not so simple. This can be seen by considering $(d-1)$-dimensional balls K. For them $H_1(K,x) = \ldots = H_{d-1}(K,x) = 0$ a.e. on $\operatorname{bd} K$ and therefore the functionals (3) always vanish. But $P_1(K,u), \ldots, P_{d-2}(K,u) > 0$ on a set of positive measure on S^{d-1}, and therefore the functionals (5) do not always vanish.

2 Tools

For $j = 1, \ldots, d-1$, let $S_j(K,\cdot)$ and $C_j(K,\cdot)$ be the j-th *area measure* and *curvature measure* of a convex body K. For the definition of these Borel-measures on the sphere S^{d-1} and on $\mathbb{E}^d$, respectively, and their properties, see [17]. We need the following results. For a sequence of convex bodies K_n converging to K, the measures $S_j(K_n,\cdot)$ converge weakly to $S_j(K,\cdot)$, and the measures $C_j(K_n,\cdot)$ converge weakly to $C_j(K,\cdot)$. This implies that for every closed set $\omega \subset S^{d-1}$

$$\limsup_{n\to\infty} S_j(K_n,\omega) \le S_j(K,\omega) \tag{7}$$

and

$$\lim_{n\to\infty} S_j(K_n,S^{d-1}) = S_j(K,S^{d-1}), \tag{8}$$

and that for every closed set $\beta \subset \mathbb{E}^d$

$$\limsup_{n\to\infty} C_j(K_n,\beta) \le C_j(K,\beta)$$

and

$$\lim_{n\to\infty} C_j(K_n,\mathbb{E}^d) = C_j(K,\mathbb{E}^d)$$

(concerning the notion of weak convergence, see, for example, [2]).

The measures $S_j(K,\cdot)$ can be decomposed into measures absolutely continuous and singular with respect to the $(d-1)$-dimensional Hausdorff measure $\mathcal{H}^{d-1}(\cdot)$ on the sphere, say, $S_j(K,\cdot) = S_j^a(K,\cdot) + S_j^s(K,\cdot)$. For the absolutely continuous part, we have

$$S_j^a(K,\omega) = \int_\omega P_j(K,u)\, d\mathcal{H}^{d-1}(u) \tag{9}$$

(see, for example, [8]). Here

$$P_j(K,u) = \frac{1}{\binom{d-1}{j}} \sum_{1\le i_1<\ldots<i_j\le d-1} \rho_{i_1}(K,u) \cdots \rho_{i_j}(K,u)$$

where $\rho_i(K,u)$ is the i-th principal radius of curvature at $u \in S^{d-1}$. That the $P_j(K,\cdot)$ exist almost everywhere on S^{d-1} was shown by A.D. Aleksandrov [1] (or see [3]). The Lebesgue-measurability of $P_j(K,\cdot)$ is implied by (9) (see also the remarks in [8]). The singular part is concentrated on a null set, i.e., there is a set $\omega_0 \subset S^{d-1}$ such that $\mathcal{H}^{d-1}(\omega_0) = 0$ and

$$S_j^s(K,\omega\backslash\omega_0) = 0 \tag{10}$$

for every Borel set $\omega \subset S^{d-1}$.

The measures $C_j(K,\cdot)$ are concentrated on $\operatorname{bd} K$. Decomposing $C_j(K,\cdot)$ into measures absolutely continuous and singular with respect to the $(d-1)$-dimensional Hausdorff measure $\mathcal{H}^{d-1}(\cdot)$ on $\operatorname{bd} K$, say, $C_j(K,\cdot) = C_j^a(K,\cdot) + C_j^s(K,\cdot)$, gives

$$C_j^a(K,\beta) = \int\limits_{\beta\cap \operatorname{bd} K} H_{d-1-j}(K,x)\, d\mathcal{H}^{d-1}(x)$$

(see, for example, [8]). Here

$$H_j(K,x) = \frac{1}{\binom{d-1}{j}} \sum_{1\le i_1<\ldots<i_j\le d-1} \kappa_{i_1}(K,x)\ \cdots\ \kappa_{i_j}(K,x)$$

where $\kappa_i(K,x)$ is the i-th principal curvature at $x \in \operatorname{bd} K$. $H_j(K,\cdot)$ is defined a.e. on $\operatorname{bd} K$ and Lebesgue-measurable. The singular part $C_j^s(K,\cdot)$ vanishes outside a set $\beta_0 \subset \operatorname{bd} K$ with $\mathcal{H}^{d-1}(\beta_0) = 0$.

3 Proof of Theorem 2

The definition of $\mathcal{D}$ implies that every $f(t) \in \mathcal{D}$ is continuous and that $f(0) = 0$. Since $f(t)$ is concave and non-negative on $[0,\infty)$, $f(t)$ is non-decreasing. Using the concavity of $f(t)$ and the fact that $f(0) = 0$, we have for every $t > 0$ and $0 < \lambda < 1$

$$f(\lambda\, t + (1-\lambda)\, 0) \ge \lambda\, f(t) + (1-\lambda)\, f(0)$$

and consequently with $s = \lambda\, t < t$

$$f(s) \ge \frac{s}{t}\, f(t).$$

This shows that $f(t)/t$ is non-increasing.

Using the fact that f is concave, we obtain by Jensen's inequality

$$\frac{1}{\mathcal{H}^{d-1}(S^{d-1})} \int_{S^{d-1}} f(P_j(K,u))\, d\mathcal{H}^{d-1}(u)$$
$$\le f\Big(\frac{1}{\mathcal{H}^{d-1}(S^{d-1})} \int_{S^{d-1}} P_j(K,u)\, d\mathcal{H}^{d-1}(u)\Big).$$

The left hand side is always finite, since by (9)

$$\int_{S^{d-1}} P_j(K,u)\, d\mathcal{H}^{d-1}(u) \le S_j(K,S^{d-1})$$

and since f is non-decreasing. Therefore we have $\mu(K) < \infty$ for every $K \in \mathcal{K}$.

Let $K \in \mathcal{K}^d$ and $\varepsilon > 0$ be chosen. Since $P_j(K, \cdot)$ is measurable a.e. on S^{d-1} and since the set ω_0, where the singular part of $S_j(K, \cdot)$ is concentrated, is a null set, we can choose by Lusin's theorem (see, for example, [16]) a closed set $\omega \subset S^{d-1}$ such that $P_j(K, \cdot)$ is continuous as a function restricted to ω, such that

$$\omega \cap \omega_0 = \emptyset \tag{11}$$

and such that

$$\mathcal{H}^{d-1}(S^{d-1} \backslash \omega) \leq \varepsilon. \tag{12}$$

Let K_n be a sequence of convex bodies converging to K. First, we show that

$$\limsup_{n\to\infty} \int_\omega f(P_j(K_n, u))\, d\mathcal{H}^{d-1}(u) \leq \int_\omega f(P_j(K, u))\, d\mathcal{H}^{d-1}(u) \tag{13}$$

holds. Let $\eta > 0$ be chosen, and set $a = \inf\{f(P_j(K, u)) : u \in \omega\}$ and $b = \sup\{f(P_j(K, u)) : u \in \omega\}$. Since $P_j(K, \cdot)$ is continuous on ω and ω is compact, $b < \infty$. f is therefore uniformly continuous on $[a, b]$, i.e. there is a $\delta > 0$ such that

$$|f(s) - f(t)| \leq \eta \quad \text{for} \quad |s - t| \leq \delta. \tag{14}$$

We choose a subdivision $a = t_1 \leq t_2 \leq \ldots \leq t_{m+1} = b$, such that

$$\max_{i=1,\ldots,m} \{t_{i+1} - t_i\} \leq \delta \tag{15}$$

and such that

$$\mathcal{H}^{d-1}(\{u \in \omega : P_j(K, u) = t_i\}) = 0$$

for $i = 2, \ldots, m$. This is possible, since $\mathcal{H}^{d-1}(\{u \in \omega : P_j(K, u) = t\}) > 0$ holds only for countably many t. Setting

$$\omega_i = \{u \in \omega : t_i \leq P_j(K, u) \leq t_{i+1}\},$$

and using the monotonicity of f we have

$$\begin{aligned} \int_\omega f(P_j(K, u))\, d\mathcal{H}^{d-1}(u) &= \sum_{i=1}^m \int_{\omega_i} f(P_j(K,u))\, d\mathcal{H}^{d-1}(u) \\ &\geq \sum_{i=1}^m f(t_i)\, \mathcal{H}^{d-1}(\omega_i). \end{aligned} \tag{16}$$

Since f is concave, by Jensen's inequality

$$\begin{aligned}&\frac{1}{\mathcal{H}^{d-1}(\omega_i)}\int_{\omega_i} f(P_j(K_n,u))\,d\mathcal{H}^{d-1}(u)\\ &\qquad\qquad \le f\Big(\frac{1}{\mathcal{H}^{d-1}(\omega_i)}\int_{\omega_i} P_j(K_n,u)\,d\mathcal{H}^{d-1}(u)\Big)\end{aligned}$$

holds for $i=1,\dots,m$, and $\mathcal{H}^{d-1}(\omega_i)\neq 0$. By (9)

$$\int_{\omega_i} P_j(K_n,u)\,d\mathcal{H}^{d-1}(u)\le S_j(K_n,\omega_i).$$

Using these inequalities and the monotonicity of f, we obtain

$$\begin{aligned}\int_{\omega} f(P_j(K_n,u))\,d\mathcal{H}^{d-1}(u) &\le \sum_{i=1}^m \int_{\omega_i} f(P_j(K_n,u))\,d\mathcal{H}^{d-1}(u)\\ &= {\sum_{i=1}^m}' \int_{\omega_i} f(P_j(K_n,u))\,d\mathcal{H}^{d-1}(u)\\ &\le {\sum_{i=1}^m}' f\left(\frac{S_j(K_n,\omega_i)}{\mathcal{H}^{d-1}(\omega_i)}\right)\mathcal{H}^{d-1}(\omega_i)\end{aligned}$$

where the $'$ indicates that we sum only over ω_i with $\mathcal{H}^{d-1}(\omega_i)\neq 0$. Since $P_j(K,\cdot)$ is continuous on ω and ω is closed, the sets ω_i are closed for $i=1,\dots,m$. This implies by (7) that

$$\limsup_{n\to\infty} S_j(K_n,\omega_i)\le S_j(K,\omega_i).$$

By (11) and (10),

$$S_j(K,\omega_i)=S_j^a(K,\omega_i).$$

Consequently, using the continuity and monotonicity of f, the fact that by (9) and the definition of ω_i

$$S_j^a(K,\omega_i)\le t_{i+1}\,\mathcal{H}^{d-1}(\omega_i),$$

(16), (15), and (14), we obtain

$$\begin{aligned}&\limsup_{n\to\infty}\int_{\omega} f(P_j(K_n,u)\,d\mathcal{H}^{d-1}(u)\\ &\qquad\qquad \le {\sum_{i=1}^m}' f\left(\frac{S_j(K,\omega_i)}{\mathcal{H}^{d-1}(\omega_i)}\right)\mathcal{H}^{d-1}(\omega_i)\end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{i=1}^{m} f(t_{i+1})\,\mathcal{H}^{d-1}(\omega_i) \\
&= \sum_{i=1}^{m} f(t_i)\,\mathcal{H}^{d-1}(\omega_i) + \sum_{i=1}^{m} \left(f(t_{i+1}) - f(t_i)\right)\,\mathcal{H}^{d-1}(\omega_i) \\
&\leq \int_{\omega} f(P_j(K,u))\,d\mathcal{H}^{d-1}(u) + \eta\,\mathcal{H}^{d-1}(\omega).
\end{aligned}
$$

Since $\eta > 0$ is arbitrary, this proves (13).

Finally, we show that

$$
\limsup_{n\to\infty} \int_{S^{d-1}} f(P_j(K_n,u))\,d\mathcal{H}^{d-1}(u) \leq \int_{S^{d-1}} f(P_j(K,u))\,d\mathcal{H}^{d-1}(u). \tag{17}
$$

Since $f(t)$ is non-decreasing and $f(t)/t$ is non-increasing, using (9) we see that for every $t > 0$,

$$
\begin{aligned}
&\int_{S^{d-1}\setminus\omega} f(P_j(K_n,u))\;d\mathcal{H}^{d-1}(u) \\
&\quad= \int_{\{u\in S^{d-1}\setminus\omega\,:\,P_j(K_n,u)\leq t\}} f(P_j(K_n,u))\,d\mathcal{H}^{d-1}(u) \\
&\qquad+ \int_{\{u\in S^{d-1}\setminus\omega\,:\,P_j(K_n,u)> t\}} \frac{f(P_j(K_n,u))}{P_j(K_n,u)}\,P_j(K_n,u)\,d\mathcal{H}^{d-1}(u) \\
&\quad\leq f(t)\,\mathcal{H}^{d-1}(S^{d-1}\setminus\omega) + \frac{f(t)}{t}\,S_j(K_n,S^{d-1}).
\end{aligned}
$$

Combined with (13), (12), and (8), this implies that for every $t > 0$

$$
\begin{aligned}
&\limsup_{n\to\infty} \int_{S^{d-1}} f(\,P_j(K_n,u))\,d\mathcal{H}^{d-1}(u) \\
&\qquad\leq \int_{S^{d-1}} f(P_j(K,u))\,d\mathcal{H}^{d-1}(u) + f(t)\,\varepsilon + \frac{f(t)}{t}\,S_j(K,S^{d-1}).
\end{aligned}
$$

Since $\varepsilon > 0$ is arbitrary and since t does not depend on ε, we therefore have for every $t > 0$

$$
\begin{aligned}
&\limsup_{n\to\infty} \int_{S^{d-1}} f(P_j(K_n,u))\,d\mathcal{H}^{d-1}(u) \\
&\qquad\leq \int_{S^{d-1}} f(P_j(K,u))\,d\mathcal{H}^{d-1}(u) + \frac{f(t)}{t}\,S_j(K,S^{d-1}).
\end{aligned} \tag{18}
$$

Using the fact that $f(t)/t$ is continuous and that $\lim_{t\to\infty} f(t)/t = 0$, we now can make $f(t)/t$ arbitrarily small by choosing t suitably large. Therefore (18) proves (17).

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