Valuations on Polytopes containing the Origin in their Interiors

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Abstract

We give a classification of non-negative or Borel measurable, $SL(d)$ invariant, homogeneous valuations on the space of d-dimensional convex polytopes containing the origin in their interiors. The only examples are volume, volume of the polar body, and the Euler characteristic.

1 Introduction and Statement of Results

In recent years, important new results on the classification of valuations on the space of convex bodies have been obtained. The starting point for these results is Hadwiger's classical characterization of quermassintegrals [7], [8], which can be stated in the following way. Let $\mathcal{K}^d$ be the space of convex bodies, i.e., of compact convex sets, in Euclidean d-dimensional space $\mathbb{E}^d$. Call a functional $\mu : \mathcal{K}^d \to \mathbb{R}$ a *valuation* if

$$\mu(K) + \mu(L) = \mu(K \cup L) + \mu(K \cap L)$$

whenever $K, L, K \cup L \in \mathcal{K}^d$.

Theorem 1.1 (Hadwiger). *A functional $\mu : \mathcal{K}^d \to \mathbb{R}$ is a continuous, rigid motion invariant valuation if and only if there are constants $c_0, c_1, \dots, c_d \in \mathbb{R}$ such that*

$$\mu(K) = c_0\, W_0(K) + \ldots + c_d\, W_d(K)$$

for every $K \in \mathcal{K}^d$.

*This paper was written during stays at University College London and Polytechnic University New York. The author thanks these universities for their hospitality and the FWF for the financial support.

Theorem 1.2 (Hadwiger). *A functional $\mu : \mathcal{K}^d \to \mathbb{R}$ is a monotonously increasing, rigid motion invariant valuation if and only if there are constants $c_0, c_1, \ldots, c_d \geq 0$ such that*

$$\mu(K) = c_0\, W_0(K) + \ldots + c_d\, W_d(K)$$

for every $K \in \mathcal{K}^d$.

Here $W_0(K), \ldots, W_d(K)$ are the quermassintegrals of K and continuity is with respect to the usual topology induced by the Hausdorff metric.

These theorems are of fundamental importance in the theory of convex bodies. They also allow deriving simple proofs of numerous results in integral geometry and geometric probability. These beautiful applications are described in Hadwiger's book [9] and Klain and Rota's recent book [14].

Excellent surveys on the history of valuations from Dehn's solution of Hilbert's third problem to approximately 1990 were given by McMullen and Schneider [21] and by McMullen [20]. Here we mention some of the more recent results.

In 1995, Klain [10] gave a new and shorter proof of Hadwiger's Theorem 1.1. Whereas Hadwiger's proof is based on dissections of polytopes and is rather complicated, Klain's proof makes use of completely different tools such as generalized zonoids and spherical harmonics. His result also contributed to the difficult problem of classifying continuous, translation invariant valuations on the space of convex bodies. Further contributions to this problem include results by Schneider [22], Klain [13], and Alesker [5]. Recently, Alesker [6] has given a complete classification of continuous, translation invariant valuations on the space of convex bodies thereby confirming a twenty year old conjecture by McMullen [19].

In the dual Brunn-Minkowski theory, Klain gave a classification of continuous, rotation invariant valuations on star-shaped sets [11], [12]. In particular, for continuous valuations invariant with respect to the special linear group $SL(d)$, i.e., the group of linear transformations with determinant 1, he showed that the only examples are linear combinations of the Euler characteristic and volume. The difficult problem of classifying continuous, rotation invariant valuations on the space of convex bodies was solved by Alesker [2], [3]. In his proof, he approximated continuous, rotation invariant valuations by polynomial valuations and then classified the latter, making use of representations of the orthogonal group. As an application of his results, he obtained a classification of tensor or polynomial valued, continuous, rigid motion covariant valuations on the space of convex bodies [4].

In [15] and in the joint paper with Reitzner [17], a classification of upper semicontinuous, equi-affine invariant valuations on the space of convex bodies is given. A functional on $\mathcal{K}^d$ is called equi-affine invariant if it is invariant with respect to translations and $SL(d)$. These functionals are linear combinations of the Euler characteristic, volume, and affine surface area. In the planar case, this result was generalized to a classification of upper semicontinuous, rigid motion invariant valuations [16].

Here we consider the following problem. Let $\mathcal{P}_o^d$ be the space of convex polytopes containing the origin in their interiors. What are the $SL(d)$ invariant valuations on $\mathcal{P}_o^d$? To state our results we need the following notions. A functional $\mu : \mathcal{P}_o^d \to \mathbb{R}$ is called *(positively) homogeneous of degree* q, $q \in \mathbb{R}$, if $\mu(t\,P) = t^q \mu(P)$ for every $P \in \mathcal{P}_o^d$ and $t > 0$. It is called *(Borel) measurable* if the pre-image of every open set is a Borel set. Denote by P^* the polar body of $P \in \mathcal{P}_o^d$.

Theorem 1.3. *Let $\mu : \mathcal{P}_o^d \to \mathbb{R}$, $d \geq 2$, be a measurable, $SL(d)$ invariant valuation which is homogeneous of degree q, $q \in \mathbb{R}$. If $q = 0$, then there is a constant $c_0 \in \mathbb{R}$ such that*

$$\mu(P) = c_0$$

for every $P \in \mathcal{P}_o^d$, if $q = d$, then there is a constant $c_0 \in \mathbb{R}$ such that

$$\mu(P) = c_0\, V(P)$$

for every $P \in \mathcal{P}_o^d$, if $q = -d$, then there is a constant $c_0 \in \mathbb{R}$ such that

$$\mu(P) = c_0\, V(P^*)$$

for every $P \in \mathcal{P}_o^d$, and in all other cases

$$\mu(P) = 0$$

for every $P \in \mathcal{P}_o^d$.

Theorem 1.4. *Let $\mu : \mathcal{P}_o^d \to [0, \infty)$, $d \geq 2$, be an $SL(d)$ invariant valuation which is homogeneous of degree q, $q \in \mathbb{R}$. If $q = 0$, then there is a constant $c_0 \geq 0$ such that*

$$\mu(P) = c_0$$

for every $P \in \mathcal{P}_o^d$, if $q = d$, then there is a constant $c_0 \geq 0$ such that

$$\mu(P) = c_0\, V(P)$$

for every $P \in \mathcal{P}_o^d$*, if* $q = -d$*, then there is a constant* $c_0 \geq 0$ *such that*

$$\mu(P) = c_0 \, V(P^*)$$

for every $P \in \mathcal{P}_o^d$*, and in all other cases*

$$\mu(P) = 0$$

for every $P \in \mathcal{P}_o^d$*.*

We remark that on the space $\mathcal{K}_o^d$ of convex bodies which contain the origin in their interiors there are upper semicontinuous, non-negative, $SL(d)$ invariant, homogeneous valuations called L_p-affine surface areas [18]. Their existence shows that Theorems 1.3 and 1.4 do not hold if we replace $\mathcal{P}_o^d$ by $\mathcal{K}_o^d$.

It is an open problem to classify the measurable or non-negative, $SL(d)$ invariant valuations on $\mathcal{P}_o^d$.

2 Proofs

Let $\mu : \mathcal{P}_o^d \to \mathbb{R}$ be an $SL(d)$ invariant valuation which is homogeneous of degree q. Set $\nu(P) = \mu(P^*)$, where P^* is the polar body of $P \in \mathcal{P}_o^d$, i.e.,

$$P^* = \{y \in \mathbb{E}^d \,|\, x \cdot y \leq 1 \text{ for all } x \in P\}.$$

Here $x \cdot y$ denotes the inner product x and y in $\mathbb{E}^d$.

The functional $\nu : \mathcal{P}_o^d \to \mathbb{R}$ has the following properties. Since μ is homogeneous of degree q,

$$\nu(t\,P) = \mu((t\,P)^*) = \mu(t^{-1} P^*) = t^{-q}\,\nu(P),$$

i.e., ν is homogeneous of degree $-q$. For $P, Q, P \cup Q \in \mathcal{P}_o^d$, we have

$$(P \cup Q)^* = P^* \cap Q^* \text{ and } (P \cap Q)^* = P^* \cup Q^*.$$

Since μ is a valuation,

$$\begin{array}{rcl} \nu(P) + \nu(Q) = & \mu(P^*) + \mu(Q^*) & = \\ & \mu(P^* \cup Q^*) + \mu(P^* \cap Q^*) & = \\ & \mu((P \cap Q)^*) + \mu((P \cup Q)^*) & = \nu(P \cap Q) + \nu(P \cup Q), \end{array}$$

i.e., ν is also a valuation. For $\phi \in SL(d)$ and $P \in \mathcal{P}_o^d$, we have

$$(\phi\,P)^* = \phi^{-t}\,P^*,$$

where ϕ^{-t} is the inverse of the transpose of ϕ. Since μ is $SL(d)$ invariant,

$$\nu(\phi P) = \mu((\phi P)^*) = \mu(\phi^{-t} P^*) = \mu(P^*) = \nu(P),$$

i.e., ν is also $SL(d)$ invariant. Thus $\nu : \mathcal{P}_o^d \to \mathbb{R}$ is an $SL(d)$ invariant valuation which is homogeneous of degree $-q$. Consequently to prove Theorems 1.3 and 1.4 it is enough to consider valuations $\mu : \mathcal{P}_o^d \to \mathbb{R}$ which are non-negative or measurable, $SL(d)$ invariant, and homogeneous of degree $q \geq 0$.

1. We begin by proving Theorems 1.3 and 1.4 in the case $d = 2$. We fix a rectangular x_1-x_2-coordinate system and denote by $\mathcal{Q}_o(x_1, x_2)$ the set of convex polygons $Q = [I_1, I_2]$ where I_1 and I_2 are closed intervals lying on the x_1-axis and x_2-axis, respectively, and containing the origin in their interiors. Here $[P_1, \dots, P_m]$ stands for convex hull of $P_1, \dots, P_m$. Let I_1 be fixed and define $\nu : \mathcal{P}_o^1 \to \mathbb{R}$ by

$$\nu(I_2) = \mu([I_1, I_2]).$$

Then ν is a valuation on $\mathcal{P}_o^1$. Since μ is $SL(2)$ invariant and homogeneous of degree q,

$$\mu(t\,[I_1, I_2]) = \mu([I_1, t^2\, I_2]) = t^q\, \mu([I_1, I_2]).$$

Thus $\nu(t^2\, I_2) = t^q\, \nu(I_2)$, i.e., ν is homogeneous of degree $p = q/2$. If μ is non-negative or measurable, then so is ν.

Next, we need a characterization of non-negative or measurable valuations ν on $\mathcal{P}_o^1$ which are homogeneous of degree $p \geq 0$. The elements of $\mathcal{P}_o^1$ are intervals $[-s, t]$ with $s, t > 0$. Since ν is a valuation, we have

$$\nu([-s_1, t_1]) + \nu([-s_2, t_2]) = \nu([-s_1, t_2]) + \nu([-s_2, t_1]). \tag{1}$$

Since ν is homogeneous of degree p,

$$\nu([-s, t]) = s^p\, \nu([-1, \frac{t}{s}]) = s^p\, f(\frac{t}{s})$$

with a suitable function $f : (0, \infty) \to \mathbb{R}$. Because of (1), the following functional equation holds for f:

$$s_1^p\, f(\frac{t_1}{s_1}) + s_2^p\, f(\frac{t_2}{s_2}) = s_1^p\, f(\frac{t_2}{s_1}) + s_2^p\, f(\frac{t_1}{s_2}).$$

By setting $s_1 = t_1 = 1$, $t_2 = y$, and $s_2 = 1/x$, we get

$$f(1) + \frac{1}{x^p}\, f(x\, y) = f(y) + \frac{1}{x^p}\, f(x).$$

Setting $g(x) = f(x) - f(1)$ therefore leads to the following functional equation for $g : (0, \infty) \to \mathbb{R}$:

$$g(x\,y) = g(x) + x^p\, g(y) \tag{2}$$

for $x, y > 0$. If $p = 0$, then this is one of Cauchy's functional equations (cf., e.g., [1]). For g bounded from below or measurable, all solutions are given by

$$g(x) = a\, \log(x)$$

with $a \in \mathbb{R}$. If $p > 0$, then using the symmetry of the left-hand side of (2) we get

$$g(x) = b\,(x^p - 1),$$

with $b \in \mathbb{R}$. For ν this gives the following. If $p = 0$, then

$$\nu([-s, t]) = a\, \log(\frac{t}{s}) + b, \tag{3}$$

and if $p > 0$, then

$$\nu([-s, t]) = a\, s^p + b\, t^p, \tag{4}$$

with suitable $a, b \in \mathbb{R}$.

Now we return to μ on $\mathcal{Q}_o(x_1, x_2)$. First, we consider the case that μ is homogeneous of degree $q = 0$. Then the corresponding ν is also homogeneous of degree $p = 0$. Let $Q = [I_1, I_2]$, where $I_1 = [-s_1, t_1]$ lies on the x_1-axis and $I_2 = [-s_2, t_2]$ lies on the x_2-axis. By (3) we have

$$\mu([I_1, I_2]) = a(s_1, t_1) \log(\frac{t_2}{s_2}) + b(s_1, t_1),$$

where $a(s_1, t_1), b(s_1, t_1)$ are suitable functions of $s_1, t_1 > 0$. Note that $a(s_1, t_1)$ and $b(s_1, t_1)$ are homogeneous of degree 0. If μ is non-negative or measurable, then so are $a(s_1, t_1)$ and $b(s_1, t_1)$. Since μ is a valuation, using (3) we obtain

$$a(s_1, t_1) = a\, \log(\frac{t_1}{s_1}) + c \quad \text{and} \quad b(s_1, t_1) = b\, \log(\frac{t_1}{s_1}) + d$$

with suitable $a, b, c, d \in \mathbb{R}$. Therefore

$$\mu([I_1, I_2]) = \left(a\, \log(\frac{t_1}{s_1}) + c\right) \log(\frac{t_2}{s_2}) + b\, \log(\frac{t_1}{s_1}) + d$$

holds for $s_1, t_1, s_2, t_2 > 0$ with $a, b, c, d \in \mathbb{R}$. Since μ is rotation invariant,

$$\mu([I_1, I_2]) = \mu([-I_2, I_1]) = \mu([-I_2, -I_1]) \tag{5}$$

with $-I_1 = [-t_1, s_1]$ and $-I_2 = [-t_2, s_2]$. Comparing coefficients in (5) implies that $a = b = c = 0$. Thus

$$\mu(Q) = c_0 \tag{6}$$

with $c_0 \in \mathbb{R}$ for $Q \in \mathcal{Q}_o(x_1, x_2)$.

Now let μ be homogeneous of degree $q > 0$. Then the corresponding ν is homogeneous of degree $p = q/2 > 0$. By (4) we have

$$\mu([I_1, I_2]) = a(s_1, t_1)\, s_2^p + b(s_1, t_1)\, t_2^p,$$

where $a(s_1, t_1), b(s_1, t_1)$ are suitable functions of $s_1, t_1 > 0$. Note that $a(s_1, t_1)$ and $b(s_1, t_1)$ are homogeneous of degree p. If μ is non-negative, then $a(s_1, t_1)$ and $b(s_1, t_1)$ are non-negative. If μ is measurable, then they are measurable. Since μ is a valuation, using (4) we obtain

$$a(s_1, t_1) = a\, s_1^p + c\, t_1^p \quad \text{and} \quad b(s_1, t_1) = b\, s_1^p + d\, t_1^p$$

with suitable $a, b, c, d \in \mathbb{R}$. Therefore

$$\mu([I_1, I_2]) = (a\, s_1^p + c\, t_1^p)\, s_2^p + (b\, s_1^p + d\, t_1^p)\, t_2^p$$

holds for $s_1, t_1, s_2, t_2 > 0$ with $a, b, c, d \in \mathbb{R}$. Combined with (5) this implies that $a = b = c = d$. Thus for $Q = [I_1, I_2] \in \mathcal{Q}_o(x_1, x_2)$, $I_1 = [-s_1, t_1]$, $I_2 = [-s_2, t_2]$,

$$\mu(Q) = a\, (s_1^p + t_1^p)(s_2^p + t_2^p). \tag{7}$$

Let $\mathcal{R}_o^2(x_1)$ be the set of convex polygons $[I_1, u, v]$ where I_1 is a closed interval on the x_1-axis containing the origin in its interior and u, v are points in the open lower and upper halfplane, respectively. Denote by $\mathcal{Q}_o^2$ the set of $SL(2)$-images of $Q \in \mathcal{Q}_o(x_1, x_2)$ and by $\mathcal{R}_o^2$ the set of $SL(2)$-images of $R \in \mathcal{R}_o^2(x_1)$. We need the following result.

Lemma 1. *Let $\mu : \mathcal{P}_o^2 \to \mathbb{R}$ be a non-negative or measurable, $SL(2)$ invariant valuation which is homogeneous of degree $q = 2\,p$ and for which (7) holds. If $p > 0$ and $p \neq 1$, then $\mu(Q) = 0$ for every $Q \in \mathcal{Q}_o^2$.*

Proof. Let $R = [I_1, s\,u, t\,v]$ where $I_1 = [-s_1, t_1]$ lies on the x_1-axis, $u = \binom{x}{-1}$, $v = \binom{y}{1}$ with $x, y \in \mathbb{R}$, $s_1, t_1, s, t > 0$. First we show that

$$\lim_{s,t \to 0} \mu([I_1, s\,u, t\,v]) \tag{8}$$

exists. Since μ is a valuation, we have for $0 < t' < t$ and $t'' > 0$ suitably large

$$\mu([I_1, s\,u, t\,v]) + \mu([I_1, -t''\,v, t'\,v]) = \mu([I_1, s\,u, t'\,v]) + \mu([I_1, -t''\,v, t\,v]).$$

Since $[I_1, -t''\,v, t'\,v], [I_1, -t''\,v, t\,v] \in \mathcal{Q}_o^2$ and since μ is $SL(2)$ invariant, we can use (7) and obtain

$$\mu([I_1, s\,u, t\,v]) - \mu([I_1, s\,u, t'\,v]) = a\,(s_1^p + t_1^p)(t^p - t'^p).$$

Similarly for $0 < s' < s$ and $s'' > 0$ suitably large,

$$\mu([I_1, s\,u, t'\,v]) - \mu([I_1, s'\,u, t'\,v]) = a\,(s_1^p + t_1^p)(s^p - s'^p).$$

Since $p > 0$, this implies that the limit (8) exists. Note that we have

$$\mu([I_1, s\,u, t\,v]) = \mu([I_1, s'\,u, t'\,v]) + a\,(s_1^p + t_1^p)(t^p - t'^p + s^p - s'^p). \tag{9}$$

For I_1 fixed, set $f(x, y) = \lim_{s,t\to 0} \mu([I_1, s\,u, t\,v])$ and $e = \binom{0}{1}$. Since μ is a valuation, we have for $r > 0$ suitably small

$$\mu([I_1, s\,u, t\,v]) + \mu([I_1, -s\,r\,e, t\,r\,e]) = \mu([I_1, s\,u, t\,r\,e]) + \mu([I_1, -s\,r\,e, t\,v]).$$

This implies that

$$f(x, y) + f(0, 0) = f(x, 0) + f(0, y). \tag{10}$$

Note that $f(0, 0) = 0$, since $[I_1, -s\,r\,e, t\,r\,e] \in \mathcal{Q}_o^2$ and since we can use (7). Set

$$\phi = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}.$$

Then $\phi \in SL(2)$. Since μ is $SL(2)$ invariant, this implies that

$$\mu([I_1, s\,u, t\,v]) = \mu([\phi I_1, s\,\phi u, t\,\phi v]) = \mu([I_1, -s\,e, t\,w])$$

where $w = \binom{x+y}{1}$. Consequently

$$f(x, y) = f(0, x + y). \tag{11}$$

Set $g(x) = f(0, x)$. Then it follows from (10) and (11) that

$$g(x) + g(y) = g(x + y).$$

This is one of Cauchy's functional equations. Since μ is measurable or non-negative, so is g. This implies that there is a $\nu(I_1) \in \mathbb{R}$ such that

$$\lim_{s,t\to 0} \mu([I_1, s\,u, t\,v]) = \nu(I_1)(x+y). \tag{12}$$

Using this we obtain the following. The functional μ is homogeneous of degree q, therefore

$$\nu(r\,I_1) = r^q \nu(I_1). \tag{13}$$

On the other hand, let

$$\phi = \begin{pmatrix} r & 0 \\ 0 & \frac{1}{r} \end{pmatrix}.$$

Then $\mu(\phi R) = \mu(R)$,

$$\phi\begin{pmatrix} x \\ -1 \end{pmatrix} = \frac{1}{r}\begin{pmatrix} r^2\,x \\ -1 \end{pmatrix} \quad \text{and} \quad \phi\begin{pmatrix} y \\ 1 \end{pmatrix} = \frac{1}{r}\begin{pmatrix} r^2\,y \\ 1 \end{pmatrix}.$$

By (12) this implies that

$$\nu(r\,I_1) = r^{-2}\,\nu(I_1).$$

Combined with (13) this shows that $\nu(I_1) = 0$. By (9) and (12) this implies that

$$\mu([I_1, s\,u, t\,v]) = a\,(s_1^p + t_1^p)(s^p + t^p). \tag{14}$$

Let T_r^s be the triangle with vertices $\binom{1}{0}$, $\binom{0}{1}$, $\binom{-s}{-s\,r}$, $r, s > 0$. Then $T_r^s = [I_1, s\,r\,u, v]$ with $I_1 = [-s_1, 1]$, $s_1 = s/(1+s\,r)$, $u = \binom{x}{-1}$, $x = 1/r$, $v = \binom{y}{1}$, $y = 0$. By (14) we have for every $r > 0$

$$\lim_{s\to 0} \mu(T_r^s) = a. \tag{15}$$

Define the triangle $T^s(x,y)$ as the convex hull of $\binom{y}{1-y}$, $\binom{x}{1-x}$, $\binom{-s}{-s}$. For $0 \le x < 1/2 < y \le 1$, we have $T^s(x,y) \in \mathcal{P}_o^2$, $T_r^s = T^s(0,1) = T^s(0,y) \cup T^s(x,1)$ with $r = 1$, and $T^s(x,y) = T^s(0,y) \cap T^s(x,1)$. Therefore

$$\mu(T^s(0,y)) + \mu(T^s(x,1)) = \mu(T^s(0,1)) + \mu(T^s(x,y)). \tag{16}$$

Let

$$\phi = \begin{pmatrix} y & x \\ 1-y & 1-x \end{pmatrix}.$$

Then $T^s(x,y) = \phi T_r^{s\,t}$ with $r = (2y-1)/(1-2x)$ and $t = (1-2x)/(y-x)$. Therefore we get by (15)

$$\lim_{s\to 0} \mu(T^s(x,y)) = \lim_{s\to 0} (y-x)^p\,\mu(T_r^{s\,t}) = a\,(y-x)^p.$$

Combined with (16) it follows that

$$a\,y^p + a\,(1-x)^p = a + a\,(y-x)^p.$$

Since $p \neq 1$ this shows that $a = 0$. $\square$

We make the following definitions. If $q = 0$, set

$$\mu_0(P) = \mu(P) - c_0,$$

if $q = 2$, set

$$\mu_0(P) = \mu(P) - c_0\,A(P),$$

and in all other cases, set

$$\mu_0(P) = \mu(P).$$

Because of (6), (7) and the $SL(2)$ invariance of μ, and Lemma 1, we obtain for any $q \geq 0$ a valuation μ_0 on $\mathcal{P}_o^2$ which vanishes for every $Q \in \mathcal{Q}_o^2$. Using Lemmas 2 and 3 below completes the proof in the case $d = 2$.

2. Now let $d \geq 3$. We use induction on the dimension, d, to prove Theorems 1.3 and 1.4. Suppose that the theorems are true in dimension $d-1$.

We fix a rectangular $x_1, \dots, x_d$-coordinate system and denote by H the $x_1, \dots, x_{d-1}$-coordinate hyperplane. We identify H and $\mathbb{E}^{d-1}$. Let $\mathcal{Q}_o(x_d)$ be the set of convex polytopes $Q = [P', I]$ where $P' \in \mathcal{P}_o^{d-1}$ and I is a closed interval lying on the x_d-axis and containing the origin in its interior. For I fixed, define $\mu' : \mathcal{P}_o^{d-1} \to \mathbb{R}$ by

$$\mu'(P') = \mu([P', I]).$$

If μ is a non-negative or measurable valuation, then so is μ' on $\mathcal{P}_o^{d-1}$. Since μ is $SL(d)$ invariant and homogeneous of degree q,

$$\mu(t\,[P', I]) = \mu([t^{\frac{d}{d-1}} P', I]) = t^q\,\mu([P', I]).$$

Therefore

$$\mu'(t^{\frac{d}{d-1}} P') = t^q\,\mu'(P'),$$

i.e., μ' is homogeneous of degree $p = q\,(d-1)/d$. Since μ is $SL(d)$ invariant, μ' is $SL(d-1)$ invariant.

If $q = 0$, by induction there exists a constant $\nu \in \mathbb{R}$ such that $\mu'(P') = \nu$ for every $P' \in \mathcal{P}_o^{d-1}$. Therefore $\mu(Q) = \nu(I)$, where ν depends on I. Note that ν is a non-negative or measurable valuation on $\mathcal{P}_o^1$ which is homogeneous of degree 0. By (3) there are $a, b \in \mathbb{R}$ such that for $Q = [P', I]$, where $P' \in \mathcal{P}_o^{d-1}$ and $I = [-s, t]$ lies on the x_d-axis, we have

$$\mu(Q) = a \log(\frac{t}{s}) + b.$$

Since μ is rotation invariant, by choosing P' symmetric with respect to the coordinate hyperplanes we see that $a = 0$. Therefore there is a $c_0 \in \mathbb{R}$ such that

$$\mu(Q) = c_0 \tag{17}$$

for every $Q \in \mathcal{Q}_o(x_d)$.

If $q = d$, by induction there exists a constant $\nu \in \mathbb{R}$ such that $\mu'(P') = \nu\, V_{d-1}(P')$ for every $P' \in \mathcal{P}_o^{d-1}$, where V_{d-1} denotes $(d-1)$-dimensional volume. Consequently, $\mu(Q) = \nu(I)\, V_{d-1}(P')$. Here ν is a non-negative or measurable valuation on $\mathcal{P}_o^1$, which is homogeneous of degree 1. By (4) there are $a, b \in \mathbb{R}$ such

$$\mu(Q) = (a\, s + b\, t) V_{d-1}(P')$$

for $Q = [P', I]$, where $P' \in \mathcal{P}_o^{d-1}$ and $I = [-s, t]$ lies on the x_d-axis. Using the rotation invariance of μ and choosing P' symmetric with respect to the coordinate hyperplanes, we see that $a = b$. Therefore there is a constant $c_0 \in \mathbb{R}$ such that

$$\mu(Q) = c_0\, V(Q) \tag{18}$$

for every $Q \in \mathcal{Q}_o(x_d)$.

If $q \neq 0, p \neq d$, then we get by induction that $\mu'(P') = 0$ for every $P' \in \mathcal{P}_o^{d-1}$. Therefore

$$\mu(Q) = 0 \tag{19}$$

for every $Q \in \mathcal{Q}_o(x_d)$.

We make the following definitions. If $q = 0$, set

$$\mu_0(P) = \mu(P) - c_0,$$

if $q = d$, set

$$\mu_0(P) = \mu(P) - c_0\, V(P),$$

and in all other cases, set

$$\mu_0(P) = \mu(P).$$

Let $\mathcal{Q}_o^d$ be the set of $SL(d)$-images of $Q \in \mathcal{Q}_o(x_d)$. Because of (17), (18), and (19), and since μ is $SL(d)$ invariant, we obtain for every $q \geq 0$ a valuation $\mu_0 : \mathcal{P}_o^d \to \mathbb{R}$ which vanishes for every $Q \in \mathcal{Q}_o^d$. Let $\mathcal{R}_o^d(x_d)$ be the set of convex polytopes $[P', u, v]$ where $P' \in \mathcal{P}_o^{d-1}$ and u, v are points in the open lower and upper halfspace bounded by H, respectively. Denote by $\mathcal{R}_o^d$ the set of $SL(d)$-images of $R \in \mathcal{R}_o^d(x_d)$. The following two lemmas conclude the proof of Theorems 1.3 and 1.4.

Lemma 2. *Let $\mu : \mathcal{P}_o^d \to \mathbb{R}$ be a non-negative or measurable, $SL(d)$ invariant valuation which is homogeneous of degree q, $q \geq 0$. If μ vanishes on $\mathcal{Q}_o^d$, then $\mu(R) = 0$ for every $R \in \mathcal{R}_o^d$.*

Proof. Let $R = [P', s\,u, t\,v]$ where $P \in \mathcal{P}_o^{d-1}$, $u = \binom{u'}{-1}$ and $v = \binom{v'}{1}$ with $u', v' \in \mathbb{E}^{d-1}$ and $s, t > 0$. Since μ is a valuation, we have for $0 < t < t'$ and $t'' > 0$ suitably small

$$\mu([P', s\,u, t\,v]) + \mu([P', -t''\,v, t'\,v]) = \mu([P', s\,u, t'\,v]) + \mu([P', -t''\,v, t\,v]).$$

Since $[P', -t''\,v, t'\,v], [P', -t''\,v, t\,v] \in \mathcal{Q}_o^d$ and since μ vanishes on $\mathcal{Q}_o^d$, this implies that $\mu([P', s\,u, t\,v])$ does not depend on $t > 0$. A similar argument shows that it does not depend on $s > 0$. Thus

$$\mu([P', s\,u, t\,v]) = \mu([P', u, v]). \tag{20}$$

For P' fixed, set

$$f(u', v') = \mu([P', u, v]).$$

Let $e = \binom{o'}{1}$ where o' denotes the origin in $\mathbb{E}^{d-1}$. Since μ is a valuation, we have for $r > 0$ suitably small

$$\mu([P', u, v]) + \mu([P', -r\,e, r\,e]) = \mu([P', u, r\,e]) + \mu([P', -r\,e, v]).$$

By (20) this implies that

$$f(u', v') + f(o', o') = f(u', o') + f(o', v'). \tag{21}$$

Note that since $[P', -r\,e, r\,e] \in \mathcal{Q}_o^d$, we have $f(o', o') = 0$. Let

$$\phi = \begin{pmatrix} 1 & \dots & 0 & u_1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 1 & u_{d-1} \\ 0 & \dots & 0 & 1 \end{pmatrix}.$$

Then $\phi\binom{u'}{-1} = \binom{o'}{-1}$ and $\phi\binom{v'}{1} = \binom{u'+v'}{1} = w$. Since μ is $SL(d)$ invariant, this implies that

$$\mu([P', u, v]) = \mu([\phi P', \phi u, \phi v]) = \mu([P', -e, w]).$$

Consequently
$$f(u',v') = f(o',u'+v'). \tag{22}$$
Set $g(u') = f(o',u')$. Then we get by (21) and (22) that
$$g(u'+v') = g(u') + g(v').$$
This is one of Cauchy's functional equations. Since μ is measurable or non-negative, this implies that there is a $z'(P') \in \mathbb{E}^{d-1}$ such that
$$\mu(R) = \mu([P',u,v]) = z'(P') \cdot (u'+v') \tag{23}$$
for every $u', v' \in \mathbb{E}^{d-1}$.

Using this we obtain the following. The functional μ is homogeneous of degree q. Since we know by (20) that $\mu([t\,P', t\,u, t\,v]) = \mu([t\,P', u, v])$, this and (23) imply that
$$z'(t\,P') = t^q z'(P'). \tag{24}$$
On the other hand, let $\phi \in SL(d)$ be the map that multiplies the first $(d-1)$ coordinates with t and the last coordinate with $t^{-(d-1)}$. Then $\mu(\phi R) = \mu(R)$,
$$\phi\begin{pmatrix} u' \\ -1 \end{pmatrix} = t^{-(d-1)}\begin{pmatrix} t^d\,u' \\ -1 \end{pmatrix} \quad \text{and} \quad \phi\begin{pmatrix} v' \\ 1 \end{pmatrix} = t^{-(d-1)}\begin{pmatrix} t^d\,v' \\ 1 \end{pmatrix}.$$
By (23) this implies that
$$z'(t\,P') = t^{-d}\,z'(P').$$
Combined with (24) this shows that $z'(P') = o'$. Because of (23) this completes the proof of the lemma. $\square$

Lemma 3. *Let $\mu : \mathcal{P}_o^d \to \mathbb{R}$ be a valuation. If μ vanishes on $\mathcal{R}_o^d$, then $\mu(P) = 0$ for every $P \in \mathcal{P}_o^d$.*

Proof. For a hyperplane H (always containing the origin), denote by H^+ and H^- the complementary closed halfspaces bounded by H. We need the following definitions. Let $\mathcal{P}_j^d$, $j = 1, \dots, d$, be the set of convex polytopes P such that there exist $P_o \in \mathcal{P}_o^d$ and hyperplanes $H_1, \dots, H_j$ with $\dim(H_1^+ \cap \dots \cap H_j^+) = d$ and
$$P = P_o \cap H_1^+ \cap \dots \cap H_j^+. \tag{25}$$
Let $\mathcal{R}_j^d$, $j = 1, \dots, d-1$, be the set of polytopes $R = [S,u,v]$ such that there exist $P_o \in \mathcal{P}_o^d$ and hyperplanes $H_1, \dots, H_{j+1}$ with $\dim(H_1^+ \cap \dots \cap H_{j+1}^+) = d$

where $S = P_o \cap H_1^+ \cap \ldots \cap H_j^+ \cap H_{j+1}$ and $u, v \in P_o \cap H_1 \cap \ldots \cap H_j$, $u, v \notin H_{j+1}$, $u \in H_{j+1}^+$, $v \in H_{j+1}^-$. Note that $\mathcal{R}_j^d \subseteq \mathcal{P}_j^d$.

Set $\mathcal{P}_0^d = \mathcal{P}_o^d$ and $\mathcal{R}_0^d = \mathcal{R}_o^d$. We define μ on $\mathcal{P}_j^d$, $j = 1, \ldots, d$, inductively, starting with $j = 1$, in the following way. For $P \in \mathcal{P}_j^d$, set

$$\mu(P) = \mu(P \cup R_{u,v}) \tag{26}$$

where $R_{u,v} = [P \cap H_j, u, v]$ with $u, v \in P_o \cap H_1 \cap \ldots \cap H_{j-1}$, $u, v \notin H_j$, $u \in H_j^+$, $v \in H_j^-$, and $u \in P$. Note that $R_{u,v} \in \mathcal{R}_{j-1}^d$ and $P \cup R_{u,v} \in \mathcal{P}_{j-1}^d$. We show that μ is well defined, that it vanishes on $\mathcal{R}_j^d$, and that it has the following additivity properties. If $P \in \mathcal{P}_{j-1}^d$ and H is a hyperplane such that $P \cap H^+, P \cap H^- \in \mathcal{P}_j^d$, then

$$\mu(P) = \mu(P \cap H^+) + \mu(P \cap H^-). \tag{27}$$

And if $P', P'', P' \cap P'', P' \cup P'' \in \mathcal{P}_j^d$ are defined by (25) with the same halfspaces $H_1^+, \ldots, H_j^+$, then

$$\mu(P') + \mu(P'') = \mu(P' \cup P'') + \mu(P' \cap P''). \tag{28}$$

The functional μ is well defined on $\mathcal{P}_0^d$, it vanishes on $\mathcal{R}_0^d$, and it is a valuation. In particular, (28) holds for $j = 0$. Suppose that μ is well defined by definition (26) on $\mathcal{P}_{k-1}^d$, that μ vanishes on $\mathcal{R}_{k-1}^d$, and that (27) (if $k > 1$) and (28) hold for $j = k - 1$. For $k \leq d$, we show that μ is well defined on $\mathcal{P}_k^d$, that μ vanishes on $\mathcal{R}_k^d$ (if $k < d$), and that (27) and (28) hold for $j = k$.

First, we show that (26) does not depend on the choice of u and v in $P_o \cap H_1 \cap \ldots \cap H_{k-1}$. Let $u', v' \in P_o \cap H_1 \cap \ldots \cap H_{k-1}$, $u', v' \notin H_k$, $u' \in P$, $v' \in H_k^-$ be chosen such that $R_{u,v} \subseteq R_{u',v'}$. Then applying (28) with $j = k - 1$ gives

$$\mu(P \cup R_{u,v}) + \mu(R_{u',v'}) = \mu(P \cup R_{u',v'}) + \mu(R_{u',v}).$$

Since $R_{u',v}, R_{u',v'} \in \mathcal{R}_{k-1}^d$ and since μ vanishes on $\mathcal{R}_{k-1}^d$, this implies that

$$\mu(P \cup R_{u,v}) = \mu(P \cup R_{u',v'}).$$

Consequently, definition (26) does not depend on the choice of u and v in $P_o \cap H_1 \cap \ldots \cap H_{k-1}$. If $j = 1$ this shows that μ is well defined on $\mathcal{P}_j^d$. For $j > 1$ we show that $\mu(P)$ as defined by (26) does not depend on the choice of H_k in the construction of $R_{u,v}$. Let $R_{u,v}$ be defined as before. Let u', v' be chosen in $P_o \cap H_2 \cap \ldots \cap H_k$, $u', v' \notin H_1$, $u' \in P$, $v' \in H_1^-$.

Let $R_{u',v'} = [P \cap H_1, u', v']$ and $S = [R_{u,v}, R_{u',v'}]$. Then applying (27) for $j = k-1$ gives

$$\mu(P \cup S) = \mu((P \cup S) \cap H_k^+) + \mu((P \cup S) \cap H_k^-).$$

We have $(P \cup S) \cap H_k^- = [P, v, v'] \cap H_k^- = [P \cap H_k, v', v]$ and $v' \in H_k$. By definition (26) $\mu([P \cap H_k, v', v]) = \mu([P \cap H_k, v', v, w])$ with $w \in H_k^+$, $w \notin H_k$. Since μ vanishes on $\mathcal{R}_j^d$, $j < k$, $\mu([P \cap H_k, v', v, w]) = 0$. Combined with $(P \cup S) \cap H_k^+ = P \cup R_{u',v'}$, this implies that $\mu(P \cup S) = \mu(P \cup R_{u',v'})$. Similarly, we have $\mu(P \cup S) = \mu(P \cup R_{u,v})$. Therefore μ is well defined on $\mathcal{P}_k^d$.

Note that since μ vanishes on $\mathcal{R}_{k-1}^d$, definition (26) implies that for $k < d$

$$\mu(R) = 0 \quad \text{for } R \in \mathcal{R}_k^d. \tag{29}$$

Next, we show that (27) holds for $j = k$. Let $P \in \mathcal{P}_{k-1}^d$ be such that there is a $P_o \in \mathcal{P}_o^d$ and hyperplanes $H_1, \ldots, H_{k-1}$ such that $P = P_o \cap H_1^+ \cap \ldots \cap H_{k-1}^+$. Choose $u, v \in P_o \cap H_1 \cap \ldots \cap H_{k-1}$, $u, v \notin H$ such that $u \in H^+$ and $v \in H^-$. Let $R_{u,v} = [P \cap H, u, v]$. Then P, $R_{u,v}$, $(P \cap H^+) \cup R_{u,v}$, $(P \cap H^-) \cup R_{u,v}$ have the hyperplanes $H_1, \ldots, H_{k-1}$ in common. Applying (28) for $j = k-1$ gives

$$\mu(P) + \mu(R_{u,v}) = \mu((P \cap H^+) \cup R_{u,v}) + \mu((P \cap H^-) \cup R_{u,v}).$$

By (29) and definition (26), this implies that (27) holds for $j = k$.

Next, we show that (28) holds for $j = k$. Choose $u, v \in H_1 \cap \ldots \cap H_{k-1}$, $u, v, \notin H_k$ such that $u \in P' \cap P''$, $v \in H_k^-$. Let $R'_{u,v} = [P' \cap H_k, u, v]$ and $R''_{u,v} = [P'' \cap H_k, u, v]$. Applying (28) for $j = k-1$ shows that

$$\begin{aligned}\mu(P' \cup R'_{u,v}) &+ \mu(P'' \cup R''_{u,v}) \\ &= \mu((P' \cup P'') \cup (R'_{u,v} \cup R''_{u,v})) + \mu((P' \cap P'') \cup (R'_{u,v} \cap R''_{u,v})).\end{aligned}$$

Because of definition (26) this implies that (28) holds for $j = k$.

We need one more additivity property. Let $P \in \mathcal{P}_d^d$ and let H be a hyperplane such that $P \cap H^+, P \cap H^- \in \mathcal{P}_d^d$. Then

$$\mu(P) = \mu(P \cap H^+) + \mu(P \cap H^-). \tag{30}$$

This can be seen in the following way.

First, let $d = 2$. Let P be bounded by H_1, H_2, and let $P \cap H^+$ and $P \cap H^-$ be bounded by H_1, H and H, H_2, respectively. Choose $u, v \in H$ such that

$$\mu(P \cap H^+) = \mu((P \cap H^+) \cup R_{u,v}^+) \quad \text{and} \quad \mu(P \cap H^-) = \mu((P \cap H^+) \cup R_{u,v}^-)$$

where $R^+_{u,v} = [P \cap H_1, u, v]$ and $R^-_{u,v} = [P \cap H_2, u, v]$. Let $S = [R^+_{u,v}, R^-_{u,v}]$. By (27), we have

$$\begin{aligned} \mu(P \cup S) &= \mu((P \cup S) \cap H^+) + \mu((P \cup S) \cap H^-) \\ &= \mu((P \cap H^+) \cup R^+_{u,v}) + \mu((P \cap H^-) \cup R^-_{u,v}) \\ &= \mu(P \cap H^+) + \mu(P \cap H^-). \end{aligned} \tag{31}$$

By (27), definition (26) and since μ vanishes on $\mathcal{R}^2_1$,

$$\mu(P \cup S) = \mu((P \cup S) \cap H_1^+) + \mu((P \cup S) \cap H_1^-) = \mu(P).$$

Combined with (31) this implies (30).

Second, let $d \geq 3$. Let $P = P_o \cap H_1^+ \cap \ldots \cap H_d^+$, $P_o \in \mathcal{P}^d_o$. Since $P \cap H^+$, $P \cap H^- \in \mathcal{P}^d_d$, we can say that $P \cap H^+$ is bounded by $H_1, H, H_3, \ldots, H_d$ and that $P \cap H^-$ is bounded by $H, H_2, H_3, \ldots, H_d$, where $H_1 \cap H_2 \cap \ldots \cap H_{d-1} \subseteq H$. Therefore we can choose $u, v \in H_1 \cap \ldots \cap H_{d-1}$ such that

$$\mu(P) = \mu(P \cup R_{u,v}),$$

where $R_{u,v} = [P \cap H_d, u, v]$, and

$$\mu(P \cap H^+) = \mu((P \cup R_{u,v}) \cap H^+) \quad \text{and} \quad \mu(P \cap H^-) = \mu((P \cup R_{u,v}) \cap H^-).$$

Applying (27) for $j = d$ shows that

$$\mu(P \cup R_{u,v}) = \mu((P \cup R_{u,v}) \cap H^+) + \mu((P \cup R_{u,v}) \cap H^-).$$

Because of definition (26) this implies (30).

We have to show that $\mu(P_o) = 0$ for every $P_o \in \mathcal{P}^d_o$. Since we can dissect P into two convex polytopes which are elements of $\mathcal{P}^d_1$, (27) implies that it suffices to prove that $\mu(P_1) = 0$ for every $P \in \mathcal{P}^d_1$. Using (27) repeatedly shows that it is enough to prove that $\mu(P) = 0$ for every $P \in \mathcal{P}^d_d$. So let $P \in \mathcal{P}^d_d$ with $P = P_o \cap C$, $P_o \in \mathcal{P}^d_o$, and $C = H_1^+ \cap \ldots \cap H_d^+$. The polytope P has a vertex at the origin and d vertices at the exposed rays of C. If P has n, $n \geq 1$, further vertices, then we can dissect P into $P_1, P_2 \in \mathcal{P}^d_d$ such that P_1 and P_2 both have fewer than $d + 1 + n$ vertices. By (30), $\mu(P) = \mu(P_1) + \mu(P_2)$. Therefore it suffices to show that $\mu(P_1) = \mu(P_2) = 0$. Using (30) repeatedly shows that it is enough to prove that $\mu(T) = 0$ for every simplex T with one vertex at the origin. Let T be such a simplex. By (29) for $k = d - 1$ and definition (26) we have $\mu(T) = 0$. This completes the prove of the lemma. $\square$

Acknowledgments

I would like to thank Matthias Reitzner for valuable comments on an earlier version of this paper and Chandni Shah for her helpful remarks.

References

[1] J. Aczel, "Lectures on functional equations and their applications," Academic Press, New York, 1966.

[2] S. Alesker, Continuous valuations on convex sets, *Geom. Funct. Anal.* **8** (1998), 402–409.

[3] S. Alesker, Continuous rotation invariant valuations on convex sets, *Ann. of Math.* (2) **149** (1999), 977–1005.

[4] S. Alesker, Description of continuous isometry covariant valuations on convex sets, *Geom. Dedicata* **74** (1999), 241–248.

[5] S. Alesker, On P. McMullen's conjecture on translation invariant valuations, *Adv. Math.* **155** (2000), 239–263.

[6] S. Alesker, Description of translation invariant valuations on convex sets with solution of P. McMullen's conjecture, *Geom. Funct. Anal.* **11** (2001), 244–272.

[7] H. Hadwiger, Additive Funktionale k-dimensionaler Eikörper. I, *Arch. Math.* **3** (1952), 470–478.

[8] H. Hadwiger, Additive Funktionale k-dimensionaler Eikörper. II, *Arch. Math.* **4** (1953), 374–379.

[9] H. Hadwiger, "Vorlesungen über Inhalt, Oberfläche und Isoperimetrie," Springer, Berlin, 1957.

[10] D. Klain, A short proof of Hadwiger's characterization theorem, *Mathematika* **42** (1995), 329–339.

[11] D. Klain, Star valuations and dual mixed volumes, *Adv. Math.* **121** (1996), 80–101.

[12] D. Klain, Invariant valuations on star-shaped sets, *Adv. Math.* **125** (1997), 95–113.

[13] D. Klain, Even valuations on convex bodies, *Trans. Amer. Math. Soc.* **352** (2000), 71–93.

[14] D. Klain and G. Rota, "Introduction to geometric probability," Cambridge University Press, Cambridge, 1997.

[15] M. Ludwig, A characterization of affine length and asymptotic approximation of convex discs, *Abh. Math. Semin. Univ. Hamb.* **69** (1999), 75–88.

[16] M. Ludwig, Upper semicontinuous valuations on the space of convex discs, *Geom. Dedicata* **80** (2000), 263–279.

[17] M. Ludwig and M. Reitzner, A characterization of affine surface area, *Adv. Math.* **147** (1999), 138–172.

[18] E. Lutwak, The Brunn-Minkowski-Firey theory. II: Affine and geominimal surface areas, *Adv. Math.* **118** (1996), 244–294.

[19] P. McMullen, Continuous translation-invariant valuations on the space of compact convex sets, *Arch. Math.* **55** (1980), 377–384.

[20] P. McMullen, Valuations and dissections, *in* "Handbook of Convex Geometry B"(P.M. Gruber and J. Wills, Eds.), pp. 933–990, North-Holland, Amsterdam, 1993.

[21] P. McMullen and R. Schneider, Valuations on convex bodies,*in* "Convexity and its applications"(P.M. Gruber and J. Wills, Eds.), pp. 170–247, Birkhäuser, 1983.

[22] R. Schneider, Simple valuations on convex bodies, *Mathematika* **43** (1996), 32–39.