Valuations on Convex Functions

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Abstract

All continuous, $\mathrm{SL}(n)$ and translation invariant valuations on the space of convex functions on $\mathbb{R}^n$ are completely classified.

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A function Z defined on a lattice $(\mathcal{L}, \vee, \wedge)$ and taking values in an abelian semigroup is called a *valuation* if

$$\mathrm{Z}(u \vee v) + \mathrm{Z}(u \wedge v) = \mathrm{Z}(u) + \mathrm{Z}(v) \tag{1}$$

for all $u, v \in \mathcal{L}$. A function Z defined on some subset $\mathcal{S}$ of $\mathcal{L}$ is called a valuation on $\mathcal{S}$ if (1) holds whenever $u, v, u \vee v, u \wedge v \in \mathcal{S}$. For $\mathcal{S}$ the set of compact convex sets, $\mathcal{K}^n$, in $\mathbb{R}^n$ with $\vee$ denoting union and $\wedge$ intersection, valuations have been studied since Dehn's solution of Hilbert's Third Problem in 1901 and interesting new ones keep arising (see, for example, [16]). The natural topology on $\mathcal{K}^n$ is induced by the Hausdorff metric and continuous, $\mathrm{SL}(n)$ and translation invariant valuations on $\mathcal{K}^n$ were first classified by Blaschke. The celebrated Hadwiger classification theorem establishes a complete classification of continuous, rigid motion invariant valuations on $\mathcal{K}^n$ and provides a characterization of intrinsic volumes. See [1–3, 6, 12–14, 20, 25] for some recent results on valuations on convex sets and [15, 17] for information on the classical theory.

More recently, valuations have been studied on function spaces. Here $\mathcal{S}$ is a space of real valued functions and $u \vee v$ is the pointwise maximum of u and v while $u \wedge v$ is the pointwise minimum. For Sobolev spaces [21, 23, 27] and L^p spaces [24, 32, 33] complete classifications for valuations intertwining the $\mathrm{SL}(n)$ were established. See also [19, 22, 29, 34]. Moreover, classical functionals for convex sets including the intrinsic volumes have been extended to the space of quasi-concave functions in [7] and [28] (see also [9, 18]). A classification of rigid motion invariant valuations on quasi-concave functions is established in [10]. For definable functions such a result was previously established in [5].

The aim of this paper is to establish a complete classification of $\mathrm{SL}(n)$ and translation invariant valuations on convex functions. Let $\mathrm{Conv}(\mathbb{R}^n)$ denote the space of convex functions $u : \mathbb{R}^n \to (-\infty, +\infty]$ which are proper, lower semicontinuous and coercive. Here a function is *proper* if it is not identically $+\infty$ and it is *coercive* if

$$\lim_{|x| \to +\infty} u(x) = +\infty \tag{2}$$

where $|x|$ is the Euclidean norm of x. The space $\mathrm{Conv}(\mathbb{R}^n)$ is one of the standard spaces in convex analysis and it is equipped with the topology associated to epi-convergence (see Section 1).

Let $n \geq 2$ throughout the paper. A functional $\mathrm{Z} : \mathrm{Conv}(\mathbb{R}^n) \to \mathbb{R}$ is $\mathrm{SL}(n)$ *invariant* if $\mathrm{Z}(u \circ \phi^{-1}) = \mathrm{Z}(u)$ for every $u \in \mathrm{Conv}(\mathbb{R}^n)$ and $\phi \in \mathrm{SL}(n)$. It is *translation invariant* if $\mathrm{Z}(u \circ \tau^{-1}) = \mathrm{Z}(u)$ for every $u \in \mathrm{Conv}(\mathbb{R}^n)$ and translation $\tau : \mathbb{R}^n \to \mathbb{R}^n$. In [8], a class of rigid motion invariant valuations on $\mathrm{Conv}(\mathbb{R}^n)$ was introduced and classification results were established. However, the setting is different from our setting, as a different topology (coming from a notion of monotone convergence) is used in [8] and monotonicity of the valuations is assumed. Variants of the functionals from [8] also appear in our classification. We say that a functional $\mathrm{Z} : \mathrm{Conv}(\mathbb{R}^n) \to \mathbb{R}$ is *continuous* if $\mathrm{Z}(u) = \lim_{k\to\infty} \mathrm{Z}(u_k)$ for every sequence $u_k \in \mathrm{Conv}(\mathbb{R}^n)$ that epi-converges to $u \in \mathrm{Conv}(\mathbb{R}^n)$.

Theorem. *A functional $\mathrm{Z} : \mathrm{Conv}(\mathbb{R}^n) \to [0, \infty)$ is a continuous, $\mathrm{SL}(n)$ and translation invariant valuation if and only if there exist a continuous function $\zeta_0 : \mathbb{R} \to [0, \infty)$ and a continuous function $\zeta_n : \mathbb{R} \to [0, \infty)$ with finite $(n-1)$-st moment such that*

$$\mathrm{Z}(u) = \zeta_0\big(\min\nolimits_{x\in\mathbb{R}^n} u(x)\big) + \int_{\mathrm{dom}\, u} \zeta_n\big(u(x)\big)\, dx \tag{3}$$

for every $u \in \mathrm{Conv}(\mathbb{R}^n)$.

Here, a function $\zeta : \mathbb{R} \to [0, \infty)$ has finite $(n-1)$-st moment if $\int_0^{+\infty} t^{n-1}\zeta(t)\,\mathrm{d}t < +\infty$ and $\mathrm{dom}\, u$ is the domain of u, that is, $\mathrm{dom}\, u = \{x \in \mathbb{R}^n : u(x) < +\infty\}$. Since $u \in \mathrm{Conv}(\mathbb{R}^n)$, the minimum of u is attained and hence finite.

If the valuation in (3) is evaluated for a (convex) indicator function I_K for $K \in \mathcal{K}^n$ (where $\mathrm{I}_K(x) = 0$ for $x \in K$ and $\mathrm{I}_K(x) = +\infty$ for $x \notin K$), then $\zeta_0(0)V_0(K)+\zeta_n(0)\, V_n(K)$ is obtained, where $V_0(K)$ is the Euler characteristic and $V_n(K)$ the n-dimensional volume of K. The proof of the theorem makes essential use of the following classification of continuous and $\mathrm{SL}(n)$ invariant valuations on $\mathcal{P}_0^n$, the space of convex polytopes which contain the origin. A functional $\mathrm{Z} : \mathcal{P}_0^n \to \mathbb{R}$ is a continuous and $\mathrm{SL}(n)$ invariant valuation if and only if there are constants $c_0, c_n \in \mathbb{R}$ such that

$$\mathrm{Z}(P) = c_0 V_0(P) + c_n V_n(P) \tag{4}$$

for every $P \in \mathcal{P}_0^n$ (see, for example, [26]). For continuous and rotation invariant valuations on $\mathcal{K}^n$ that have polynomial behavior with respect to translations, a classification was established by Alesker [2] but a classification of continuous and rotation invariant valuations on $\mathcal{P}_0^n$ is not known. It is also an open problem to establish a classification of continuous and rigid motion invariant valuations on $\mathrm{Conv}(\mathbb{R}^n)$.

1 The Space of Convex Functions

We collect some properties of convex functions and of the space $\mathrm{Conv}(\mathbb{R}^n)$. A basic reference is the book by Rockafellar & Wets [30] (see also [4, 11]). In particular, epi-convergence is discussed and some properties of epi-convergent sequences of convex functions are established. For these results, conjugate functions are introduced. We also discuss piecewise affine functions and give a self-contained proof that they are dense in $\mathrm{Conv}(\mathbb{R}^n)$.

To every convex function $u : \mathbb{R}^n \to (-\infty, +\infty]$, there can be assigned several convex sets. For $t \in (-\infty, +\infty]$, the *sublevel sets*

$$\{u < t\} = \{x \in \mathbb{R}^n : u(x) < t\}, \quad \{u \le t\} = \{x \in \mathbb{R}^n : u(x) \le t\},$$

are convex. The domain, $\operatorname{dom} u$, of u is convex and the *epigraph* of u,

$$\operatorname{epi} u = \{(x, y) \in \mathbb{R}^n \times \mathbb{R} : u(x) \le y\},$$

is a convex subset of $\mathbb{R}^n \times \mathbb{R}$.

The lower semicontinuity of a convex function $u : \mathbb{R}^n \to (-\infty, +\infty]$ is equivalent to its epigraph being closed and to all sublevel sets, $\{u \le t\}$, being closed. Such functions are also called *closed*. The growth condition (2) is equivalent to the boundedness of all sublevel sets $\{u \le t\}$. Hence, $\{u \le t\} \in \mathcal{K}^n$ for $u \in \operatorname{Conv}(\mathbb{R}^n)$ and $t \ge \min_{x \in \mathbb{R}^n} u(x)$.

For convex functions $u, v \in \operatorname{Conv}(\mathbb{R}^n)$, the pointwise minimum $u \wedge v$ corresponds to the union of their epigraphs and therefore to the union of their sublevel sets. Similarly, the pointwise maximum $u \vee v$ corresponds to the intersection of the epigraphs and sublevel sets. Hence for all $t \in \mathbb{R}$

$$\{u \wedge v \le t\} = \{u \le t\} \cup \{v \le t\}, \qquad \{u \vee v \le t\} = \{u \le t\} \cap \{v \le t\},$$

where for $u \vee v \in \operatorname{Conv}(\mathbb{R}^n)$ all sublevel sets are either empty or in $\mathcal{K}^n$. For $u \in \operatorname{Conv}(\mathbb{R}^n)$,

$$\operatorname{relint}\{u \le t\} \subseteq \{u < t\} \tag{5}$$

for every $t > \min_{x \in \mathbb{R}^n} u(x)$, where relint is the relative interior (see [8, Lemma 3.2]).

1.1 Epi-convergence

A sequence $u_k : \mathbb{R}^n \to (-\infty, \infty]$ is *epi-convergent* to $u : \mathbb{R}^n \to (-\infty, \infty]$ if for all $x \in \mathbb{R}^n$ the following conditions hold:

(i) For every sequence x_k that converges to x,

$$u(x) \le \liminf_{k \to \infty} u_k(x_k). \tag{6}$$

(ii) There exists a sequence x_k that converges to x such that

$$u(x) = \lim_{k \to \infty} u_k(x_k). \tag{7}$$

In this case we also write $u = \operatorname{epi-lim}_{k \to \infty} u_k$ and $u_k \xrightarrow{epi} u$.

Equation (6) means, that u is an asymptotic common lower bound for the sequence u_k. Consequently, (7) states that this bound is optimal. The name epi-convergence is due to the fact, that this convergence is equivalent to the convergence of the corresponding epigraphs in the Painlevé-Kuratowski sense. Another name for epi-convergence is Γ-convergence (see [11, Theorem 4.16] and [30, Proposition 7.2]). We consider $\operatorname{Conv}(\mathbb{R}^n)$ with the topology associated to epi-convergence.

Immediately from the definition of epi-convergence we get the following result (see, for example, [11, Proposition 6.1.]).

Lemma 1. *If $u_k : \mathbb{R}^n \to (-\infty, \infty]$ is a sequence that epi-converges to $u : \mathbb{R}^n \to (-\infty, \infty]$, then also every subsequence u_{k_i} of u_k epi-converges to u.*

For the following result, see, for example, [30, Proposition 7.4 and Theorem 7.17].

Lemma 2. *If u_k is a sequence of convex functions that epi-converges to a function u, then u is convex and lower semicontinuous. Moreover, if $\operatorname{dom} u$ has non-empty interior, then u_k converges uniformly to u on every compact set that does not contain a boundary point of $\operatorname{dom} u$.*

We also require the following connection to pointwise convergence (see, for example, [11, Example 5.13]).

Lemma 3. *Let $u_k : \mathbb{R}^n \to \mathbb{R}$ be a sequence of finite convex functions and $u : \mathbb{R}^n \to \mathbb{R}$ a finite convex function. Then u_k is epi-convergent to u, if and only if u_k converges pointwise to u.*

The last statement is no longer true if the functions may attain the value $+\infty$. In that case

$$\text{epi-lim}_{k\to\infty} u_k(x) \le \lim_{k\to\infty} u_k(x),$$

for all $x \in \mathbb{R}^n$ such that both limits exist.

We want to connect epi-convergence of functions from $\operatorname{Conv}(\mathbb{R}^n)$ with the convergence of their sublevel sets. The natural topology on $\mathcal{K}^n$ is induced by the Hausdorff distance. For $K, L \subset \mathbb{R}^n$, we write

$$K + L = \{x + y : x \in K, y \in L\}$$

for their *Minkowski sum*. Let $B \subset \mathbb{R}^n$ be the closed, n-dimensional unit ball. For $K, L \in \mathcal{K}^n$, the Hausdorff distance is

$$\delta(K, L) = \inf\{\varepsilon > 0 : K \subset L + \varepsilon B,\, L \subset K + \varepsilon B\}.$$

We write $K_i \to K$ as $i \to \infty$, if $\delta(K_i, K) \to 0$ as $i \to \infty$. For the next result we need the following description of Hausdorff convergence on $\mathcal{K}^n$ (see, for example, [31, Theorem 1.8.8]).

Lemma 4. *The convergence $\lim_{i\to\infty} K_i = K$ in $\mathcal{K}^n$ is equivalent to the following conditions taken together:*

(i) Each point in K is the limit of a sequence $(x_i)_{i\in\mathbb{N}}$ with $x_i \in K_i$ for $i \in \mathbb{N}$.

(ii) The limit of any convergent sequence $(x_{i_j})_{j\in\mathbb{N}}$ with $x_{i_j} \in K_{i_j}$ for $j \in \mathbb{N}$ belongs to K.

Each sublevel set of a function from $\operatorname{Conv}(\mathbb{R}^n)$ is either empty or in $\mathcal{K}^n$. We say that $\{u_k \le t\} \to \emptyset$ as $k \to \infty$ if there exists $k_0 \in \mathbb{N}$ such that $\{u_k \le t\} = \emptyset$ for $k \ge k_0$. We include the proof of the following simple result, for which we did not find a suitable reference.

Lemma 5. *Let $u_k, u \in \operatorname{Conv}(\mathbb{R}^n)$. If $u_k \xrightarrow{epi} u$ as $k \to \infty$, then $\{u_k \le t\} \to \{u \le t\}$ as $k \to \infty$ for every $t \in \mathbb{R}$ with $t \ne \min_{x\in\mathbb{R}^n} u(x)$.*

Proof. First, let $t > \min_{x \in \mathbb{R}^n} u(x)$. For $x \in \operatorname{relint}\{u \leq t\}$, it follows from (5) that $s = u(x) < t$. Since u_k epi-converges to u, there exists a sequence x_k that converges to x such that $u_k(x_k)$ converges to $u(x)$. Therefore, there exist $\varepsilon > 0$ and $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$

$$u_k(x_k) \leq s + \varepsilon \leq t.$$

Thus, $x_k \in \{u_k \leq t\}$, which shows that x is a limit of a sequence of points from $\{u_k \leq t\}$. It is easy to see that this implies (i) of Lemma 4.

Now, let $(x_{i_j})_{j \in \mathbb{N}}$ be a convergent sequence in $\{u_{i_j} \leq t\}$ with limit $x \in \mathbb{R}^n$. By Lemma 1, the subsequence u_{i_j} epi-converges to u. Therefore

$$u(x) \leq \liminf_{j \to \infty} u_{i_j}(x_{i_j}) \leq t$$

which gives (ii) of Lemma 4.

Second, let $t < \min_{x \in \mathbb{R}^n} u(x) = u_{\min}$. Since $\{u \leq t\} = \emptyset$, we have to show that there exists $k_0 \in \mathbb{N}$ such that for every $k \geq k_0$ and $x \in \mathbb{R}^n$,

$$u_k(x) > t.$$

Assume that there does not exist such an index k_0. Then there are infinitely many points x_{i_j} such that $u_{i_j}(x_{i_j}) \leq t$. Note, that

$$x_{i_j} \in \{u_{i_j} \leq t\} \subseteq \{u_{i_j} \leq u_{\min} + 1\}.$$

By Lemma 1, we know that $u_{i_j} \xrightarrow{epi} u$ and therefore we can apply the previous argument to obtain that $\{u_{i_j} \leq u_{\min} + 1\} \to \{u \leq u_{\min} + 1\}$, which shows that the sequence x_{i_j} is bounded. Hence, there exists a convergent subsequence $x_{i_{j_k}}$ with limit $x \in \mathbb{R}^n$. Applying Lemma 1 again, we obtain that $u_{i_{j_k}}$ is epi-convergent to u and therefore

$$u(x) \leq \liminf_{k \to \infty} u_{i_{j_k}}(x_{i_{j_k}}) \leq t.$$

This is a contradiction. Hence $\{u_k \leq t\}$ must be empty eventually. $\square$

1.2 Conjugate functions and the cone property

We require a uniform lower bound for an epi-convergent sequence of functions from $\operatorname{Conv}(\mathbb{R}^n)$. This is established by showing that all epigraphs are contained in a suitable cone that is given by the function $a\,|x| + b$ with $a > 0$ and $b \in \mathbb{R}$. To establish this *uniform cone property* of an epi-convergent sequence, we use conjugate functions.

For a convex function $u : \mathbb{R}^n \to (-\infty, +\infty]$, its *conjugate function* $u^* : \mathbb{R}^n \to (-\infty, +\infty]$ is defined as

$$u^*(y) = \sup_{x \in \mathbb{R}^n} \big(\langle y, x\rangle - u(x)\big), \quad y \in \mathbb{R}^n.$$

Here $\langle y, x\rangle$ is the inner product of $x, y \in \mathbb{R}^n$. If u is a closed convex function, then also u^* is a closed convex function and $u^{**} = u$. Conjugation reverses inequalities, that is, if $u \leq v$, then $u^* \geq v^*$.

The *infimal convolution* of two closed convex functions $u_1, u_2 : \mathbb{R}^n \to (-\infty, +\infty]$ is defined by

$$(u_1 \,\Box\, u_2)(x) = \inf_{x=x_1+x_2} \big(u_1(x_1) + u_2(x_2)\big), \quad x \in \mathbb{R}^n.$$

This just corresponds to the Minkowski addition of the epigraphs of u_1 and u_2, that is

$$\operatorname{epi}(u_1 \,\Box\, u_2) = \operatorname{epi} u_1 + \operatorname{epi} u_2. \tag{8}$$

We remark that for two closed convex functions u_1, u_2 the infimal convolution $u_1 \,\Box\, u_2$ need not be closed, even when it is convex. If $u_1 \Box u_2 > -\infty$ pointwise, then

$$(u_1 \,\Box\, u_2)^* = u_1^* + u_2^*. \tag{9}$$

For $t > 0$ and a closed convex function u define the function u_t by

$$u_t(x) = t\, u\left(\tfrac{x}{t}\right).$$

For the convex conjugate of u_t, we have

$$u_t^*(y) = \sup_{x\in\mathbb{R}^n} \left(\langle y, x\rangle - t\, u\left(\tfrac{x}{t}\right)\right) = \sup_{x\in\mathbb{R}^n} \big(\langle y, tx\rangle - t\, u(x)\big) = t\, u^*(y) \tag{10}$$

(see, for example, [31], Section 1.6.2).

The next result shows a fundamental relationship between convex functions and their conjugates. It was first established by Wijsman (see [30, Theorem 11.34]).

Lemma 6. *If $u_k, u \in \operatorname{Conv}(\mathbb{R}^n)$, then*

$$u_k \xrightarrow{epi} u \quad \Longleftrightarrow \quad u_k^* \xrightarrow{epi} u^*.$$

The *cone property* was established in [9, Lemma 2.5] for functions in $\operatorname{Conv}(\mathbb{R}^n)$.

Lemma 7. *For $u \in \operatorname{Conv}(\mathbb{R}^n)$, there exist constants $a, b \in \mathbb{R}$ with $a > 0$ such that*

$$u(x) > a|x| + b$$

for every $x \in \mathbb{R}^n$.

Next, we extend this result to an epi-convergent sequence of functions in $\operatorname{Conv}(\mathbb{R}^n)$ and obtain a *uniform cone property.*

Lemma 8. *Let $u_k, u \in \operatorname{Conv}(\mathbb{R}^n)$. If $u_k \xrightarrow{epi} u$, then there exist constants $a, b \in \mathbb{R}$ with $a > 0$ such that*

$$u_k(x) > a\,|x| + b \quad \text{and} \quad u(x) > a\,|x| + b$$

for every $k \in \mathbb{N}$ and $x \in \mathbb{R}^n$.

Proof. By Lemma 7, there exist constants $c > 0$ and d such that

$$u(x) > c|x| + d = l(x).$$

Switching to conjugates gives $u^* < l^*$. Note that

$$l^*(y) = \Big(\sup_{x \in \mathbb{R}^n} \big(\langle y, x \rangle - c\,|x| \big) \Big) - d$$

and

$$\sup_{x \in \mathbb{R}^n} \big(\langle y, x \rangle - c\,|x| \big) = \begin{cases} 0 & \text{if } |y| \leq c \\ +\infty & \text{if } |y| > c. \end{cases}$$

Hence $l^* = \mathrm{I}_{cB} - d$, where cB is the closed centered ball with radius c. Set $a = c/2 > 0$. Hence aB is a compact subset of $\operatorname{int}\operatorname{dom} u^*$. Therefore, Lemma 6 and Lemma 2 imply that u_k^* converges uniformly to u^* on aB. Since $u^* < -d$ on aB, there exists a constant b such that $u_k^*(y) < -b$ for every $y \in aB$ and $k \in \mathbb{N}$ and therefore

$$u_k^* < \mathrm{I}_{aB} - b,$$

for every $k \in \mathbb{N}$. Consequently

$$u_k(x) > a|x| + b$$

for every $k \in \mathbb{N}$ and $x \in \mathbb{R}^n$. □

Note, that Lemma 5 and Lemma 8 are no longer true if $u \equiv +\infty$. For example, consider $u_k(x) = \mathrm{I}_{k\,r(B + k^2 x_0)}$ for some $r > 0$ and $x_0 \in \mathbb{R}^n \backslash \{0\}$. Then u_k epi-converges to u but every set $\{u_k \leq t\}$ for $t \geq 0$ is a ball of radius kr. In this case, the sublevel sets are not even bounded. Moreover, it is clear that there does not exist a uniform pointed cone that contains all the sets $\operatorname{epi} u_k$.

1.3 Piecewise affine functions

A polyhedron is the intersection of finitely many closed halfspaces. A function $u \in \mathrm{Conv}(\mathbb{R}^n)$ is called *piecewise affine*, if there exist finitely many n-dimensional convex polyhedra $C_1, \ldots, C_m$ with pairwise disjoint interiors such that $\bigcup_{i=1}^m C_i = \mathbb{R}^n$ and the restriction of u to each C_i is an affine function. The set of piecewise affine and convex functions will be denoted by $\mathrm{Conv}_{\text{p.a.}}(\mathbb{R}^n)$. We call $u \in \mathrm{Conv}(\mathbb{R}^n)$ a *finite element* of $\mathrm{Conv}(\mathbb{R}^n)$ if $u(x) < +\infty$ for every $x \in \mathbb{R}$. Note that piecewise affine and convex functions are finite elements of $\mathrm{Conv}(\mathbb{R}^n)$. We want to show that $\mathrm{Conv}_{\text{p.a.}}(\mathbb{R}^n)$ is dense in $\mathrm{Conv}(\mathbb{R}^n)$ and use the Moreau-Yosida approximation in our proof. That $\mathrm{Conv}_{\text{p.a.}}(\mathbb{R}^n)$ is dense in $\mathrm{Conv}(\mathbb{R}^n)$ can also be deduced from more general results (see, [4, Corollary 3.42]).

Let $u \in \mathrm{Conv}(\mathbb{R}^n)$ and $t > 0$. Set $q(x) = \frac{1}{2}|x|^2$ and recall that $q_t(x) = \frac{t}{2}\, q(\frac{x}{t})$. The *Moreau-Yosida approximation* of u is defined as

$$e_t u = u \,\square\, q_t,$$

or equivalently

$$e_t u(x) = \inf_{y \in \mathbb{R}^n} \big(u(y) + \tfrac{1}{2t}|x-y|^2\big) = \inf_{x_1+x_2=x} \big(u(x_1) + \tfrac{1}{2t}|x_2|^2\big).$$

See, for example, [30, Chapter 1, Section G]. We require the following simple properties of the Moreau-Yosida approximation.

Lemma 9. *For $u \in \mathrm{Conv}(\mathbb{R}^n)$, the Moreau-Yosida approximation $e_t u$ is a finite element of $\mathrm{Conv}(\mathbb{R}^n)$ for every $t > 0$. Moreover, $e_t u(x) \le u(x)$ for $x \in \mathbb{R}^n$ and $t > 0$.*

Proof. Fix $t > 0$. Since

$$\inf_{x_1+x_2=x} \big(u(x_1) + \tfrac{1}{2t}|x_2|^2\big) \le u(x) + \tfrac{1}{2t}|0|^2$$

for $x \in \mathbb{R}^n$, we have $e_t u(x) \le u(x)$ for all $x \in \mathbb{R}^n$. Since u is proper, there exists $x_0 \in \mathbb{R}^n$ such that $u(x_0) < +\infty$. This shows that

$$e_t u(x) = \inf_{x_1+x_2=x} \big(u(x_1) + \tfrac{1}{2t}|x_2|^2\big) \le u(x_0) + \tfrac{1}{2t}|x-x_0|^2 < +\infty$$

for $x \in \mathbb{R}^n$, which shows that $e_t u$ is finite. Using (8) we obtain that

$$\operatorname{epi} e_t u = \operatorname{epi} u + \operatorname{epi} q_t.$$

It is therefore easy to see, that $e_t u$ is a convex function such that $\lim_{|x|\to+\infty} e_t u(x) = +\infty$. □

Lemma 10. *For every $u \in \mathrm{Conv}(\mathbb{R}^n)$, $\text{epi-lim}_{t\to 0^+} e_t u = u$.*

Proof. By Lemma 6, we have $e_t u \xrightarrow{epi} u$ if and only if $(e_t u)^* \xrightarrow{epi} u^*$. By the definition of e_t, (9) and (10), we have

$$(e_t u)^* = (u \,\square\, q_t)^* = u^* + t q^*.$$

Therefore, we need to show that $u^* + tq^* \xrightarrow{epi} u^*$. For $q(x) = \frac{1}{2}|x|^2$, we have $q = q^*$. Since epi-convergence is equivalent to pointwise convergent if the functions are finite, it follows that $\text{epi-lim}_{t\to 0^+} tq^* = 0$. It is now easy to see that $\text{epi-lim}_{t\to 0^+} \big(u^* + tq^*\big) = u^*$ and therefore $\text{epi-lim}_{t\to 0^+} (e_t u)^* = u^*$. □

Lemma 11. $\mathrm{Conv}_{\mathrm{p.a.}}(\mathbb{R}^n)$ *is dense in* $\mathrm{Conv}(\mathbb{R}^n)$.

Proof. By Lemma 3, epi-convergence coincides with pointwise convergence on finite functions in $\mathrm{Conv}(\mathbb{R}^n)$. Therefore, it is easy to see that $\mathrm{Conv}_{\mathrm{p.a.}}(\mathbb{R}^n)$ is epi-dense in the finite elements of $\mathrm{Conv}(\mathbb{R}^n)$. Now for arbitrary $u \in \mathrm{Conv}(\mathbb{R}^n)$ it follows from Lemma 9 that $e_t u$ is a finite element of $\mathrm{Conv}(\mathbb{R}^n)$. Since Lemma 10 gives that $\text{epi-lim}_{t\to 0^+} e_t u = u$, the finite elements of $\mathrm{Conv}(\mathbb{R}^n)$ are a dense subset of $\mathrm{Conv}(\mathbb{R}^n)$. Since denseness is transitive, the piecewise affine functions are a dense subset of $\mathrm{Conv}(\mathbb{R}^n)$. □

2 Valuations on Convex Functions

The functionals that appear in the theorem are discussed. It is shown that they are continuous, $\mathrm{SL}(n)$ and translation invariant valuations on $\mathrm{Conv}(\mathbb{R}^n)$.

Lemma 12. *For* $\zeta \in C(\mathbb{R})$, *the map*

$$u \mapsto \zeta\big(\min_{x\in\mathbb{R}^n} u(x)\big) \tag{11}$$

is a continuous, $\mathrm{SL}(n)$ *and translation invariant valuation on* $\mathrm{Conv}(\mathbb{R}^n)$.

Proof. Let $u \in \mathrm{Conv}(\mathbb{R}^n)$. Since

$$\min_{x\in\mathbb{R}^n} u(x) = \min_{x\in\mathbb{R}^n} u(\tau x) = \min_{x\in\mathbb{R}^n} u(\phi^{-1}x),$$

for every $\phi \in \mathrm{SL}(n)$ and translation $\tau : \mathbb{R}^n \to \mathbb{R}^n$, (11) defines an $\mathrm{SL}(n)$ and translation invariant map. If $u, v \in \mathrm{Conv}(\mathbb{R}^n)$ are such that $u \wedge v \in \mathrm{Conv}(\mathbb{R}^n)$, then clearly

$$\min_{x\in\mathbb{R}^n} (u \wedge v)(x) = \min\{\min_{x\in\mathbb{R}^n} u(x), \min_{x\in\mathbb{R}^n} v(x)\}.$$

By [8, Lemma 3.7] we have

$$\min_{x\in\mathbb{R}^n} (u \vee v)(x) = \max\{\min_{x\in\mathbb{R}^n} u(x), \min_{x\in\mathbb{R}^n} v(x)\}.$$

Hence, a function $\zeta \in C(\mathbb{R})$ composed with the minimum of a function $u \in \mathrm{Conv}(\mathbb{R}^n)$ defines a valuation on $\mathrm{Conv}(\mathbb{R}^n)$. The continuity of (11) follows from Lemma 5. □

Let $\zeta \in C(\mathbb{R})$ be non-negative. For $u \in \mathrm{Conv}(\mathbb{R}^n)$, define

$$\mathrm{Z}_\zeta(u) = \int_{\mathrm{dom}\, u} \zeta(u(x))\,\mathrm{d}x.$$

We want to investigate conditions on ζ such that Z_ζ defines a continuous valuation on $\mathrm{Conv}(\mathbb{R}^n)$.

It is easy to see, that in order for $\mathrm{Z}_\zeta(u)$ to be finite for every $u \in \mathrm{Conv}(\mathbb{R}^n)$, it is necessary for ζ to have finite $(n-1)$-st moment. Indeed, if $u(x) = |x|$, then

$$\mathrm{Z}_\zeta(u) = \int_{\mathbb{R}^n} \zeta(|x|)\,\mathrm{d}x = n\, v_n \int_0^{+\infty} t^{n-1}\zeta(t)\,\mathrm{d}t, \tag{12}$$

where v_n is the volume of the n-dimensional unit ball. We will see in Lemma 14, that this condition is also sufficient. For this, we require the following result.

Lemma 13. *Let u_k be a sequence in* $\operatorname{Conv}(\mathbb{R}^n)$ *with epi-limit* $u \in \operatorname{Conv}(\mathbb{R}^n)$. *If* $\zeta \in C(\mathbb{R})$ *is non-negative with finite* $(n-1)$*-st moment, then, for every* $\varepsilon > 0$, *there exist* $t_0 \in \mathbb{R}$ *and* $k_0 \in \mathbb{N}$ *such that*

$$\int_{\operatorname{dom} u \cap \{u>t\}} \zeta(u(x))\,\mathrm{d}x < \varepsilon \quad \text{and} \quad \int_{\operatorname{dom} u_k \cap \{u_k>t\}} \zeta(u_k(x))\,\mathrm{d}x < \varepsilon$$

for every $t \geq t_0$ *and* $k \geq k_0$.

Proof. Without loss of generality, let $\min_{x\in\mathbb{R}^n} u(x) = u(0)$. By the definition of epi-convergence, there exists a sequence x_k in $\mathbb{R}^n$ such that $x_k \to 0$ and $u_k(x_k) \to u(0)$. Therefore, there exists $k_0 \in \mathbb{N}$ such that $|x_k| < 1$ and $u_k(x_k) < u(0) + 1$ for every $k \geq k_0$. By Lemma 8, there exist constants $a > 0$ and $\bar{b} \in \mathbb{R}$, such that

$$u(x), u_k(x) > a|x| + \bar{b},$$

for every $x \in \mathbb{R}^n$ and $k \in \mathbb{N}$. Setting $\widetilde{u}_k(x) = u_k(x - x_k)$, we have

$$\widetilde{u}_k(x) > a|x - x_k| + \bar{b} \geq a|x| - a|x_k| + \bar{b} \geq a|x| + (\bar{b} - a),$$

for every $k \geq k_0$. Hence, with $b = \bar{b} - a$, we have

$$u(x), \widetilde{u}_k(x) > a|x| + b, \tag{13}$$

for every $x \in \mathbb{R}^n$ and $k \geq k_0$.

We write $x = r\omega$ with $r \in [0, +\infty)$ and $\omega \in \mathbb{S}^{n-1}$. For $u(r\omega) \geq 1$, we obtain from (13) that

$$r^{n-1} < \left(\frac{u(r\omega)}{a} - \frac{b}{a}\right)^{n-1} \leq c\, u(r\omega)^{n-1}, \quad r^{n-1} < c\, \widetilde{u}_k(r\omega)^{n-1}, \tag{14}$$

for every $r \in [0, +\infty)$, $\omega \in \mathbb{S}^{n-1}$ and $k \geq k_0$, where c only depends on a, b and the dimension n. Now choose $\bar{t}_0 \geq \max\{1, 2(u(0)+1) - b\}$. Then for all $t \geq \bar{t}_0$

$$\frac{t - u(0)}{t - b} \geq \frac{1}{2}, \quad \frac{t - (u(0)+1)}{t - b} \geq \frac{1}{2}. \tag{15}$$

For $\omega \in \mathbb{S}^{n-1}$, let $v_\omega(r) = u(r\omega)$. The function v_ω is non-decreasing and convex on $[0, +\infty)$. So, in particular, the left and right derivatives, $v'_{\omega,l}, v'_{\omega,r}$ of v_ω exist and for the subgradient $\partial v_\omega(r) = [v'_{\omega,l}, v'_{\omega,r}]$, it follows from $r < \bar{r}$ that $\eta \leq \bar{\eta}$ for $\eta \in \partial v_\omega(r)$ and $\bar{\eta} \in \partial v_\omega(\bar{r})$.

For $t \geq \bar{t}_0$, set

$$D_\omega(t) = \{r \in [0, +\infty) \,:\, t < u(r\omega) < +\infty\}.$$

For every $\omega \in \mathbb{S}^{n-1}$, the set $D_\omega(t)$ is either empty or there exists

$$r_\omega(t) = \inf D_\omega(t) \leq \tfrac{t-b}{a} \tag{16}$$

and $v_\omega(r_\omega(t)) = t$. Therefore, if $D_\omega(t)$ is non-empty, we have

$$t - u(0) \leq \xi\, r_\omega(t)$$

for $\xi \in \partial v_\omega(r_\omega(t))$. Hence, it follows from (16) and (15) that

$$\vartheta \geq \xi \geq \frac{t-u(0)}{r_\omega(t)} \geq \frac{a(t-u(0))}{t-b} \geq \frac{a}{2}, \tag{17}$$

for all $r \in D_\omega(t)$, $\vartheta \in \partial v_\omega(r)$ and $\xi \in \partial v_\omega(r_\omega(t))$. Similarly, setting $\widetilde{v}_{k,\omega}(r) = \widetilde{u}_k(r\omega)$ and

$$\widetilde{D}_{k,\omega}(t) = \{r \in [0,+\infty) \,:\, t < u_k(r\omega) < +\infty\},$$

it is easy to see that $\widetilde{v}_{k,\omega}$ is convex on $[0,+\infty)$ and monotone increasing on $\widetilde{D}_{k,\omega}(t)$ for all $k \geq k_0$. By the choice of $\bar{t}_0$ and (15), for $t \geq \bar{t}_0$

$$\vartheta \geq \frac{a}{2},$$

for all $r \in D_{k,\omega}(t)$, $k \geq k_0$ and $\vartheta \in \partial \widetilde{v}_{k,\omega}(r)$. Recall, that as a convex function v_ω is locally Lipschitz and differentiable almost everywhere on the interior of its domain. Using polar coordinates, (14) and the substitution $v_\omega(r) = s$, we obtain from (17) that

$$\begin{aligned}
\int_{\operatorname{dom} u \cap \{u>t\}} \zeta(u(x)) \,\mathrm{d}x &= \int_{\mathbb{S}^{n-1}} \int_{D_\omega(t)} r^{n-1} \zeta(v_\omega(r)) \,\mathrm{d}r \,\mathrm{d}\omega \\
&\leq c \int_{\mathbb{S}^{n-1}} \int_{D_\omega(t)} v_\omega(r)^{n-1} \zeta(v_\omega(r)) \,\mathrm{d}r \,\mathrm{d}\omega \\
&\leq \frac{2\,n\,v_n\,c}{a} \int_t^{+\infty} s^{n-1} \zeta(s) \,\mathrm{d}s
\end{aligned} \tag{18}$$

for every $t \geq \bar{t}_0$. In the same way,

$$\int_{\operatorname{dom} u_k \cap \{u_k>t\}} \zeta(u_k(x)) \,\mathrm{d}x = \int_{\operatorname{dom} \widetilde{u}_k \cap \{\widetilde{u}_k>t\}} \zeta(\widetilde{u}_k(x)) \,\mathrm{d}x \leq \frac{2\,n\,v_n\,c}{a} \int_t^{+\infty} s^{n-1} \zeta(s) \,\mathrm{d}s,$$

for every $t \geq \bar{t}_0$ and $k \geq k_0$ with the same constant c as in (18). The statement now follows, since ζ is non-negative and has finite $(n-1)$-st moment. □

Lemma 14. *Let $\zeta \in C(\mathbb{R})$ be non-negative. Then $\mathrm{Z}_\zeta(u) < +\infty$ for every $u \in \mathrm{Conv}(\mathbb{R}^n)$ if and only if ζ has finite $(n-1)$-st moment.*

Proof. As already pointed out in (12), it is necessary for ζ to have finite $(n-1)$-st moment in order for Z_ζ to be finite.

Now let $u \in \mathrm{Conv}(\mathbb{R}^n)$ be arbitrary, let ζ have finite $(n-1)$-st moment and let $u_{\min} = \min_{x \in \mathbb{R}^n} u(x)$. By Lemma 13, there exists $t \in \mathbb{R}$ such that

$$\int_{\operatorname{dom} u \cap \{u>t\}} \zeta(u(x)) \,\mathrm{d}x \leq 1.$$

It follows that

$$\begin{aligned}
\mathrm{Z}_\zeta(u) &= \int_{\operatorname{dom} u} \zeta(u(x))\,\mathrm{d}x \\
&= \int_{\{u\le t\}} \zeta(u(x))\,\mathrm{d}x + \int_{\operatorname{dom} u\cap\{u>t\}} \zeta(u(x))\,\mathrm{d}x \\
&\le \max\nolimits_{s\in[u_{\min},t]} \zeta(s)\, V_n(\{u\le t\}) + 1
\end{aligned}$$

and hence $\mathrm{Z}_\zeta(u) < \infty$. $\square$

Lemma 15. *For $\zeta \in C(\mathbb{R})$ non-negative and with finite $(n-1)$-st moment, the functional Z_ζ is continuous on $\mathrm{Conv}(\mathbb{R}^n)$.*

Proof. Let $u \in \mathrm{Conv}(\mathbb{R}^n)$ and let u_k be a sequence in $\mathrm{Conv}(\mathbb{R}^n)$ such that $u_k \xrightarrow{epi} u$. Set $u_{\min} = \min_{x\in\mathbb{R}^n} u(x)$. By Lemma 13, it is enough to show that

$$\int_{\{u_k\le t\}} \zeta(u_k(x))\,\mathrm{d}x \to \int_{\{u\le t\}} \zeta(u(x))\,\mathrm{d}x$$

for every fixed $t > u_{\min}$. Lemma 5 implies that $\{u_k \le t\} \to \{u \le t\}$ in the Hausdorff metric. By Lemma 8, there exists $b \in \mathbb{R}$ such that $u(x), u_k(x) > b$ for $x \in \mathbb{R}^n$ and $k \in \mathbb{N}$. Set $c = \max_{s\in[b,t]} \zeta(s) \ge 0$. We distinguish the following cases.

First, let $\dim(\operatorname{dom} u) < n$. In this case $V_n(\{u \le t\}) = 0$ and since volume is continuous on convex sets, $V_n(\{u_k \le t\}) \to 0$. Hence,

$$0 \le \int_{\{u_k\le t\}} \zeta(u_k(x))\,\mathrm{d}x \le c\, V_n(\{u_k \le t\}) \to 0.$$

Second, let $\dim(\operatorname{dom} u) = n$. In this case, $\{u \le t\}$ is a set in $\mathcal{K}^n$ with non-empty interior. Therefore, for $\varepsilon > 0$ there exist $k_0 \in \mathbb{N}$ and $C \in \mathcal{K}^n$ such that for every $k \ge k_0$ the following hold:

$$\begin{gathered}
C \subset \operatorname{int}(\{u \le t\}) \cap \{u_k \le t\}, \\
V_n(\{u \le t\} \cap C^c) \le \frac{\varepsilon}{3c}, \\
V_n(\{u_k \le t\} \cap C^c) \le \frac{\varepsilon}{3c},
\end{gathered}$$

where C^c is the complement of C. Note, that $u(x), u_k(x) \in [b,t]$ for $x \in C$ and $k \ge k_0$. Since $C \subset \operatorname{int}\operatorname{dom} u$, Lemma 2 implies that u_k converges to u uniformly on C. Since ζ is continuous, the restriction of ζ to $[b,t]$ is uniformly continuous. Hence, $\zeta \circ u_k$ converges uniformly to $\zeta \circ u$ on C. Therefore, there exists $k_1 \ge k_0$ such that

$$|\zeta(u(x)) - \zeta(u_k(x))| \le \frac{\varepsilon}{3V_n(C)},$$

for all $x \in C$ and $k \geq k_1$. This gives

$$\begin{aligned}
&\Big|\int_{\{u\le t\}} \zeta(u(x))\,\mathrm{d}x - \int_{\{u_k\le t\}} \zeta(u_k(x))\,\mathrm{d}x\Big| \\
&\le \int_C |\zeta(u(x)) - \zeta(u_k(x))|\,\mathrm{d}x + \int_{\{u\le t\}\cap C^c} \zeta(u(x))\,\mathrm{d}x + \int_{\{u_k\le t\}\cap C^c} \zeta(u_k(x))\,\mathrm{d}x \\
&\le V_n(C)\frac{\varepsilon}{3V_n(C)} + c\frac{\varepsilon}{3c} + c\frac{\varepsilon}{3c} = \varepsilon,
\end{aligned}$$

for $k \geq k_1$. The statement now follows, since $\varepsilon > 0$ was arbitrary. □

Lemma 16. *For $\zeta \in C(\mathbb{R})$ non-negative and with finite $(n-1)$-st moment, the functional Z_ζ is an $\mathrm{SL}(n)$ and translation invariant valuation on $\mathrm{Conv}(\mathbb{R}^n)$.*

Proof. It is easy to see that Z_ζ is $\mathrm{SL}(n)$ and translation invariant. It remains to show the valuation property. Let $u, v \in \mathrm{Conv}(\mathbb{R}^n)$ be such that $u \wedge v \in \mathrm{Conv}(\mathbb{R}^n)$. We have

$$\mathrm{Z}_\zeta(u\wedge v) = \int_{\mathrm{dom}\, v\cap\{v<u\}} \zeta(v(x))\,\mathrm{d}x + \int_{\mathrm{dom}\, v\cap\{u=v\}} \zeta(v(x))\,\mathrm{d}x + \int_{\mathrm{dom}\, u\cap\{u<v\}} \zeta(u(x))\,\mathrm{d}x,$$

$$\mathrm{Z}_\zeta(u\vee v) = \int_{\mathrm{dom}\, u\cap\{v<u\}} \zeta(u(x))\,\mathrm{d}x + \int_{\mathrm{dom}\, u\cap\{u=v\}} \zeta(u(x))\,\mathrm{d}x + \int_{\mathrm{dom}\, v\cap\{u<v\}} \zeta(v(x))\,\mathrm{d}x.$$

Hence,

$$\mathrm{Z}_\zeta(u\wedge v) + \mathrm{Z}_\zeta(u\vee v) = \mathrm{Z}_\zeta(u) + \mathrm{Z}_\zeta(v)$$

and the valuation property is proved. □

3 Valuations on Cone and Indicator Functions

Let $\mathcal{K}_0^n$ be the set of compact convex sets which contain the origin. For $K \in \mathcal{K}_0^n$, we define the convex function $\ell_K : \mathbb{R}^n \to [0,\infty]$ via

$$\mathrm{epi}\,\ell_K = \mathrm{pos}(K \times \{1\}),$$

where pos denotes the positive hull. This means that the epigraph of ℓ_K is a cone with apex at the origin and $\{\ell_K \le t\} = t\,K$ for all $t \geq 0$. It is easy to see that ℓ_K is an element of $\mathrm{Conv}(\mathbb{R}^n)$ for all $K \in \mathcal{K}_0^n$. We have $\mathrm{dom}\,\ell_K = \mathbb{R}^n$ if and only if K contains the origin in its interior. If $P \in \mathcal{P}_0^n$ contains the origin in its interior, then $\ell_P \in \mathrm{Conv}_{\mathrm{p.a.}}(\mathbb{R}^n)$. For $K \in \mathcal{K}_0^n$ and $t \in \mathbb{R}$, we call the function $\ell_K + t$ a *cone function* and we call the function $\mathrm{I}_K + t$ an *indicator function*. Cone and indicator functions play a special role in our proof.

The next result shows that to classify continuous and translation invariant valuations on $\mathrm{Conv}(\mathbb{R}^n)$, it is enough to know the behavior of these valuations on cone functions. The main argument of the following lemma is due to [23, Lemma 8], where it was used for functions on Sobolev spaces.

Lemma 17. *Let $\langle A, +\rangle$ be a topological abelian semigroup with cancellation law and let* $\mathrm{Z}_1, \mathrm{Z}_2 : \mathrm{Conv}(\mathbb{R}^n) \to \langle A, +\rangle$ *be continuous, translation invariant valuations. If* $\mathrm{Z}_1(\ell_P + t) = \mathrm{Z}_2(\ell_P + t)$ *for every* $P \in \mathcal{P}_0^n$ *and* $t \in \mathbb{R}$*, then* $\mathrm{Z}_1 \equiv \mathrm{Z}_2$ *on* $\mathrm{Conv}(\mathbb{R}^n)$.

Proof. By Lemma 11 and the continuity of Z_1 and Z_2, it suffices to show that Z_1 and Z_2 coincide on $\mathrm{Conv}_{\text{p.a.}}(\mathbb{R}^n)$. So let $u \in \mathrm{Conv}_{\text{p.a.}}(\mathbb{R}^n)$ and set $U = \operatorname{epi} u$. Note, that U is a convex polyhedron in $\mathbb{R}^{n+1}$ and that none of the facet hyperplanes of U is parallel to the x_{n+1}-axis. Here, we say that a hyperplane H in $\mathbb{R}^{n+1}$ is a *facet hyperplane* of U if its intersection with the boundary of U has positive n-dimensional Hausdorff measure. Furthermore, we call U *singular* if U has n facet hyperplanes whose intersection contains a line parallel to $\{x_{n+1} = 0\}$. Since Z_1 and Z_2 are continuous, we can assume that U is not singular.

Since U is not singular and $u \in \mathrm{Conv}_{\text{p.a.}}(\mathbb{R}^n)$, there exists a unique vertex, $\bar{p}$ of U with smallest x_{n+1} coordinate. We use induction on the number m of facet hyperplanes of U that are not passing through $\bar{p}$. If $m = 0$, then there exist $P \in \mathcal{P}_0^n$ and $t \in \mathbb{R}$ such that u is a translate of $\ell_P + t$. Since Z_1 and Z_2 are translation invariant, it follows that $\mathrm{Z}_1(u) = \mathrm{Z}_2(u)$.

Now let U have $m > 0$ facet hyperplanes that are not passing through $\bar{p}$ and assume that Z_1 and Z_2 coincide for all functions with at most $(m-1)$ such facet hyperplanes. Let $p_0 = (x_0, u(x_0)) \in \mathbb{R}^{n+1}$ where $x_0 \in \mathbb{R}^n$ is a vertex of U with maximal x_{n+1}-coordinate and let $H_1, \ldots, H_j$ be the facet hyperplanes of U through p_0 such that the corresponding facets of U have infinite n-dimensional volume. Note, that $H_1, \ldots, H_j$ do not contain $\bar{p}$ and therefore there is at least one such hyperplane. Define $\bar{U}$ as the polyhedron bounded by the intersection of all facet hyperplanes of U with the exception of $H_1, \ldots, H_j$. Since U is not singular, there exists a function $\bar{u} \in \mathrm{Conv}_{\text{p.a.}}(\mathbb{R}^n)$ with $\operatorname{dom} \bar{u} = \mathbb{R}^n$ such that $\bar{U} = \operatorname{epi} \bar{u}$. Note, that $\bar{U}$ has at most $(m-1)$ facet hyperplanes not containing $\bar{p}$. Hence, by the induction hypothesis

$$\mathrm{Z}_1(\bar{u}) = \mathrm{Z}_2(\bar{u}).$$

Let $\overline{H}_1, \ldots, \overline{H}_i$ be the facet hyperplanes of $\bar{U}$ that contain p_0 such that the corresponding facets of $\bar{U}$ have infinite n-dimensional volume. Choose suitable hyperplanes $\overline{H}_{i+1}, \ldots, \overline{H}_k$ not parallel to the x_{n+1}-axis and containing p_0 so that the hyperplanes $\overline{H}_1, \ldots, \overline{H}_k$ bound a polyhedral cone with apex p_0 that is contained in $\bar{U}$, has $\overline{H}_1, \ldots, \overline{H}_i$ among its facet hyperplanes and contains $\{x_0\} \times [u(x_0), +\infty)$. Define ℓ as the piecewise affine function determined by this polyhedral cone. Notice, that ℓ is a translate of $\ell_P + u(x_0)$, where $P \in \mathcal{P}_0^n$ is the projection onto the first n coordinates of the intersection of the polyhedral cone with $\{x_{n+1} = u(x_0) + 1\}$. Hence, Z_1 and Z_2 coincide on ℓ. Set $\bar{\ell} = u \vee \ell$. The epigraph of $\bar{\ell}$ is again a polyhedral cone with apex p_0. Hence $\bar{\ell}$ is a translate of $\ell_{\bar{P}} + u(x_0)$ with $\bar{P} \in \mathcal{P}_0^n$ since it is bounded by hyperplanes containing p_0 that are not parallel to the x_{n+1}-axis. Therefore, Z_1 and Z_2 also coincide on $\bar{\ell}$. We now have

$$u \wedge \ell = \bar{u}, \quad u \vee \ell = \bar{\ell}.$$

From the valuation property of Z_i, $i = 1, 2$, we obtain

$$\mathrm{Z}_1(u) + \mathrm{Z}_1(\ell) = \mathrm{Z}_1(\bar{u}) + \mathrm{Z}_1(\bar{\ell}) = \mathrm{Z}_2(\bar{u}) + \mathrm{Z}_2(\bar{\ell}) = \mathrm{Z}_2(u) + \mathrm{Z}_2(\ell),$$

which completes the proof. □

Next, we study the behavior of a continuous and $\mathrm{SL}(n)$ invariant valuation on cone and indicator functions.

Lemma 18. *If* $\mathrm{Z}:\mathrm{Conv}(\mathbb{R}^n)\to\mathbb{R}$ *is a continuous and* $\mathrm{SL}(n)$ *invariant valuation, then there exist continuous functions* $\psi_0,\psi_n,\zeta_0,\zeta_n:\mathbb{R}\to\mathbb{R}$ *such that*

$$\begin{aligned}\mathrm{Z}(\ell_P+t) &= \psi_0(t)+\psi_n(t)V_n(P),\\ \mathrm{Z}(\mathrm{I}_P+t) &= \zeta_0(t)+\zeta_n(t)V_n(P)\end{aligned}$$

for every $P\in\mathcal{P}_0^n$ *and* $t\in\mathbb{R}$.

Proof. For $t\in\mathbb{R}$, define $\mathrm{Z}_t:\mathcal{P}_0^n\to\mathbb{R}$ as

$$\mathrm{Z}_t(P)=\mathrm{Z}(\ell_P+t).$$

It is easy to see that Z_t defines a continuous, $\mathrm{SL}(n)$ invariant valuation on $\mathcal{P}_0^n$ for every $t\in\mathbb{R}$. Therefore, by (4), for every $t\in\mathbb{R}$ there exist constants $c_{0,t},c_{n,t}\in\mathbb{R}$ such that

$$\mathrm{Z}(\ell_P+t)=\mathrm{Z}_t(P)=c_{0,t}+c_{n,t}V_n(P),$$

for every $P\in\mathcal{P}_0^n$. This defines two functions $\psi_0(t)=c_{0,t}$ and $\psi_n(t)=c_{n,t}$. Taking $P\in\mathcal{P}_0^n$ with $\dim P<n$, we have $V_n(P)=0$. By the continuity of Z,

$$t\mapsto \mathrm{Z}(\ell_P+t)=\psi_0(t)$$

is continuous, which implies that ψ_0 is a continuous function. Similarly, taking $Q\in\mathcal{P}_0^n$ with $V_n(Q)>0$, we see that

$$t\mapsto\psi_n(t)=\frac{\mathrm{Z}(\ell_Q+t)-\psi_0(t)}{V_n(Q)},$$

can be expressed as the difference of two continuous functions and is therefore continuous itself. Using $P\mapsto\mathrm{Z}(\mathrm{I}_P+t)$ we get the corresponding results for the functions ζ_0 and ζ_n. □

For a continuous and $\mathrm{SL}(n)$ invariant valuation $\mathrm{Z}:\mathrm{Conv}(\mathbb{R}^n)\to\mathbb{R}$, we call the functions ψ_0 and ψ_n from Lemma 18 the *cone growth functions* of Z. The functions ζ_0 and ζ_n are its *indicator growth functions*. By Lemma 17, we immediately get the following result.

Lemma 19. *Every continuous,* $\mathrm{SL}(n)$ *and translation invariant valuation* $\mathrm{Z}:\mathrm{Conv}(\mathbb{R}^n)\to\mathbb{R}$ *is uniquely determined by its cone growth functions.*

In order to classify valuations, we want to determine how the cone growth functions and the indicator growth functions are related.

Lemma 20. *For* $k\geq 1$, *let* $\mathrm{Z}:\mathrm{Conv}(\mathbb{R}^k)\to\mathbb{R}$ *be a continuous, translation invariant valuation and let* $\psi\in C(\mathbb{R})$. *If*

$$\mathrm{Z}(\ell_P+t)=\psi(t)V_k(P)\tag{19}$$

for every $P\in\mathcal{P}_0^k$ *and* $t\in\mathbb{R}$, *then*

$$\mathrm{Z}(\mathrm{I}_{[0,1]^k}+t)=\frac{(-1)^k}{k!}\frac{\mathrm{d}^k}{\mathrm{d}t^k}\psi(t)$$

for every $t\in\mathbb{R}$. *In particular,* ψ *is* k*-times differentiable.*

Proof. To explain the idea of the proof, we first consider the case $k = 1$. For $h > 0$, let $u_h = \ell_{[0,1/h]}$, that is, $u^h(x) = +\infty$ for $x < 0$ and $u^h(x) = h\,x$ for $x \geq 0$. Define $v^h : \mathbb{R} \to [0, +\infty]$ by $v^h = u^h + \mathrm{I}_{[0,1]}$. Since Z is a translation invariant valuation and by (19), we obtain

$$\mathrm{Z}(v^h + t) = \mathrm{Z}(u^h + t) - \mathrm{Z}(u^h + h + t) = \frac{1}{h}\Big(\psi(t) - \psi(t+h)\Big)$$

for $t \in \mathbb{R}$. As $h \to 0$, the epi-limit of $v^h + t$ is $\mathrm{I}_{[0,1]} + t$. Since Z is continuous, we thus obtain

$$\mathrm{Z}(\mathrm{I}_{[0,1]} + t) = \lim_{h \to 0^+} \frac{1}{h}\Big(\psi(t) - \psi(t+h)\Big)$$

for $t \in \mathbb{R}$. Hence ψ is differentiable from the right at every $t \in \mathbb{R}$. Since $v^h + t - h \xrightarrow{epi} \mathrm{I}_{[0,1]} + t$ as $h \to 0$, we also obtain

$$\mathrm{Z}(\mathrm{I}_{[0,1]} + t) = \lim_{h \to 0^+} \big(\mathrm{Z}(u^h + t - h) - \mathrm{Z}(u^h + t)\big) = \lim_{h \to 0^+} \frac{1}{h}\Big(\psi(t-h) - \psi(t)\Big).$$

Hence ψ is also differentiable from the left at every $t \in \mathbb{R}$ and $\mathrm{Z}(\mathrm{I}_{[0,1]} + t) = -\psi'(t)$. This concludes the proof for $k = 1$.

Next, let $\{e_1, \dots, e_k\}$ denote the standard basis of $\mathbb{R}^k$ and set $e_0 = 0$. For $h = (h_1, \dots, h_k)$ with $0 < h_1 \leq \dots \leq h_k$ and $0 \leq i < k$, define the function u_i^h through its sublevel sets as

$$\{u_i^h < 0\} = \emptyset, \quad \{u_i^h \leq s\} = [0, e_0] + \dots + [0, e_i] + \operatorname{conv}\{0, s\, e_{i+1}/h_{i+1}, \dots, s\, e_k/h_k\},$$

for every $s \geq 0$. Let $u_k^h = \mathrm{I}_{[0,1]^k}$. Note, that u_i^h does not depend on h_j for $0 \leq j \leq i$. We use induction on i to show that $u_i^h \in \operatorname{Conv}(\mathbb{R}^k)$ and that

$$\mathrm{Z}(u_i^h + t) = \frac{(-1)^i}{k!\, h_{i+1} \cdots h_k} \psi^{(i)}(t),$$

for every $t \in \mathbb{R}$ and $0 \leq i \leq k$, where $\psi^{(i)}(t) = \frac{\mathrm{d}^i}{\mathrm{d}t^i}\psi(t)$.

For $i = 0$, set $P_h = \operatorname{conv}\{0, e_1/h_1, \dots, e_k/h_k\} \in \mathcal{P}_0^k$ and note that $u_0^h = \ell_{P_h} \in \operatorname{Conv}(\mathbb{R}^k)$. Hence, by the assumption on Z, we have

$$\mathrm{Z}(u_0^h + t) = \mathrm{Z}(\ell_{P_h} + t) = \psi(t) V_k(P_h) = \frac{1}{k!\, h_1 \cdots h_k} \psi(t).$$

Now assume that the statement holds true for $i \geq 0$. Define the function v_{i+1}^h by

$$\{v_{i+1}^h \leq s\} = \{u_i^h \leq s\} \cap \{x_{i+1} \leq 1\},$$

for every $s \in \mathbb{R}$. Since $\operatorname{epi} v_{i+1}^h = \operatorname{epi} u_i^h \cap \{x_{i+1} \leq 1\}$, it is easy to see that $v_{i+1}^h \in \operatorname{Conv}(\mathbb{R}^k)$. As $h_{i+1} \to 0$, we have epi-convergence of v_{i+1}^h to u_{i+1}^h. Lemma 2 implies that u_{i+1}^h is a convex function and hence $u_{i+1}^h \in \operatorname{Conv}(\mathbb{R}^k)$. Now, let τ_{i+1} be the translation $x \mapsto x + e_{i+1}$. Note that

$$\{v_{i+1}^h \leq s\} \cup \{(u_i^h \circ \tau_{i+1}^{-1} + h_{i+1}) \leq s\} = \{u_i^h \leq s\},$$

$$\{v_{i+1}^h \le s\} \cap \{(u_i^h \circ \tau_{i+1}^{-1} + h_{i+1}) \le s\} \subset \{x_{i+1} = 1\},$$

for every $s \in \mathbb{R}$. Since Z is a continuous, translation invariant valuation and $\mathrm{Z}(\ell_P + t) = 0$ for $P \in \mathcal{P}_0^k$ with $\dim(P) < k$, Lemma 17 and its proof imply that Z vanishes on all functions $u \in \mathrm{Conv}(\mathbb{R}^k)$ with $\operatorname{dom} u \subset H$, where H is a hyperplane in $\mathbb{R}^k$. Hence,

$$\mathrm{Z}(v_{i+1}^h \vee (u_i^h \circ \tau_{i+1}^{-1} + h_{i+1})) = 0.$$

Thus, by the valuation property

$$\mathrm{Z}(u_i^h + t) = \mathrm{Z}((v_{i+1}^h + t) \wedge (u_i^h \circ \tau_{i+1}^{-1} + h_{i+1} + t)) = \mathrm{Z}(v_{i+1}^h + t) + \mathrm{Z}(u_i^h \circ \tau_{i+1}^{-1} + h_{i+1} + t).$$

Using the induction assumption and the translation invariance of Z, we obtain

$$\mathrm{Z}(v_{i+1}^h + t) = \frac{(-1)^{i+1}}{k!\, h_{i+2} \cdots h_k} \frac{\psi^{(i)}(t + h_{i+1}) - \psi^{(i)}(t)}{h_{i+1}}.$$

As $h_{i+1} \to 0$, the continuity of Z shows that

$$\mathrm{Z}(u_{i+1}^h + t) = \frac{(-1)^{i+1}}{k!\, h_{i+2} \cdots h_k} \lim_{h_{i+1} \to 0^+} \frac{\psi^{(i)}(t + h_{i+1}) - \psi^{(i)}(t)}{h_{i+1}}.$$

Hence $\psi^{(i)}$ is differentiable from the right. Similarly, we have $v_{i+1}^h + t - h_{i+1} \stackrel{epi}{\longrightarrow} u_{i+1}^h$ as $h_{i+1} \to 0$ and thus

$$\mathrm{Z}(u_{i+1}^h + t) = \lim_{h_{i+1} \to 0^+} \mathrm{Z}(v_{i+1}^h + t - h_{i+1}) = \frac{(-1)^{i+1}}{k!\, h_{i+2} \cdots h_k} \lim_{h_{i+1} \to 0^+} \frac{\psi^{(i)}(t) - \psi^{(i)}(t - h_{i+1})}{h_{i+1}},$$

which shows that $\psi^{(i)}$ is differentiable from the left and therefore,

$$\mathrm{Z}(u_{i+1}^h + t) = \frac{(-1)^{i+1}}{k!\, h_{i+2} \cdots h_k} \psi^{(i+1)}(t),$$

for every $t \in \mathbb{R}$. ∎

Lemma 21. *If* $\mathrm{Z} : \mathrm{Conv}(\mathbb{R}^n) \to \mathbb{R}$ *is a continuous,* $\mathrm{SL}(n)$ *and translation invariant valuation, then the growth functions* ψ_0 *and* ζ_0 *coincide and*

$$\zeta_n(t) = \frac{(-1)^n}{n!} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \psi_n(t),$$

for every $t \in \mathbb{R}$.

Proof. Since $\ell_{\{0\}} = \mathrm{I}_{\{0\}}$, Lemma 18 implies that

$$\psi_0(t) = \mathrm{Z}(\ell_{\{0\}} + t) = \mathrm{Z}(\mathrm{I}_{\{0\}} + t) = \zeta_0(t),$$

for every $t \in \mathbb{R}$.

Now define $\bar{\mathrm{Z}} : \mathrm{Conv}(\mathbb{R}^n) \to \mathbb{R}$ as

$$\bar{\mathrm{Z}}(u) = \mathrm{Z}(u) - \zeta_0\big(\min_{x\in\mathbb{R}^n} u(x)\big).$$

By Lemma 12, the functional $\bar{\mathrm{Z}}$ is a continuous, $\mathrm{SL}(n)$ and translation invariant valuation that satisfies

$$\bar{\mathrm{Z}}(\ell_P + t) = \psi_n(t) V_n(P)$$

and

$$\bar{\mathrm{Z}}(\mathrm{I}_P + t) = \zeta_n(t) V_n(P),$$

for every $P \in \mathcal{P}_0^n$ and $t \in \mathbb{R}$. Hence, by Lemma 20,

$$\zeta_n(t) = \zeta_n(t) V_n([0,1]^n) = \bar{\mathrm{Z}}(\mathrm{I}_{[0,1]^n} + t) = \frac{(-1)^n}{n!} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \psi_n(t),$$

for every $t \in \mathbb{R}$. □

Lemma 22. *If* $\mathrm{Z} : \mathrm{Conv}(\mathbb{R}^n) \to \mathbb{R}$ *is a continuous,* $\mathrm{SL}(n)$ *and translation invariant valuation, then its cone growth function* ψ_n *satisfies*

$$\lim_{t\to\infty} \psi_n(t) = 0.$$

Proof. Let $\{e_1, e_2, \dots, e_n\}$ be the standard basis of $\mathbb{R}^n$ and let

$$P = \mathrm{conv}\{0, \tfrac{e_1+e_2}{2}, e_2, e_3, \dots, e_n\}, \qquad Q = \mathrm{conv}\{0, e_2, e_3, \dots, e_n\}.$$

For $s > 0$, define $u_s \in \mathrm{Conv}(\mathbb{R}^n)$ by its epigraph as $\mathrm{epi}\, u_s = \mathrm{epi}\, \ell_P \cap \{x_1 \le \frac{s}{2}\}$. Note, that for $t \ge 0$ this gives $\{u_s \le t\} = tP \cap \{x_1 \le \frac{s}{2}\}$. Let τ_s be the translation $x \mapsto x + s\,\frac{e_1+e_2}{2}$ and define $\ell_{P,s}(x) = \ell_P(x) \circ \tau_s^{-1} + s$ and similarly $\ell_{Q,s}(x) = \ell_Q(x) \circ \tau_s^{-1} + s$. We will now show that

$$u_s \wedge \ell_{P,s} = \ell_P \qquad u_s \vee \ell_{P,s} = \ell_{Q,s},$$

or equivalently

$$\mathrm{epi}\, u_s \cup \mathrm{epi}\, \ell_{P,s} = \mathrm{epi}\, \ell_P \qquad \mathrm{epi}\, u_s \cap \mathrm{epi}\, \ell_{P,s} = \mathrm{epi}\, \ell_{Q,s},$$

which is the same as

$$\{u_s \le t\} \cup \{\ell_{P,s} \le t\} = \{\ell_P \le t\} \qquad \{u_s \le t\} \cap \{\ell_{P,s} \le t\} = \{\ell_{Q,s} \le t\} \tag{20}$$

for every $t \in \mathbb{R}$. Indeed, it is easy to see, that (20) holds for all $t < s$. Therefore, fix an arbitrary $t \ge s$. We have

$$\{\ell_{P,s} \le t\} = \{\ell_P + s \le t\} + s\,\tfrac{e_1+e_2}{2} = (t-s)P + s\,\tfrac{e_1+e_2}{2}.$$

This can be rewritten as

$$\{\ell_{P,s} \le t\} = tP \cap \{x_1 \ge \tfrac{s}{2}\}.$$

Hence

$$\{u_s \le t\} \cup \{\ell_{P,s} \le t\} = \big(tP \cap \{x_1 \le \tfrac{s}{2}\}\big) \cup \big(tP \cap \{x_1 \ge \tfrac{s}{2}\}\big) = tP = \{\ell_P \le t\},$$

and

$$\begin{aligned}\{u_s \le t\} \cap \{\ell_{P,s} \le t\} &= tP \cap \{x_1 = \tfrac{s}{2}\} = \big((t-s)P \cap \{x_1 = 0\}\big) + s\,\tfrac{e_1+e_2}{2}\\ &= (t-s)\,Q + s\,\tfrac{e_1+e_2}{2} = \{\ell_Q + s \le t\} + s\,\tfrac{e_1+e_2}{2} = \{\ell_{Q,s} \le t\}.\end{aligned}$$

From the valuation property of Z we now get

$$\mathrm{Z}(u_s) + \mathrm{Z}(\ell_{P,s}) = \mathrm{Z}(\ell_P) + \mathrm{Z}(\ell_{Q,s}).$$

By Lemma 18 and since $V_n(Q) = 0$, we have

$$\mathrm{Z}(u_s) + \psi_n(s)V_n(P) + \psi_0(s) = \psi_n(0)V_n(P) + \psi_0(0) + \psi_0(s).$$

As $s \to \infty$, we obtain $u_s \xrightarrow{epi} \ell_P$ and therefore

$$\psi_n(0)V_n(P) + \psi_0(0) - \psi_n(s)V_n(P) = \mathrm{Z}(u_s) \xrightarrow{s\to\infty} \mathrm{Z}(\ell_P) = \psi_n(0)V_n(P) + \psi_0(0).$$

Since $V_n(P) > 0$, this shows that $\psi_n(s) \to 0$. □

Lemma 21 shows that for a continuous, $\mathrm{SL}(n)$ and translation invariant valuation Z the indicator growth functions ζ_0 and ζ_n coincide with its cone growth function ψ_0 and up to a constant factor with the n-th derivative of its cone growth function ψ_n, respectively. Since Lemma 22 shows that $\lim_{t\to\infty} \psi_n(t) = 0$, the cone growth functions ψ_0 and ψ_n are completely determined by the indicator growth functions of Z. Hence Lemma 19 immediately implies the following result.

Lemma 23. *Every continuous,* $\mathrm{SL}(n)$ *and translation invariant valuation* $\mathrm{Z} : \mathrm{Conv}(\mathbb{R}^n) \to \mathbb{R}$ *is uniquely determined by its indicator growth functions.*

We also require the following result.

Lemma 24. *Let* $\zeta \in C(\mathbb{R})$ *have constant sign on* $[t_0, \infty)$ *for some* $t_0 \in \mathbb{R}$*. If there exist* $n \in \mathbb{N}$*,* $c_n \in \mathbb{R}$ *and* $\psi \in C^n(\mathbb{R})$ *with* $\lim_{t\to+\infty} \psi(t) = 0$ *such that*

$$\zeta(t) = c_n \frac{\mathrm{d}^n}{\mathrm{d}t^n}\psi(t)$$

for $t \ge t_0$*, then*

$$\Big| \int_0^{+\infty} t^{n-1}\zeta(t)\,\mathrm{d}t \Big| < +\infty.$$

Proof. Since we can always consider $\widetilde{\psi}(t) = \pm c_n\, \psi(t)$ instead of $\psi(t)$, we assume that $c_n = 1$ and $\zeta \geq 0$. To prove the statement, we use induction on n and start with the case $n = 1$. For $t_1 > t_0$,

$$\int_{t_0}^{t_1} \zeta(t)\,\mathrm{d}t = \int_{t_0}^{t_1} \psi'(t)\,\mathrm{d}t = \psi(t_1) - \psi(t_0).$$

Hence, it follows from the assumption for ψ that

$$\int_{t_0}^{+\infty} \zeta(t)\,\mathrm{d}t = \lim_{t_1 \to +\infty} \psi(t_1) - \psi(t_0) = -\psi(t_0) < +\infty.$$

This proves the statement for $n = 1$.

Let $n \geq 2$ and assume that the statement holds true for $n - 1$. Since $\zeta \geq 0$, the function $\psi^{(n-1)}$ is increasing. Therefore, the limit

$$c = \lim_{t \to +\infty} \psi^{(n-1)}(t) \in (-\infty, +\infty]$$

exists. Moreover, $\psi^{(n-1)}$ has constant sign on $[\bar{t}_0, +\infty)$ for some $\bar{t}_0 \geq t_0$. By the induction hypothesis,

$$\left| \int_0^{+\infty} t^{n-2} \psi^{(n-1)}(t)\,\mathrm{d}t \right| < +\infty,$$

which is only possible if $c = 0$. In particular, $\psi^{(n-1)}(t) \leq 0$ for all $t \geq \bar{t}_0$.

Using integration by parts, we obtain

$$\int_{t_0}^{t_1} t^{n-1} \psi^{(n)}(t)\,\mathrm{d}t = t_1^{n-1} \psi^{(n-1)}(t_1) - t_0^{n-1} \psi^{(n-1)}(t_0) - (n-1) \int_{t_0}^{t_1} t^{n-2} \psi^{(n-1)}(t)\,\mathrm{d}t. \quad (21)$$

Since $t^{n-1}\psi^{(n)}(t) \geq 0$ for $t \geq \max\{0, t_0\}$, we have

$$d = \int_{t_0}^{+\infty} t^{n-1} \psi^{(n)}(t)\,\mathrm{d}t \in (-\infty, +\infty].$$

Hence, (21) implies that $t_1^{n-1}\psi^{(n-1)}(t_1)$ converges to

$$d + t_0^{n-1} \psi^{(n-1)}(t_0) + (n-1) \int_{t_0}^{+\infty} t^{(n-2)} \psi^{(n-1)}(t)\,\mathrm{d}t.$$

Since $t_1^{n-1}\psi^{(n-1)}(t_1) \leq 0$ for $t_1 \geq \max\{\bar{t}_0, 0\}$, we conclude that d is not $+\infty$. □

4 Proof of the Theorem

If $\zeta_0 : \mathbb{R} \to [0, \infty)$ is continuous and $\zeta_n : \mathbb{R} \to [0, \infty)$ is continuous with finite $(n-1)$-st moment, then Lemmas 12 and 16 show that

$$u \mapsto \zeta_0\big(\min_{x \in \mathbb{R}^n} u(x)\big) + \int_{\operatorname{dom} u} \zeta_n\big(u(x)\big)\, dx$$

defines a non-negative, continuous, $\mathrm{SL}(n)$ and translation invariant valuation on $\mathrm{Conv}(\mathbb{R}^n)$.

Conversely, let $\mathrm{Z} : \mathrm{Conv}(\mathbb{R}^n) \to [0, \infty)$ be a continuous, $\mathrm{SL}(n)$ and translation invariant valuation on $\mathrm{Conv}(\mathbb{R}^n)$ with indicator growth functions ζ_0 and ζ_n. For a polytope $P \in \mathcal{P}_0^n$ with $\dim P < n$, Lemma 18 implies that

$$0 \leq \mathrm{Z}(\mathrm{I}_P + t) = \zeta_0(t)$$

for every $t \in \mathbb{R}$. Hence, ζ_0 is a non-negative and continuous function. Similarly, for $Q \in \mathcal{P}_0^n$ with $V_n(Q) > 0$, we have

$$0 \leq \mathrm{Z}(\mathrm{I}_{sQ} + t) = \zeta_0(t) + s^n \zeta_n(t) V_n(Q),$$

for every $t \in \mathbb{R}$ and $s > 0$. Therefore, also ζ_n is a non-negative and continuous function. By Lemmas 21, 22 and 24, the growth function ζ_n has finite $(n-1)$-st moment. Finally, for $u = \mathrm{I}_P + t$ with $P \in \mathcal{P}_0^n$ and $t \in \mathbb{R}$, we obtain that

$$\mathrm{Z}(u) = \zeta_0(t) + \zeta_n V_n(P) = \zeta_0(\min_{x \in \mathbb{R}^n} u(x)) + \int_{\mathrm{dom}\, u} \zeta_n(u(x)) \,\mathrm{d}x.$$

By the first part of the proof,

$$u \mapsto \zeta_0(\min_{x \in \mathbb{R}^n} u(x)) + \int_{\mathrm{dom}\, u} \zeta_n(u(x)) \,\mathrm{d}x$$

defines a non-negative, continuous, $\mathrm{SL}(n)$ and translation invariant valuation on $\mathrm{Conv}(\mathbb{R}^n)$. Thus Lemma 23 completes the proof of the theorem.

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