MINKOWSKI AREAS AND VALUATIONS

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Abstract

All continuous $\mathrm{GL}(n)$ covariant valuations that map convex bodies to convex bodies are completely classified. This establishes a characterization of moment and projection operators and shows that Holmes-Thompson area is the unique Minkowski area that is also a bivaluation.

On finite dimensional Banach spaces, there are essentially different ways to choose a notion of surface area that is independent of the choice of a Euclidean coordinate system. In particular, *Minkowski areas* have this property. The precise definition of Minkowski area requires a set of conditions, which will be given in Section 2. Important examples of Minkowski areas are due to Busemann [**12**], Gromov [**19**] and Holmes & Thompson [**28**]. Central questions in the geometry of finite dimensional Banach spaces are those of finding the *right* notion of Minkowski area and, more generally, of *invariant area* (see Section 2 for the definition), and of finding distinctive properties of different such areas (see [**9**] and [**55**] for more information on area in Banach and Finsler spaces and see [**7**, **8**, **50**] for some recent results). As will be shown, if the critical *valuation* property is required and if invariant areas are defined for spaces with not necessarily symmetric unit balls, it turns out that Holmes-Thompson area is the only answer.

We need the following definitions. Let $\mathcal{K}^n$ be the space of convex bodies (that is, of compact convex sets) in $\mathbb{R}^n$ equipped with the Hausdorff metric and $\mathcal{K}_0^n$ the subspace of convex bodies in $\mathbb{R}^n$ that contain the origin in their interiors. A function z defined on a certain subset $\mathcal{C}$ of $\mathcal{K}^n$ and taking values in an abelian semigroup is called a *valuation* if

$$z(K) + z(L) = z(K \cup L) + z(K \cap L), \tag{1}$$

whenever $K, L, K \cup L, K \cap L \in \mathcal{C}$. A function $z : \mathcal{K}^n \times \mathcal{K}_0^n \to \langle \mathbb{R}, + \rangle$ is called a *bivaluation* if it is a valuation in both arguments. Valuations on convex bodies are a classical concept going back to Dehn's solution in 1900 of Hilbert's Third Problem. Starting with Hadwiger's celebrated classification of rigid motion invariant valuations and characterization

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of elementary mixed volumes, valuations have become a critical notion (see [**27**, **31**, **47**] and see [**1**, **2**, **3**, **4**, **5**, **6**, **11**, **15**, **29**, **30**, **49**, **51**, **52**, **53**] for some of the more recent contributions).

Invariant areas have strong invariance properties with respect to the general linear group, $\mathrm{GL}(n)$. In recent years, such *affine functions* on convex bodies have attracted increased interest (see, for example, [**10**, **14**, **18**, **21**, **25**, **26**, **41**, **42**, **43**, **45**, **46**, **56**, **57**]). Many of these affine functions on convex bodies can be completely characterized by their invariance and valuation properties (see, for example, [**22**, **23**, **24**, **33**, **34**, **35**, **36**, **37**, **38**, **40**]). The following result shows that the valuation property also completely characterizes Holmes-Thompson area.

Theorem 1. *A functional $z : \mathcal{K}^n \times \mathcal{K}_0^n \to \mathbb{R}$ is a bivaluation and an invariant area if and only if there is a constant $c > 0$ such that*

$$z(K, B) = c\, V(K, \dots, K, \Pi\, B^*)$$

for every $K \in \mathcal{K}^n$ and $B \in \mathcal{K}_0^n$.

Here $V(K_1, \dots, K_n)$ denotes the mixed volume of $K_1, \dots, K_n \in \mathcal{K}^n$ and B^* the polar body of $B \in \mathcal{K}_0^n$ (see Section 1). A convex body K is uniquely determined by its support function

$$h(K, v) = \max\{v \cdot x : x \in K\} \text{ for } v \in \mathbb{R}^n,$$

where $v \cdot x$ is the standard inner product of v and x. The *projection body*, $\Pi\, K$, of K is the convex body whose support function is given by

$$h(\Pi\, K, u) = V_{n-1}(K|u^\perp) \text{ for } u \in S^{n-1},$$

where V_{n-1} denotes $(n-1)$-dimensional volume, $K|u^\perp$ the image of the orthogonal projection of K onto the subspace orthogonal to u and S^{n-1} the unit sphere in $\mathbb{R}^n$. Projection bodies are an important tool in geometric tomography (see [**16**]) and have found intriguing applications in recent years (see [**54**, **57**]).

The invariant area obtained in Theorem 1 is the Holmes-Thompson area, which is a Minkowski area. Thus Theorem 1 implies that every invariant area that is also a bivaluation is a Minkowski area and that the Holmes-Thompson area is (up to multiplication with a positive constant) the unique Minkowski area that is a bivaluation.

Theorem 1 is a consequence of a new classification of valuations. An operator $\mathrm{Z} : \mathcal{K}_0^n \to \langle \mathcal{K}^n, + \rangle$ is called a *Minkowski valuation* if (1) holds and addition on $\mathcal{K}^n$ is Minkowski addition (defined for $K, L \in \mathcal{K}^n$ by $K + L = \{x + y : x \in K,\, y \in L\}$). An operator $\mathrm{Z} : \mathcal{K}_0^n \to \langle \mathcal{K}^n, + \rangle$ is called $\mathrm{GL}(n)$ *covariant*, if for some $q \in \mathbb{R}$,

$$\mathrm{Z}(\phi K) = |\det \phi|^q\, \phi\, \mathrm{Z}\, K \quad \text{for every } \phi \in \mathrm{GL}(n) \text{ and } K \in \mathcal{K}_0^n. \tag{2}$$

In [**36**], a classification of $\mathrm{GL}(n)$ covariant Minkowski valuations on the space of convex bodies containing the origin was obtained. Here we show that the space $\mathcal{K}_0^n$ allows additional important $\mathrm{GL}(n)$ covariant Minkowski valuations and establish a complete classification of such valuations.

Theorem 2. *An operator* $\mathrm{Z} : \mathcal{K}_0^n \to \langle \mathcal{K}^n, + \rangle$, *where* $n \geq 3$, *is a continuous non-trivial* $\mathrm{GL}(n)$ *covariant valuation if and only if either there are constants* $c_0 \geq 0$ *and* $c_1 \in \mathbb{R}$ *such that*

$$\mathrm{Z}\, K = c_0\, \mathrm{M}\, K + c_1 m(K)$$

for every $K \in \mathcal{K}_0^n$ *or there is a constant* $c_0 \geq 0$ *such that*

$$\mathrm{Z}\, K = c_0\, \Pi\, K^*$$

for every $K \in \mathcal{K}_0^n$.

A valuation is called *trivial* if it is a linear combination of the identity and central reflection. The *moment body*, $\mathrm{M}\, K$, of K is defined by

$$h(\mathrm{M}\, K, u) = \int_K |u \cdot x|\, dx, \ \text{ for } \ u \in S^{n-1}.$$

When divided by the volume of K, the moment body of K is called *centroid body* and is a classical and important notion going back to at least Dupin (see [**16**]). The vector $m(K) = \int_K x\, dx$ is called the *moment vector* of K.

In the proof of Theorem 2, a classification of Minkowski valuations on the space of convex polytopes containing the origin in their interiors is established. These results are contained in Sections 4 and 5. The definition of invariant areas and Minkowski areas is given in Section 2. The proof of Theorem 2 is given in Section 6 and makes essential use of the new classification of Minkowski valuations.

1. Notation and background material

General references on convex bodies are the books by Gardner [**16**], Gruber [**20**], Schneider [**48**], and Thompson [**55**]. We work in Euclidean n-space, $\mathbb{R}^n$, and write $x = (x_1, \ldots, x_n)$ for $x \in \mathbb{R}^n$. We denote by $e_1, \ldots, e_n$ the vectors of the standard basis of $\mathbb{R}^n$.

For $K \in \mathcal{K}^n$, it follows immediately from the definition of the support function that for every $s > 0$ and $\phi \in \mathrm{GL}(n)$,

$$h(K, s\, x) = s\, h(K, x) \ \text{ and } \ h(\phi K, x) = h(K, \phi^t x), \tag{3}$$

where ϕ^t is the transpose of ϕ. Support functions are sublinear, that is, for all $x, y \in \mathbb{R}^n$,

$$h(K, x + y) \leq h(K, x) + h(K, y).$$

Let $x, y \in \mathbb{R}^n$ be given and let K_t for $t \geq 0$ be convex bodies. If $h(K_t, \pm y) = o(1)$ as $t \to 0$, then the sublinearity of support functions implies that

$$h(K_t, x) - h(K_t, -y) \le h(K_t, x+y) \le h(K_t, x) + h(K_t, y).$$

Hence

$$h(K_t, x+y) = h(K_t, x) + o(1) \quad \text{as } t \to 0. \tag{4}$$

If $h(K_t, \pm y) = o(1)$ uniformly (in some parameter) as $t \to 0$, then also (4) holds uniformly.

For $K, L \in \mathcal{K}^n$, the mixed volume $V(K, \dots, K, L) = V_1(K, L)$ is defined by

$$V_1(K, L) = \frac{1}{n} \lim_{\varepsilon \to 0+} \frac{V_n(K + \varepsilon L) - V_n(K)}{\varepsilon},$$

where V_n denotes n-dimensional volume. The following properties are immediate consequences of the definition (cf. [**48**]). For $K, L \in \mathcal{K}^n$,

$$V_1(\phi K, \phi L) = |\det \phi| \, V_1(K, L) \quad \text{for every } \phi \in \mathrm{GL}(n) \tag{5}$$

and

$$V_1(K, r\,L) = r\, V_1(K, L) \quad \text{for every } r \ge 0. \tag{6}$$

If $K, L_1, L_2 \in \mathcal{K}^n$, then

$$V_1(K, L_1 + L_2) = V_1(K, L_1) + V_1(K, L_2). \tag{7}$$

If $P \in \mathcal{K}^n$ is a polytope with facets $F_1, \dots, F_m$ lying in hyperplanes with outer normal unit vectors $u_1, \dots, u_m$ and $L \in \mathcal{K}^n$,

$$V_1(P, L) = \frac{1}{n} \sum_{i=1}^m h(L, u_i) V_{n-1}(F_i). \tag{8}$$

Let $\mathcal{K}_c^n$ denote the set of origin-symmetric convex bodies in $\mathbb{R}^n$. An immediate consequence of the equality case of Minkowski's inequality (cf. [**48**]) is the following result: If $L_1, L_2 \in \mathcal{K}_c^n$ and

$$V_1(K, L_1) = V_1(K, L_2) \quad \text{for every } K \in \mathcal{K}^n, \tag{9}$$

then $L_1 = L_2$.

Let $\mathcal{P}_0^n$ denote the set of convex polytopes in $\mathbb{R}^n$ that contain the origin in their interiors. A function defined on $\mathcal{P}_0^n$ is called *measurable* if it is Borel measurable, that is, the pre-image of any open set is a Borel set in the space $\mathcal{P}_0^n$ equipped with the Hausdorff metric. We require the following results on valuations on $\mathcal{P}_0^1$. Let $\nu : \mathcal{P}_0^1 \to \mathbb{R}$ be a measurable valuation that is homogeneous of degree p, that is, $\nu(t\,I) = t^p\, \nu(I)$ for all $t > 0$ and $I \in \mathcal{P}_0^1$. If $p = 0$, then there are $a, b \in \mathbb{R}$ such that

$$\nu([-s, t]) = a \log(\frac{t}{s}) + b \tag{10}$$

for every $s, t > 0$. If $p \ne 0$, then there are $a, b \in \mathbb{R}$ such that

$$\nu([-s, t]) = a\, s^p + b\, t^p \tag{11}$$

for every $s, t > 0$ (cf. [**34**], equations (3) and (4)). These results follow from the fact that every measurable solution f of the Cauchy functional equation, $f(x+y) = f(x) + f(y)$, is linear.

An operator $\mathrm{Z} : \mathcal{P}_0^n \to \langle \mathcal{K}^n, + \rangle$ is called $\mathrm{GL}(n)$ *covariant of weight* q, if

$$\mathrm{Z}\,\phi P = |\det \phi|^q\, \phi\, \mathrm{Z}\, P \quad \text{for every } \phi \in \mathrm{GL}(n) \text{ and } P \in \mathcal{P}_0^n.$$

An operator $\mathrm{Z} : \mathcal{P}_0^n \to \langle \mathcal{K}^n, + \rangle$ is called $\mathrm{GL}(n)$ *contravariant of weight* q, if

$$\mathrm{Z}\,\phi P = |\det \phi|^q\, \phi^{-t}\, \mathrm{Z}\, P \quad \text{for every } \phi \in \mathrm{GL}(n) \text{ and } P \in \mathcal{P}_0^n, \tag{12}$$

where ϕ^{-t} is the inverse of the transpose of ϕ. Let $\mathrm{Z} : \mathcal{P}_0^n \to \langle \mathcal{K}^n, + \rangle$ be a valuation which is $\mathrm{GL}(n)$ contravariant of weight q. We associate with Z an operator Z^* in the following way. For $P \in \mathcal{P}_0^n$, the polar body, P^*, of P is defined by

$$P^* = \{x \in \mathbb{R}^n : x \cdot y \le 1 \quad \text{for every } y \in P\}.$$

We define the operator $\mathrm{Z}^* : \mathcal{P}_0^n \to \langle \mathcal{K}^n, + \rangle$ by setting $\mathrm{Z}^*\, P = \mathrm{Z}\, P^*$ for $P \in \mathcal{P}_0^n$ and show that

(13) $\mathrm{Z}^* : \mathcal{P}_0^n \to \langle \mathcal{K}^n, + \rangle$ is a valuation $\mathrm{GL}(n)$ covariant of weight $-q$.

It follows immediately from the definition of polar body that for $P \in \mathcal{P}_0^n$,

$$(\phi P)^* = \phi^{-t} P^* \quad \text{for every } \phi \in \mathrm{GL}(n) \tag{14}$$

and that for $P, Q, P \cup Q \in \mathcal{P}_0^n$,

$$(P \cup Q)^* = P^* \cap Q^* \quad \text{and} \quad (P \cap Q)^* = P^* \cup Q^*. \tag{15}$$

Since Z is a valuation, for $P, Q \in \mathcal{P}_0^n$ with $P \cup Q \in \mathcal{P}_0^n$ it follows from (15) that

$$\begin{aligned} \mathrm{Z}^*\, P + \mathrm{Z}^*\, Q &= \mathrm{Z}\, P^* + \mathrm{Z}\, Q^* \\ &= \mathrm{Z}(P^* \cup Q^*) + \mathrm{Z}(P^* \cap Q^*) \\ &= \mathrm{Z}(P \cap Q)^* + \mathrm{Z}(P \cup Q)^* \\ &= \mathrm{Z}^*(P \cap Q) + \mathrm{Z}^*(P \cup Q). \end{aligned}$$

Thus Z^* is also a valuation. Since Z is $\mathrm{GL}(n)$ contravariant of weight q, it follows from (14) that

$$\mathrm{Z}^*(\phi P) = \mathrm{Z}(\phi P)^* = \mathrm{Z}(\phi^{-t} P^*) = |\det \phi|^{-q} \phi\, \mathrm{Z}^*\, P.$$

Thus (13) holds.

For $n = 2$, we also associate with $\mathrm{Z} : \mathcal{P}_0^2 \to \langle \mathcal{K}^2, + \rangle$ an operator $\mathrm{Z}^\perp$. Suppose that Z is a valuation which is $\mathrm{GL}(2)$ contravariant of weight q. Let $\rho_{\pi/2}$ denote the rotation by the angle $\pi/2$ and $\rho_{-\pi/2}$ its inverse. Define $\mathrm{Z}^\perp : \mathcal{P}_0^2 \to \langle \mathcal{K}^2, + \rangle$ by setting $\mathrm{Z}^\perp\, P = \rho_{-\pi/2}\, \mathrm{Z}\, P$ for $P \in \mathcal{P}_0^2$. We show that

(16) $\mathrm{Z}^\perp : \mathcal{P}_0^2 \to \langle \mathcal{K}^2, + \rangle$ is a valuation $\mathrm{GL}(2)$ covariant of weight $q-1$.

Since Z is a valuation, we have for $P, Q \in \mathcal{P}_0^2$ with $P \cup Q \in \mathcal{P}_0^2$,

$$\begin{aligned} \mathrm{Z}^\perp P + \mathrm{Z}^\perp Q &= \rho_{-\pi/2}\,\mathrm{Z}\,P + \rho_{-\pi/2}\,\mathrm{Z}\,Q \\ &= \rho_{-\pi/2}(\mathrm{Z}(P) + \mathrm{Z}(Q)) \\ &= \rho_{-\pi/2}(\mathrm{Z}(P \cup Q) + \mathrm{Z}(P \cap Q)) \\ &= \mathrm{Z}^\perp(P \cup Q) + \mathrm{Z}^\perp(P \cap Q). \end{aligned}$$

Thus $\mathrm{Z}^\perp$ is also a valuation. For $\phi \in \mathrm{GL}(2)$ and $P \in \mathcal{P}_0^2$,

$$\mathrm{Z}^\perp(\phi P) = |\det \phi|^q\,\rho_{-\pi/2}\,\phi^{-t}\rho_{\pi/2}\,\mathrm{Z}^\perp\,P = |\det \phi|^{q-1}\phi\,\mathrm{Z}^\perp\,P.$$

Hence $\mathrm{Z}^\perp$ is $\mathrm{GL}(2)$ covariant of weight $q-1$. Thus (16) holds.

In the proof of Theorem 2, we make use of the following results.

Theorem 1.1 ([**34**]). *A functional $z : \mathcal{P}_0^n \to \langle \mathbb{R}, + \rangle$, where $n \geq 2$, is a measurable valuation so that for some $q \in \mathbb{R}$,*

$$z(\phi P) = |\det \phi|^q\, z(P) \tag{17}$$

holds for every $\phi \in \mathrm{GL}(n)$ if and only if there is a constant $c \in \mathbb{R}$ such that

$$z(P) = c \quad \text{or} \quad z(P) = c\,V_n(P) \quad \text{or} \quad z(P) = c\,V_n(P^*)$$

for every $P \in \mathcal{P}_0^n$.

Theorem 1.2 ([**32**]). *Let $z : \mathcal{P}_0^2 \to \langle \mathbb{R}^2, + \rangle$ be a measurable valuation which is $\mathrm{GL}(2)$ covariant of weight q. If $q = 1$, then there is a constant $c \in \mathbb{R}$ such that*

$$z(P) = c\,m(P)$$

for every $P \in \mathcal{P}_0^2$. If $q = -2$, then there is a constant $c \in \mathbb{R}$ such that

$$z(P) = c\,\rho_{-\pi/2}\,m(P^*)$$

for every $P \in \mathcal{P}_0^2$. In all other cases, $z(P) = \{0\}$ for every $P \in \mathcal{P}_0^2$.

Theorem 1.3 ([**32**]). *Let $z : \mathcal{P}_0^n \to \langle \mathbb{R}^n, + \rangle$, where $n \geq 3$, be a measurable valuation which is $\mathrm{GL}(n)$ covariant of weight q. If $q = 1$, then there is a constant $c \in \mathbb{R}$ such that*

$$z(P) = c\,m(P)$$

for every $P \in \mathcal{P}_0^n$. In all other cases, $z(P) = \{0\}$ for every $P \in \mathcal{P}_0^n$.

2. Background material on invariant areas

Let $(\mathbb{R}^n, \|\cdot\|)$ be a normed space and $B = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$ its unit ball. An axiomatic approach to Minkowski areas following Busemann's ideas is presented in Thompson's book [**55**]. This approach has two important components. First, the notion of surface area is independent of the choice of a Euclidean coordinate system and has certain regularity properties. We call functionals with these properties *invariant areas.* Second, a Minkowski area assigns a notion of area in an *intrinsic* way. In [**55**], notions of volume for all Banach spaces in all dimensions are discussed. Here we restrict our attention to n-dimensional spaces and extend the definition of area to spaces with not necessarily origin-symmetric unit balls. For a hyperplane H, let $\mathcal{K}(H)$ denote the set of convex bodies contained in H.

Definition. *A functional $z : \mathcal{K}^n \times \mathcal{K}_0^n \to [0,\infty)$ is an* invariant area *if*

(i) $z(K,B) = z(\phi K, \phi B)$ *for all* $\phi \in \mathrm{GL}(n)$.
(ii) $z(K,B) = z(K+x, B)$ *for all* $x \in \mathbb{R}^n$.
(iii) z *is continuous in both variables.*
(iv) *for every hyperplane H, there exists a constant $c_H > 0$ such that* $z(\cdot, B) = c_H V_{n-1}$ *on* $\mathcal{K}(H)$.
(v) *for every polytope $P \in \mathcal{K}^n$ with facets $F_1, \dots, F_m$,* $z(F_1, B) \le \sum_{i=2}^m z(F_i, B)$.

An invariant area $z : \mathcal{K}^n \times \mathcal{K}_0^n \to [0,\infty)$ is a *Minkowski area* if it is *intrinsic*, that is, for every $(n-1)$-dimensional subspace H and every $B, B' \in \mathcal{K}_0^n$ satisfying $B' \cap H = B \cap H$, we have $z(K,B) = z(K,B')$ for every $K \in \mathcal{K}(H)$.

3. Extension

Let $\overline{\mathcal{P}}_0^n$ denote the set of convex polytopes P which are either in $\mathcal{P}_0^n$ or are the intersection of a polytope $P_0 \in \mathcal{P}_0^n$ and a polyhedral cone with apex at the origin. Here a polyhedral cone with apex at the origin is the intersection of finitely many closed halfspaces containing the origin in their boundaries. As a first step, we extend the valuation $\mathrm{Z} : \mathcal{P}_0^n \to \langle \mathcal{K}^n, + \rangle$ to a valuation on $\overline{\mathcal{P}}_0^n$.

We need the following definitions. For $A_1, \dots, A_k \subset \mathbb{R}^n$, we denote the convex hull of $A_1, \dots, A_k$ by $[A_1, \dots, A_k]$. For the convex hull of $A \subset \mathbb{R}^n$ and $u_1, \dots, u_k \in \mathbb{R}^n$, we write $[A, u_1, \dots, u_k]$. For $A \subset \mathbb{R}^n$, let

$$A^\perp = \{x \in \mathbb{R}^n : x \cdot y = 0 \quad \text{for every} \;\; y \in A\}.$$

For a hyperplane H containing the origin, let H^+ and H^- denote the complementary closed halfspaces bounded by H. Let $\mathcal{P}_0(H)$ denote the set of convex polytopes in H that contain the origin in their interiors relative to H. For $v \in \mathbb{R}^n$, let $\langle v \rangle$ denote the linear hull of v and $\mathcal{P}_0(v)$ the set of intervals in $\langle v \rangle$ that contain the origin in their interiors.

Let $C_1(\mathbb{R}^n)$ be the set of continuous functions $f : \mathbb{R}^n \to \mathbb{R}$ that are positively homogeneous of degree 1, that is, $f(t\,x) = t\,f(x)$ for all $t \ge 0$ and $x \in \mathbb{R}^n$. Since $h(K_1 + K_2, \cdot) = h(K_1, \cdot) + h(K_2, \cdot)$ for $K_1, K_2 \in \mathcal{K}^n$, we see that if $\mathrm{Z} : \mathcal{P}_0^n \to \langle \mathcal{K}^n, + \rangle$ is a valuation, then the operator Y on $\mathcal{P}_0^n$ defined by $\mathrm{Y}\,P = h(\mathrm{Z}\,P, \cdot)$ is a valuation taking values in $C_1(\mathbb{R}^n)$.

An operator $\mathrm{Y} : \mathcal{P}_0^n \to C_1(\mathbb{R}^n)$ is called $\mathrm{GL}(n)$ covariant of weight q if

$$\mathrm{Y}(\phi P) = |\det \phi|^q \ \mathrm{Y}\, P \circ \phi^t$$

for every $\phi \in \mathrm{GL}(n)$ and $P \in \mathcal{P}_0^n$. It is called $\mathrm{GL}(n)$ contravariant of weight q if

$$\mathrm{Y}(\phi P) = |\det \phi|^q \ \mathrm{Y}\, P \circ \phi^{-1}$$

for every $\phi \in \mathrm{GL}(n)$ and $P \in \mathcal{P}_0^n$.

Let H be a hyperplane containing the origin and $\mathrm{Y} : \mathcal{P}_0^n \to C_1(\mathbb{R}^n)$. We say that

$$\mathrm{Y} \text{ has the } \textit{Cauchy property} \text{ for } \mathcal{P}_0(H)$$

if for every $\varepsilon > 0$ and centered ball B, there exists $\delta > 0$ (depending only on ε and B) so that

$$\max_{x \in S^{n-1}} \big| \mathrm{Y}[P, u, v](x) - \mathrm{Y}[P, u', v'](x) \big| < \varepsilon \tag{18}$$

for every $u, u' \in H^- \backslash H$ and $v, v' \in H^+ \backslash H$ with $|u|, |u'|, |v|, |v'| < \delta$ and for every $P \in \mathcal{P}_0(H)$ with $P \subset B$. Let $w \in H^+ \backslash H$. We say that Y has the Cauchy property for $\mathcal{P}_0(H)$ with respect to $\langle w \rangle$, if (18) holds for $u, u', v, v' \in \langle w \rangle$.

Lemma 3.1. *If* $\mathrm{Y} : \mathcal{P}_0^n \to \langle C_1(\mathbb{R}^n), + \rangle$ *is a valuation so that for every hyperplane* H *containing the origin,*

$$\mathrm{Y} \textit{ has the Cauchy property for } \mathcal{P}_0(H), \tag{19}$$

then Y *can be extended to a valuation on* $\overline{\mathcal{P}}{}_0^n$. *Moreover, if* Y *is* $\mathrm{GL}(n)$ *covariant on* $\mathcal{P}_0^n$, *so is the extended valuation on* $\overline{\mathcal{P}}{}_0^n$. *If* Y *is* $\mathrm{GL}(n)$ *contravariant on* $\mathcal{P}_0^n$, *so is the extended valuation on* $\overline{\mathcal{P}}{}_0^n$.

Proof. For $j = 1, \dots, n$, let $\mathcal{P}_j^n$ be the set of convex polytopes P containing the origin such that there exist $P_0 \in \mathcal{P}_0^n$ and hyperplanes $H_1, \dots, H_i$, where $i \le j$, containing the origin with linearly independent normal vectors and either

$$P = P_0 \cap H_1^+ \cap \dots \cap H_i^+ \tag{20}$$

or

$$P = P_0 \cap H_1^+ \cap \dots \cap H_{i-1}^+ \cap H_i. \tag{21}$$

Note that $H_1^+ \cap \dots \cap H_i^+$ is the support cone at 0 of the polytope P defined by (20). For a hyperplane H and $j = 1, \dots, n-1$, let $\mathcal{P}_j(H)$ be the set of convex polytopes P such that there exist $P_0 \in \mathcal{P}_0^n$ and hyperplanes $H_1, \dots, H_i$, where $i \le j$, containing the origin with linearly independent normal vectors such that

$$P = P_0 \cap H_1^+ \cap \dots \cap H_i^+ \cap H. \tag{22}$$

Note that $\mathcal{P}_j(H) \subset \mathcal{P}^n_{j+1}$. If Y is defined on $\mathcal{P}^n_j$ for $j \in \{1, \ldots, n-1\}$ and H is a hyperplane containing the origin, we say that

$$\text{Y has the Cauchy property for } \mathcal{P}_j(H),$$

if, for every $\varepsilon > 0$ and centered ball B, there exists $\delta > 0$ (depending only on ε and B) so that for every $P \in \mathcal{P}_j(H)$ defined by (22) with $P \subset B$, we have

$$\max_{x \in S^{n-1}} \big| \mathrm{Y}[P, u, v](x) - \mathrm{Y}[P, u', v'](x) \big| < \varepsilon$$

for every $u, u', v, v' \in H_1 \cap \cdots \cap H_i$ with $u, u' \in H^- \backslash H$ and $v, v' \in H^+ \backslash H$ such that $|u|, |u'|, |v|, |v'| < \delta$.

Define Y on $\mathcal{P}^n_j$ for $j = 1, \ldots, n$ inductively, starting with $j = 1$, in the following way. For P given by (20) or (21) with j hyperplanes, set

$$\mathrm{Y}\, P = \lim_{u,v \to 0} \mathrm{Y}[P, u, v], \tag{23}$$

where $u, v \in H_1 \cap \cdots \cap H_{j-1}$ and $u \in H^-_j \backslash H_j$ and $v \in H^+_j \backslash H_j$.

We show that the limit in (23) exists uniformly on S^{n-1} and does not depend on the choice of H_j among $H_1, \ldots, H_j$ and that for every hyperplane H containing the origin,

$$\text{Y has the Cauchy property for } \mathcal{P}_j(H). \tag{24}$$

In addition, we show Y has the following additivity properties:
If $P \in \mathcal{P}^n_{j-1}$ and H is a hyperplane such that $P \cap H^+, P \cap H^- \in \mathcal{P}^n_j$, then

$$\mathrm{Y}\, P + \mathrm{Y}(P \cap H) = \mathrm{Y}(P \cap H^+) + \mathrm{Y}(P \cap H^-). \tag{25}$$

If $P, Q, P \cap Q, P \cup Q \in \mathcal{P}^n_j$, where $j < n$, are defined by (20) by the same halfspaces $H^+_1, \ldots, H^+_j$, then

$$\mathrm{Y}\, P + \mathrm{Y}\, Q = \mathrm{Y}(P \cup Q) + \mathrm{Y}(P \cap Q). \tag{26}$$

The operator Y is well defined and a valuation on $\mathcal{P}^n_0$ and properties (24) and (26) hold for $j = 0$ by assumption. Let $1 \le k \le n$. Suppose that Y is well defined by (23) on $\mathcal{P}^n_{k-1}$. Further suppose that we have (24) and (26) for $0 \le j < k$ and that we have (25) for $1 \le j < k$.

First, we show that the limit in (23) exists uniformly on S^{n-1}. For $P \in \mathcal{P}^n_k$ given by (21), this follows from (24) for $j = k - 1$. So, let $P \in \mathcal{P}^n_k$ be given by (20) with $i = k$. Note that $[P, u, v] = [P, u]$ for $|v|$ suitably small. Suppose that $\bar{u} \in H_1 \cap \cdots \cap H_{k-1}$ with $\bar{u} \in H^-_k \backslash H_k$ is chosen such that $[P, u] \subseteq [P, \bar{u}]$ and $-\bar{u} \in P$. Then applying (26) with $j = k - 1$ gives

$$\mathrm{Y}[P, u] + \mathrm{Y}[P \cap H_k, \bar{u}, -\bar{u}] = \mathrm{Y}[P, \bar{u}] + \mathrm{Y}[P \cap H_k, u, -\bar{u}].$$

Consequently, it follows from (24) for $j = k-1$ that (23) exists uniformly on S^{n-1} and that $\mathrm{Y}\, P$ is continuous on S^{n-1}. For $k > 1$ and $P \in \mathcal{P}^n_k$ given by (20) with $i = k$, we show that $\mathrm{Y}\, P$ as defined by (23) does not depend on the choice of the hyperplane H_k among $H_1, \dots, H_k$. Let $u \in H_1 \cap \dots \cap H_{k-1}$ and $u \in H_k^- \backslash H_k$. Choose $w \in H_2 \cap \dots \cap H_k$ with $w \in H_1^- \backslash H_1$. Then applying (25) for $j = k - 1$ gives

$$\mathrm{Y}[P, u, w] + \mathrm{Y}[P \cap H_k, w] = \mathrm{Y}[P, w] + \mathrm{Y}[P \cap H_k, u, w].$$

Hence, by (24) for $j = k - 1$, there exists $\delta > 0$ such that on S^{n-1},

$$|\,\mathrm{Y}[P, u, w] - \mathrm{Y}[P, w]| = |\,\mathrm{Y}[P \cap H_k, u, w] - \mathrm{Y}[P \cap H_k, w]| < \varepsilon,$$

whenever $|u| < \delta$ and $[P \cap H_k, w]$ is contained in a suitable ball. Hence

$$\lim_{u,w \to 0} \mathrm{Y}[P, u, w] = \lim_{w \to 0} \mathrm{Y}[P, w].$$

Thus Y is well defined on $\mathcal{P}_k^n$.

Next, we show that (24) holds for $j = k$. Let $\varepsilon > 0$ and a centered ball B be chosen. Suppose that $P \in \mathcal{P}_k(H)$ and that

$$P = P_0 \cap H_1^+ \cap \dots \cap H_k^+ \cap H$$

where $H, H_1, \dots, H_k$ have linearly independent normal vectors. Choose $z \in H \cap H_1 \cap \dots \cap H_{k-1}$ with $z \in H_k^- \backslash H_k$. Then $[P, z] \in \mathcal{P}_{k-1}(H)$ and by (24) for $j = k-1$, there exists $\delta > 0$ so that for $u_i, v_i \in H_1 \cap \dots \cap H_{k-1}$ with $u_i \in H^- \backslash H$ and $v_i \in H^+ \backslash H$ for $i = 1, 2$, we have on S^{n-1},

$$|\,\mathrm{Y}[P, z, u_1, v_1] - \mathrm{Y}[P, z, u_2, v_2]| < \varepsilon, \tag{27}$$

whenever $|u_i|, |v_i| < \delta$ and $[P, z]$ is contained in B. By (23), letting $z \to 0$ in (27) shows that (24) holds for $j = k$.

Next, we show that (25) holds for $j = k$. Let $P \in \mathcal{P}_{k-1}^n$, that is, there exist $P_0 \in \mathcal{P}_0^n$ and $H_1, \dots, H_{k-1}$ such that $P = P_0 \cap H_1^+ \cap \dots \cap H_{k-1}^+$. Choose $u \in H_1 \cap \dots \cap H_{k-1}$, such that $u \in P \cap H^+ \backslash H$ and $-u \in P \cap H^-$. The four polytopes P, $[P \cap H, u, -u]$, $[P \cap H^+, -u]$, $[P \cap H^-, u]$ have the hyperplanes $H_1, \dots, H_{k-1}$ in common. Applying (26) for $j = k - 1$ gives

$$\mathrm{Y}\, P + \mathrm{Y}[P \cap H, u, -u] = \mathrm{Y}[P \cap H^+, -u] + \mathrm{Y}[P \cap H^-, u].$$

By (24) and definition (23), this implies that (25) holds for $j = k$.

Finally, we show that (26) holds for $j = k$. Choose $u \in H_1 \cap \dots \cap H_{k-1}$ with $u \notin H_k$ such that $-u \in P \cap Q$. Applying (26) for $j = k - 1$ shows that

$$\mathrm{Y}[P, u] + \mathrm{Y}[Q, u] = \mathrm{Y}[P \cup Q, u] + \mathrm{Y}[P \cap Q, u].$$

Because of definition (23) this implies that (26) holds for $j = k$.

The induction is now complete and Y is extended to $\mathcal{P}_n^n$. Let $P \in \mathcal{P}_n^n$ and let H is a hyperplane such that $P \cap H^+, P \cap H^- \in \mathcal{P}_n^n$. We show that

$$\mathrm{Y}\, P + \mathrm{Y}(P \cap H) = \mathrm{Y}(P \cap H^+) + \mathrm{Y}(P \cap H^-). \tag{28}$$

Since (25) holds, it suffices to show (28) for a polytope P whose support cone at 0 is bounded by n facets with linearly independent normal vectors.

First, let $n = 2$. Let $P = P_0 \cap H_1^+ \cap H_2^+$, where H_1 and H_2 are lines containing the origin and $P_0 \in \mathcal{P}_0^2$. Further, let $P \cap H^+ = H_1^+ \cap H^+ \cap P_0$ and $P \cap H^- = H_2^+ \cap H^- \cap P_0$. For $u \in H \cap (H_1^- \backslash H_1) \cap (H_2^- \backslash H_2)$, it follows from (25) that

$$\mathrm{Y}[P, u] + \mathrm{Y}[P \cap H, u] = \mathrm{Y}[P \cap H^+, u] + \mathrm{Y}[P \cap H^-, u].$$

By (23), this implies that

$$\lim_{u \to 0} \mathrm{Y}[P, u] + \mathrm{Y}[P \cap H] = \mathrm{Y}(P \cap H^+) + \mathrm{Y}(P \cap H^-). \tag{29}$$

On the other hand, it follows from (25) that

$$\mathrm{Y}[P, u] + \mathrm{Y}[P \cap H_1, w] = \mathrm{Y}[P, w] + \mathrm{Y}[P \cap H_1, u, w], \tag{30}$$

where $w \in H_1$ depends on u. Let B be a centered ball such that $[P \cap H_1, w] \subset B$ for $|u| < 1$ and let $\varepsilon > 0$. Since Y has the Cauchy property for $\mathcal{P}_0(H_1)$, it follows from (23) that on S^1

$$|\,\mathrm{Y}[P \cap H_1, w] - \mathrm{Y}[P \cap H_1, u, w]| < \varepsilon$$

for $|u|$ sufficiently small. Thus, by (23), we obtain from (30) that $\lim_{u \to 0} \mathrm{Y}[P, u] = \mathrm{Y}\, P$. Combined with (29) this implies (28).

Second, let $n \geq 3$. Let $P = P_0 \cap H_1^+ \cap \cdots \cap H_n^+$, where $P_0 \in \mathcal{P}_0^n$. Since $P \cap H^+$, $P \cap H^- \in \mathcal{P}_n^n$, we can say that the support cone at 0 of $P \cap H^+$ is bounded by $H_1, H, H_3, \ldots, H_n$ and that the support cone at 0 of $P \cap H^-$ is bounded by $H, H_2, \ldots, H_n$, where $H_1 \cap H_2 \cap \cdots \cap H_{n-1} \subseteq H$. Therefore $\mathrm{Y}\, P = \lim_{u \to 0} \mathrm{Y}[P, u]$ and

$$\mathrm{Y}(P \cap H^+) = \lim_{u \to 0} \mathrm{Y}[P \cap H^+, u], \;\; \mathrm{Y}(P \cap H^-) = \lim_{u \to 0} \mathrm{Y}[P \cap H^-, u],$$

where $u \in H_1 \cap H_2 \cap \cdots \cap H_{n-1}$ and $u \in H_n^- \backslash H_n$. Applying (25) for $j = n$ shows that

$$\mathrm{Y}[P, u] + \mathrm{Y}[P \cap H, u] = \mathrm{Y}[P \cap H^+, u] + \mathrm{Y}[P \cap H^-, u].$$

Because of definition (23) this implies (28).

As a last step, we extend Y to a valuation on $\overline{\mathcal{P}}_0^n$. For $P \in \overline{\mathcal{P}}_0^n$, there are $P_1, \ldots, P_m \in \mathcal{P}_n^n$ such that $P = P_1 \cup \cdots \cup P_m$. It is proved in [**39**] that defining $h(\mathrm{Z}\, P, \cdot)$ by the inclusion-exclusion principle with $h(\mathrm{Z}\, P_i, \cdot)$, where $i = 1, \ldots, m$, leads to a well defined extension of Z on $\overline{\mathcal{P}}_0^n$. Clearly, the extension is $\mathrm{GL}(n)$ covariant ($\mathrm{GL}(n)$ contravariant) if Z is $\mathrm{GL}(n)$ covariant ($\mathrm{GL}(n)$ contravariant) on $\mathcal{P}_n^n$. This completes the proof of the lemma. q.e.d.

The following result is an immediate consequence of Lemma 3.1. Let $C_1^+(\mathbb{R}^n)$ denote the subset of non-negative functions in $C_1(\mathbb{R}^n)$.

Lemma 3.2. *If* $\mathrm{Y} : \mathcal{P}_0^n \to \langle\, C_1^+(\mathbb{R}^n), +\rangle$ *is a valuation so that for every hyperplane* H *containing the origin and* $w \in H^+\backslash H$*, the operator* Y *has the Cauchy property for* $\mathcal{P}_0(H)$ *with respect to* $\langle w\rangle$ *and*

$$\lim_{u,v\to 0} \mathrm{Y}[P,u,v] = 0 \quad \text{for every } P \in \mathcal{P}_0(H),$$

where $u, v \in \langle w\rangle$ *with* $u \in H^+\backslash H$ *and* $v \in H^-\backslash H$*, then* Y *can be extended to a valuation on* $\overline{\mathcal{P}}_0^n$*. Moreover, if* Y *is* $\mathrm{GL}(n)$ *covariant on* $\mathcal{P}_0^n$*, so is the extended valuation on* $\overline{\mathcal{P}}_0^n$*. If* Y *is* $\mathrm{GL}(n)$ *contravariant on* $\mathcal{P}_0^n$*, so is the extended valuation on* $\overline{\mathcal{P}}_0^n$*.*

Proof. Let H be a hyperplane containing the origin and $P \in \mathcal{P}_0(H)$. Since Y is a non-negative valuation, for $u \in H^-\backslash H$ and $v \in H^+\backslash H$,

$$\begin{aligned} \mathrm{Y}[P,u,v] &= \mathrm{Y}[P,u,-t\,u] + \mathrm{Y}[P,-s\,v,v] - \mathrm{Y}[P,-s\,v,-t\,u] \\ &\le \mathrm{Y}[P,u,-t\,u] + \mathrm{Y}[P,-s\,v,v], \end{aligned}$$

when $t, s > 0$ are chosen suitably small. Thus (18) holds and Lemma 3.1 implies the existence of an extension. q.e.d.

4. The classification on $\mathcal{P}_0^2$

In Section 4.2, we prove the following result.

Proposition 4.1. *An operator* $\mathrm{Z} : \mathcal{P}_0^2 \to \langle \mathcal{K}^2, +\rangle$ *is a measurable valuation which is* $\mathrm{GL}(2)$ *covariant of weight* $q \ge 0$ *if and only if there are* $c_0, c_1 \ge 0$ *and* $c_2 \in \mathbb{R}$ *such that*

$$\mathrm{Z}\,P = \begin{cases} c_1\,\mathrm{M}\,P + c_2\,m(P) & \text{for } q = 1 \\ c_0\,P + c_1\,(-P) & \text{for } q = 0 \\ \{0\} & \text{otherwise} \end{cases}$$

for every $P \in \mathcal{P}_0^2$*.*

By (16), Proposition 4.1 has the following immediate consequence.

Proposition 4.2. *An operator* $\mathrm{Z} : \mathcal{P}_0^2 \to \langle \mathcal{K}^2, +\rangle$ *is a measurable valuation which is* $\mathrm{GL}(2)$ *contravariant of weight* $q \ge 1$ *if and only if there are* $c_0, c_1 \ge 0$ *and* $c_2 \in \mathbb{R}$ *such that*

$$\mathrm{Z}\,P = \begin{cases} \rho_{\pi/2}(c_1\,\mathrm{M}\,P + c_2 m(P)) & \text{for } q = 2 \\ \rho_{\pi/2}(c_0 P + c_1(-P)) & \text{for } q = 1 \\ \{0\} & \text{otherwise} \end{cases}$$

for every $P \in \mathcal{P}_0^2$*.*

By (13), Proposition 4.2 is equivalent to the classification of GL(2) covariant valuations of weight $q \le -1$. Hence Propositions 4.1 and 4.2 imply the following theorem. Note that the case $q \in (-1, 0)$ remains open.

Theorem 4.3. *An operator* $\mathrm{Z} : \mathcal{P}_0^2 \to \langle \mathcal{K}^2, + \rangle$ *is a measurable valuation which is* $\mathrm{GL}(2)$ *covariant of weight* $q \in (-\infty, -1] \cup [0, \infty)$ *if and only if there are* $c_0, c_1 \ge 0$ *and* $c_2 \in \mathbb{R}$ *such that*

$$\mathrm{Z}\, P = \begin{cases} c_1 \,\mathrm{M}\, P + c_2\, m(P) & \text{for } q = 1 \\ c_0 P + c_1(-P) & \text{for } q = 0 \\ \rho_{\pi/2}(c_0 P^* + c_1(-P^*)) & \text{for } q = -1 \\ \rho_{\pi/2}(c_1 \,\mathrm{M}\, P^* + c_2 m(P^*)) & \text{for } q = -2 \\ \{0\} & \text{otherwise} \end{cases}$$

for every $P \in \mathcal{P}_0^2$.

In the proof of Proposition 4.1 the following result is used.

Theorem 4.4 ([**36**])**.** *Let* $\mathrm{Z} : \overline{\mathcal{P}}_0^2 \to \langle \mathcal{K}^2, + \rangle$ *be a valuation which is* $\mathrm{GL}(n)$ *covariant of weight* q*. If* $q = 1$*, then there are constants* $c_1 \ge 0$ *and* $c_2 \in \mathbb{R}$ *such that*

$$\mathrm{Z}\, P = c_1 \,\mathrm{M}\, P + c_2\, m(P)$$

for every $P \in \overline{\mathcal{P}}_0^2$*. If* $q = 0$*, then there are constants* $a_0, b_0 \ge 0$ *and* $a_i, b_i \in \mathbb{R}$ *with* $a_i + b_0 + b_1 \ge 0$ *and* $a_0 + a_1 + b_i \ge 0$ *for* $i = 1, 2$ *such that*

$$\mathrm{Z}\, P = a_0\, P + b_0(-P) + \sum\nolimits_{i=1,2} (a_i\, \mathrm{E}_i(P) + b_i\, \mathrm{E}_i(-P))$$

for every $P \in \overline{\mathcal{P}}_0^2$*. In all other cases,* $\mathrm{Z}\, P = \{0\}$ *for every* $P \in \overline{\mathcal{P}}_0^2$*.*

Here E_1 and E_2 are certain operators that map each $P \in \mathcal{P}_0^2$ to $\{0\}$.

4.1. An auxiliary result. We derive a lemma that is also used for $n \ge 3$. For $v = (v', 1)$ with $v' \in \mathbb{R}^{n-1}$, define the map $\phi_v \in \mathrm{SL}(n)$ by $\phi_v e_n = v$ and $\phi_v e_i = e_i$ for $i = 1, \dots, n-1$. For $r > 0$, define the map $\phi_r \in \mathrm{GL}(n)$ by $\phi_r e_n = r\, e_n$ and $\phi_r e_i$ to e_i for $i = 1, \dots, n-1$. We say that a function $z : \mathcal{P}_0^n \to \langle \mathbb{R}, + \rangle$ is ϕ_r homogeneous of degree p, if

$$z(\phi_r[P, s\, u, t\, v]) = r^p z([P, s\, u, t\, v])$$

for all $P \in \mathcal{P}_0(e_n^\perp)$, for all $r > 0$ and for all $u = (u', -1)$ and $v = (v', 1)$ with $u', v' \in \mathbb{R}^{n-1}$. Let B be a centered ball.

Lemma 4.5. *Suppose $z : \mathcal{P}_0^n \to \langle \mathbb{R}, + \rangle$ is a measurable valuation which is ϕ_r homogeneous of degree $p \neq 1$ and $v = (v', 1)$ with $v' \in \mathbb{R}^{n-1}$. If*

$$\lim_{s,t \to 0} z([P, -s\, e_n, t\, e_n])$$

exists uniformly for all $P \in \mathcal{P}_0(e_n^\perp)$ with $P \subset B$ and

$$\lim_{s,t \to 0} z(\phi_v [P, -s\, e_n, t\, e_n]) = z([P, -s\, e_n, t\, e_n])$$

uniformly for all $P \in \mathcal{P}_0(e_n^\perp)$ with $P \subset B$, then

$$\lim_{s,t \to 0} z([P, s\, u, t\, v]) = \lim_{s,t \to 0} z([P, -s\, e_n, t\, e_n])$$

uniformly for all $P \in \mathcal{P}_0(e_n^\perp)$ with $P \subset B$ and all $u = (u', -1)$ with $u' \in \mathbb{R}^{n-1}$.

Proof. First, we show that

$$\lim_{s,t \to 0} z([P, s\, u, t\, v]) \tag{31}$$

exists uniformly for $P \subset B$. Since z is a valuation and since u and v lie in complementary halfspaces, we have for $s, t > 0$ suitably small, $0 < t' < t$, and $t'' > 0$ suitably large with respect to s,

$$\begin{aligned} z([P, s\, u, t\, v]) &+ z([P, -t''\, v, t'\, v]) \\ &= z([P, s\, u, t'\, v]) + z([P, -t''\, v, t\, v]). \end{aligned} \tag{32}$$

Since $[P, -t''\, v, t\, v] = \phi_v [P, I]$ for $I = [-t''\, e_n, t\, e_n]$, we have by assumption

$$z([P, -t''\, v, t\, v]) = z(\phi_v [P, I]) = z([P, I]) + o(1) \tag{33}$$

as $t, t'' \to 0$ uniformly for $P \subset B$. Since $\lim_{s,t \to 0} z([P, I])$ exists uniformly for $P \subset B$, we obtain from (32) and (33) that for $t', t'' = O(t)$

$$\begin{aligned} z([P, s\, u, t\, v]) &- z([P, s\, u, t'\, v]) \\ &= z([P, [-t''\, e_n, t\, e_n]]) - z([P, [-t''\, e_n, t'\, e_n]]) + o(1) = o(1) \end{aligned}$$

as $t \to 0$ uniformly for $P \subset B$. Similarly, for $s, t' > 0$ suitably small, $0 < s' < s$,

$$z([P, s\, u, t'\, v]) - z([P, s'\, u, t'\, v]) = o(1)$$

as $s \to 0$ uniformly for $P \subset B$. Thus the limit (31) exists uniformly for $P \subset B$.

For $P \in \mathcal{P}_0(e_n^\perp)$ fixed, we set

$$f(u', v') = \lim_{s,t \to 0} z([P, s\, u, t\, v]).$$

Note that $f(0, 0) = \lim_{s,t \to 0} z([P, I])$. Since z is a valuation, we have for $r > 0$ suitably small

$$z([P, s\, u, t\, v]) + z([P, -rs\, e_n, rt\, e_n]) = z([P, s\, u, rt\, e_n]) + z([P, -rs\, e_n, t\, v]).$$

Taking the limit as $s, t \to 0$ gives

$$f(u', v') + f(0,0) = f(u', 0) + f(0, v'). \tag{34}$$

Since $\phi_v[P, s\,(u' + v', -1), t\, e_n] = [P, s\, u, t\, v]$, by assumption

$$z([P, s\, u, t\, v]) = z([P, s\,(u' + v', -1), t\, e_n]) + o(1).$$

Thus $f(u', v') = f(u' + v', 0)$. Setting $g(u') = f(u', 0) - f(0, 0)$, it follows from (34) that

$$g(u' + v') = g(u') + g(v')$$

for $u', v' \in \mathbb{R}^{n-1}$. This is the Cauchy functional equation. Since z is measurable, so is g, and there is a vector $w(P) \in \mathbb{R}^{n-1}$ such that $g(u') = w(P) \cdot u'$. Thus

$$\lim_{s,t\to 0} z([P, s\, u, t\, v]) = w(P) \cdot (u' + v') + f(0,0). \tag{35}$$

Using ϕ_r, it follows from (35) and the assumption that

$$\frac{1}{r}\, w(P) \cdot (u' + v') + f(0,0) = r^p (w(P) \cdot (u' + v') + f(0,0)).$$

Since this holds for all $0 < r \le 1$, we obtain $w(P) = 0$ (and $f(0,0) = 0$ for $p \ne 0$). This implies the statement of the lemma. q.e.d.

4.2. Proof of Proposition 4.1.

Let $\mathrm{Z} : \mathcal{P}_0^2 \to \langle \mathcal{K}^2, + \rangle$ be a measurable valuation which is GL(2) covariant of weight q.

Lemma 4.6. *For $q \ne -1$, there exist $a, b \in \mathbb{R}$ such that for all $s_i, t_i > 0$*

$$\begin{aligned} h(\mathrm{Z}[I_1, I_2], e_1) &= (a\, s_1^{q+1} + b\, t_1^{q+1})(s_2^q + t_2^q) \\ h(\mathrm{Z}[I_1, I_2], e_2) &= (a\, s_2^{q+1} + b\, t_2^{q+1})(s_1^q + t_1^q) \end{aligned}$$

where $I_i = [-s_i\, e_i, t_i\, e_i]$.

Proof. For $P \in \mathcal{P}_0^2$, set $z_i(P) = h(\mathrm{Z}\, P, e_i)$. If $\phi \in \mathrm{GL}(2)$ is the transformation that leaves e_1 fixed and multiplies e_2 by $r > 0$, then (2) implies $z_2([I_1, r\, I_2]) = r^{q+1} z_2([I_1, I_2])$. Thus $z_2([I_1, \cdot\,])$ is homogeneous of degree $q+1$ and it follows from (11) that there are $a(I_1), b(I_1) \in \mathbb{R}$ such that

$$z_2([I_1, I_2]) = a(I_1)\, s_2^{q+1} + b(I_1)\, t_2^{q+1}.$$

If $\phi \in \mathrm{GL}(2)$ is the transformation that leaves e_2 fixed and multiplies e_1 by $r > 0$, then (2) implies $z_2([r\, I_1, I_2]) = r^q z_2([I_1, I_2])$. Thus a and b are valuations that are homogeneous of degree q. For $q \ne 0$, it follows from (11) that there are constants $a, b, c, d \in \mathbb{R}$ such that

$$z_2([I_1, I_2]) = (a\, s_2^{q+1} + b\, t_2^{q+1})(c\, s_1^q + d\, t_1^q) \tag{36}$$

for all $s_i, t_i > 0$. For $q = 0$, it follows from (10) and (11) that there are constants $a, b, c, d \in \mathbb{R}$ such that

$$z_2([I_1, I_2]) = (a\, s_2 + b\, t_2)(c \log(\frac{t_1}{s_1}) + d) \tag{37}$$

for all $s_i, t_i > 0$. Define the transformation $\psi \in \mathrm{GL}(2)$ by $\psi e_1 = -e_1$ and $\psi e_2 = -e_2$. By (2) and (3), $z_2(\psi[I_1, I_2]) = z_2([-I_1, I_2]) = z_2([I_1, I_2])$. Combined with (36) and (37), this gives the formula for z_2. The formula for z_1 is obtained by applying $\rho_{\pi/2}$. q.e.d.

Lemma 4.7. *For $q \geq 0$, the operator $P \mapsto h(\mathrm{Z}\, P, \cdot)$ for $P \in \mathcal{P}_0^2$ has the Cauchy property for $\mathcal{P}_0(e_1)$.*

Proof. For $Q \in \mathcal{P}_0^2$, set $z_i(Q) = h(\mathrm{Z}\, Q, e_i)$ and let B be a centered ball. First, we show that there exist constants $a, b \in \mathbb{R}$ so that for all $I_1 = [-s_1\, e_1, t_1\, e_1]$, where $s_1, t_1 > 0$, for all $s, t > 0$ and for all $u = (u', -1)$ and $v = (v', 1)$, where $u', v' \in \mathbb{R}$, we have

$$z_2([I_1, s\, u, t\, v]) = (a\, s^{q+1} + b\, t^{q+1})(s_1^q + t_1^q) \tag{38}$$

whenever $[s\, u, t\, v]$ intersects I_1.

Let $I = [-s\, e_2, t\, e_2]$ and define $\phi_v, \phi_r \in \mathrm{GL}(2)$ as in Lemma 4.5. By Lemma 4.6, $\lim_{s,t \to 0} z_2([I_1, I])$ exists uniformly for $I_1 \subset B$. By (2), we have $z_2(\phi_r [I_1, I]) = r^{q+1} z_2([I_1, I])$ and

$$z_2([I_1, -s\, v, t\, v]) = z_2(\phi_v [I_1, I]) = z_2([I_1, I]).$$

Thus we apply Lemma 4.5 and obtain by Lemma 4.6 that

$$\lim_{s,t \to 0} z_2([I_1, s\, u, t\, v]) = 0. \tag{39}$$

Since z_2 is a valuation, we have for all $s', t' > 0$ sufficiently small

$$z_2([I_1, s\, u, t\, v]) + z_2([I_1, -t'v, t'v]) = z_2([I_1, s\, u, t'v]) + z_2([I_1, -t'v, t\, v])$$
$$z_2([I_1, s\, u, t'v]) + z_2([I_1, s'u, -s'u]) = z_2([I_1, s\, u, -s'u]) + z_2([I_1, s'u, t'v]).$$

Hence we have for all $s', t' > 0$ sufficiently small

$$\begin{aligned} z_2([I_1, s\, u, t\, v]) &= z_2([I_1, -t'\, v, t\, v]) - z_2([I_1, -t'\, v, t'\, v])) \\ &\quad + z_2([I_1, s\, u, -s'\, u] + z_2([I_1, s'\, u, t'\, v]) - z_2([I_1, s'\, u, -s'\, u]). \end{aligned}$$

Taking the limit as $s', t' \to 0$ and using (39) and Lemma 4.6 gives (38).

It follows from (2) that

$$h(\mathrm{Z}[I_1, -s\, v, t\, v], e_1) = h(\mathrm{Z}[I_1, I], \phi_v^t e_1) = h(\mathrm{Z}[I_1, I], e_1 + v'\, e_2).$$

Hence, Lemma 4.6, (38) and (4) imply

$$z_1([I_1, -s\, v, t\, v]) = (a\, s_1^{q+1} + b\, t_1^{q+1})(s^q + t^q) + o(1)$$

as $s,t \to 0$ uniformly for $I_1 \subset B$. Note that $\lim_{s,t\to 0} z_1([I_1,I])$ exists uniformly for $I_1 \subset B$ and that by (2), $z_1(\phi_r[I_1,u,v]) = r^q\, z_1([I_1,u,v])$. Thus we apply Lemma 4.5 and obtain by Lemma 4.6 that for $q>0$

$$\lim_{s,t\to 0} z_1([I_1, s\,u, t\,v]) = 0 \tag{40}$$

uniformly for $I_1 \subset B$ and for $q=0$,

$$\lim_{s,t\to 0} z_1([I_1, s\,u, t\,v]) = 2(a\,s_1 + b\,t_1) \tag{41}$$

uniformly for $I_1 \subset B$.

Let $x=(x_1,x_2)\in S^1$. From (4) and (39), we obtain that

$$h(\mathrm{Z}[I_1, s\,u, t\,v], x) = z_1([I_1, s\,u, t\,v])\, x_1 + o(1)$$

as $s,t \to 0$ uniformly for $I_1 \subset B$. Combined with (40) and (41), this completes the proof of the lemma. q.e.d.

Set $\mathrm{Y}\,P = h(\mathrm{Z}\,P,\cdot)$ for $P \in \mathcal{P}_0^2$. Then $\mathrm{Y} : \mathcal{P}_0^2 \to \langle C_1(\mathbb{R}^2),+\rangle$ is a valuation which is $\mathrm{GL}(2)$ covariant of weight q. For H a hyperplane containing the origin, Lemma 4.7 shows that Y has the Cauchy property for $\mathcal{P}_0(H)$. Hence Lemma 3.1 implies that we can extend Y and therefore Z to $\overline{\mathcal{P}}{}_0^2$. Thus Theorem 4.4 implies the statement of the proposition.

5. The classification on $\mathcal{P}_0^n$ for $n \ge 3$

The aim of this section is to establish the following result.

Theorem 5.1. *An operator $\mathrm{Z} : \mathcal{P}_0^n \to \langle \mathcal{K}^n,+\rangle$, where $n\ge 3$, is a measurable valuation which is $\mathrm{GL}(n)$ covariant of weight q if and only if there are $c_0, c_1 \ge 0$ and $c_2 \in \mathbb{R}$ such that*

$$\mathrm{Z}\,P = \begin{cases} c_1\,\mathrm{M}\,P + c_2\,m(P) & \textit{for } q=1\\ c_0\,P + c_1(-P) & \textit{for } q=0\\ c_0\,\Pi\,P^* & \textit{for } q=-1\\ \{0\} & \textit{otherwise}\end{cases}$$

for every $P \in \mathcal{P}_0^n$.

The proof of Theorem 5.1 is split into two cases. First, we derive in Section 5.1 the following classification of $\mathrm{GL}(n)$ covariant valuations.

Proposition 5.2. *An operator $\mathrm{Z} : \mathcal{P}_0^n \to \langle \mathcal{K}^n,+\rangle$, where $n\ge 3$, is a measurable valuation which is $\mathrm{GL}(n)$ covariant of weight $q>-1$ if and only if there are $c_0, c_1 \ge 0$ and $c_2 \in \mathbb{R}$ such that*

$$\mathrm{Z}\,P = \begin{cases} c_1\,\mathrm{M}\,P + c_2\,m(P) & \textit{for } q=1\\ c_0\,P + c_1(-P) & \textit{for } q=0\\ \{0\} & \textit{otherwise}\end{cases}$$

for every $P \in \mathcal{P}_0^n$.

In Section 5.2, we derive the following classification of $\mathrm{GL}(n)$ contravariant valuations.

Proposition 5.3. *An operator* $\mathrm{Z} : \mathcal{P}_0^n \to \langle \mathcal{K}^n, + \rangle$*, where* $n \geq 3$*, is a measurable valuation which is* $\mathrm{GL}(n)$ *contravariant of weight* $q \geq 1$ *if and only if there is a constant* $c \geq 0$ *such that*

$$\mathrm{Z}\, P = \begin{cases} c\, \Pi\, P & \text{for } q = 1 \\ \{0\} & \text{otherwise} \end{cases}$$

for every $P \in \mathcal{P}_0^n$.

By (13), Proposition 5.3 is equivalent to the classification of $\mathrm{GL}(n)$ covariant valuations of weight $q \leq -1$. Hence Propositions 5.2 and 5.3 imply Theorem 5.1, while Theorem 2 is an immediate consequence of Theorem 5.1.

In the proof of Theorem 5.1 the following results are used.

Theorem 5.4 ([**36**])**.** *Let* $\mathrm{Z} : \overline{\mathcal{P}}_0^n \to \langle \mathcal{K}^n, + \rangle$*, where* $n \geq 3$*, be a valuation which is* $\mathrm{GL}(n)$ *covariant of weight* q*. If* $q = 1$*, then there are* $c_1 \geq 0$ *and* $c_2 \in \mathbb{R}$ *such that*

$$\mathrm{Z}\, P = c_1 \, \mathrm{M}\, P + c_2\, m(P)$$

for every $P \in \overline{\mathcal{P}}_0^n$*. If* $q = 0$*, then there are* $c_0, c_1 \geq 0$ *such that*

$$\mathrm{Z}\, P = c_0\, P + c_1(-P)$$

for every $P \in \overline{\mathcal{P}}_0^n$*. In all other cases,* $\mathrm{Z}\, P = \{0\}$ *for every* $P \in \overline{\mathcal{P}}_0^n$.

Theorem 5.5 ([**36**])**.** *Let* $\mathrm{Z} : \overline{\mathcal{P}}_0^n \to \langle \mathcal{K}^n, + \rangle$*, where* $n \geq 3$*, be a valuation which is* $\mathrm{GL}(n)$ *contravariant of weight* q*. If* $q = 1$*, then there are* $c_1, c_2, c_3 \in \mathbb{R}$ *with* $c_1 \geq 0$ *and* $c_1 + c_2 + c_3 \geq 0$ *such that*

$$\mathrm{Z}\, P = c_1 \, \Pi\, P + c_2\, \Pi_0\, P + c_3(-\, \Pi_0\, P)$$

for every $P \in \overline{\mathcal{P}}_0^n$*. In all other cases,* $\mathrm{Z}\, P = \{0\}$ *for every* $P \in \overline{\mathcal{P}}_0^n$.

Here Π_0 is a certain operator with the property that $\Pi_0\, P = \{0\}$ for $P \in \mathcal{P}_0^n$.

5.1. Proof of Proposition 5.2. Let $\mathrm{Z} : \mathcal{P}_0^n \to \langle \mathcal{K}^n, + \rangle$, where $n \geq 3$, be a measurable valuation which is $\mathrm{GL}(n)$ covariant of weight q. For $n > 3$, assume that Proposition 5.2 holds in dimension $(n-1)$.

Lemma 5.6. *For* $q > -1$*, there exist* $a, b \in \mathbb{R}$ *such that*

$$h(\mathrm{Z}[P, I], e_n) \quad = \quad \begin{cases} a\, s + b\, t & \text{for } q = 0 \\ (a\, s^2 + b\, t^2) V_{n-1}(P) & \text{for } q = 1 \\ 0 & \text{otherwise} \end{cases}$$

for every $I = [-s\, e_n, t\, e_n]$ *with* $s, t > 0$ *and* $P \in \mathcal{P}_0(e_n^\perp)$.

Proof. Set $z([P,I]) = h(\mathrm{Z}[P,I], e_n)$. If $\phi \in \mathrm{GL}(n)$ leaves $e_n^\perp$ fixed and multiplies e_n by $r > 0$, then (2) implies $z([P, r\,I]) = r^{q+1} z([P,I])$. Thus $z([P,\cdot])$ is a valuation which is homogeneous of degree $q+1$. If $\phi \in \mathrm{GL}(n)$ is a transformation that leaves e_n fixed, then (2) implies $z(\phi[P,I]) = |\det\phi|^q z([P,I])$. Thus $z([\cdot, I])$ is a valuation for which (17) holds in dimension $(n-1)$. The statements follow from Theorem 1.1 and (11). q.e.d.

Lemma 5.7. *For $q > -1$ and $q \notin \{0,1\}$, we have $\mathrm{Z}[P,I] = \{0\}$ for every $I \in \mathcal{P}_0(e_n)$ and $P \in \mathcal{P}_0(e_n^\perp)$.*

Proof. By Lemma 5.6 and the $\mathrm{GL}(n)$ covariance of Z,

$$h(\mathrm{Z}[P,I], e_n) = h(\mathrm{Z}[P,I], -e_n) = 0. \tag{42}$$

For $I \in \mathcal{P}_0(e_n)$, define $\mathrm{Z}_I : \mathcal{P}_0^{n-1} \to \langle \mathcal{K}^{n-1}, +\rangle$ by setting $h(\mathrm{Z}_I\, P, x) = h(\mathrm{Z}[P,I], x)$ for $x \in \mathbb{R}^{n-1}$ and $P \in \mathcal{P}_0^{n-1}$, where we identify $e_n^\perp$ and $\mathbb{R}^{n-1}$. Note that Z_I is a valuation on $\mathcal{P}_0^{n-1}$ that is $\mathrm{GL}(n-1)$ covariant of weight q.

Let $n = 3$. Define $\mathrm{Z}_I^s : \mathcal{P}_0^2 \to \langle \mathcal{K}_c^2, +\rangle$ by $\mathrm{Z}_I^s\, P = \mathrm{Z}_I(P) + \mathrm{Z}_I(-P)$. Since Z_I^s is $\mathrm{GL}(2)$ covariant of weight q, Lemma 4.6 implies that there are $a(I) \in \mathbb{R}$ such that

$$\begin{aligned} h(\mathrm{Z}_I^s[I_1, I_2], e_1) &= a(I)(s_1^{q+1} + t_1^{q+1})(s_2^q + t_2^q) \\ h(\mathrm{Z}_I^s[I_1, I_2], e_2) &= a(I)(s_1^q + t_1^q)(s_2^{q+1} + t_2^{q+1}) \end{aligned}$$

where $I_i = [-s_i\, e_i, t_i\, e_i]$ and $s_i, t_i > 0$ for $i = 1, 2$. Define $Q \in \mathcal{P}_0^3$ as the convex hull of I_1, I_2 and I. Let ψ be the linear transformation so that $\psi\, e_1 = e_3$ and $\psi\, e_2 = e_2$ and $\psi\, e_3 = e_1$. Since Z is $\mathrm{GL}(3)$ covariant, it follows from (42) that

$$h(\mathrm{Z}(\psi Q), e_1) = h(\mathrm{Z}\, Q, e_3) = 0.$$

Let $I = [-s\, e_3, t\, e_3]$ with $s, t > 0$ and set $s_i = s$ and $t_i = t$ for $i = 1, 2$. We conclude that

$$h(\mathrm{Z}_I^s[I_1, I_2], e_1) = 0.$$

Hence $a(I) = 0$. Consequently, Lemma 3.2 shows that we can extend Z_I^s to $\overline{\mathcal{P}}_0^2$. Theorem 4.4 shows that $\mathrm{Z}_I^s\, P = \{0\}$ for all $P \in \mathcal{P}_0^2$. Since $\mathrm{Z}_I(P) + \mathrm{Z}_I(-P) = \{0\}$ for all $P \in \mathcal{P}_0^2$, we see that Z_I is vector-valued. Consequently, Theorem 1.2 implies that $\mathrm{Z}_I\, P = \{0\}$ for all $P \in \mathcal{P}_0^2$. Hence the lemma holds true for $n = 3$.

Let $n > 3$. Since Proposition 5.2 holds in dimension $(n-1)$, we obtain that $\mathrm{Z}_I\, P = \{0\}$ for all $P \in \mathcal{P}_0(e_n^\perp)$. Hence the statement of the lemma holds true for every $n > 3$. q.e.d.

Lemma 5.8. *If $q > -1$ and $q \notin \{0,1\}$, then $\mathrm{Z}\, P = \{0\}$ for all $P \in \mathcal{P}_0^n$.*

Proof. Set $\mathrm{Z}_s P = \mathrm{Z}\, P + \mathrm{Z}(-P)$ and $\mathrm{Y}\, P = h(\mathrm{Z}_s P, \cdot)$. Note that $\mathrm{Y} : \mathcal{P}_0 \to C_1^+(\mathbb{R}^n)$ is a $\mathrm{GL}(n)$ covariant valuation. By Lemma 5.7 and Lemma 3.2 we can extend Y and thus Z_s to $\overline{\mathcal{P}}_0^n$. Theorem 5.4 implies that $\mathrm{Z}_s P = \{0\}$ for all $P \in \mathcal{P}_0$. Hence Z is vector-valued and Theorem 1.3 implies the statement of the lemma. q.e.d.

Lemma 5.9. *There exist $a, b \in \mathbb{R}$ so that for all $P \in \mathcal{P}_0(e_n^\perp)$, for all $s, t > 0$, and for all $u = (u', -1)$ and $v = (v', 1)$ with $u', v' \in \mathbb{R}^{n-1}$,*

$$h(\mathrm{Z}[P, s\,u, t\,v], e_n) = \begin{cases} a\,s + b\,t & \text{for } q = 0 \\ (a\,s^2 + b\,t^2) V_{n-1}(P) & \text{for } q = 1 \end{cases}$$

whenever $[s\,u, t\,v]$ intersects P.

Proof. Set $z([P, s\,u, t\,v]) = h(\mathrm{Z}[P, s\,u, t\,v], e_n)$. Let B be a centered ball, let $I = [-s\,e_n, t\,e_n]$, and let $\phi_v, \phi_r \in \mathrm{GL}(n)$ be defined as in Lemma 4.5. By Lemma 5.6, $\lim_{s,t\to 0} z([P, I])$ exists uniformly for $I \subset B$. It follows from (2) that $z(\phi_r[P, u, v]) = r^{q+1} z([P, u, v])$ and

$$z([P, -s\,v, t\,v]) = z(\phi_v[P, I]) = z([P, I]).$$

Thus we apply Lemma 4.5 and obtain by Lemma 5.6 that

$$\lim_{s,t\to 0} z([P, s\,u, t\,v]) = 0. \tag{43}$$

Since for all $s', t' > 0$ sufficiently small

$$\begin{aligned} z([P, s\,u, t\,v]) = {} & z([P, -t'\,v, t\,v]) - z([P, -t'\,v, t'\,v]) + \\ & + z([P, s\,u, -s'\,u]) + z([P, s'\,u, t'\,v]) - z([P, s'\,u, -s'\,u]), \end{aligned}$$

taking the limit as $s', t' \to 0$ and using (43) and Lemma 5.6 gives the statement of the lemma. q.e.d.

Lemma 5.10. *There exist $a, b \in \mathbb{R}$ such that for $x \in e_n^\perp$*

$$h(\mathrm{Z}[P, I], x) = \begin{cases} a\, h(P, x) + b\, h(-P, x) & \text{for } q = 0 \\ a(s + t)\, h(\mathrm{M}\, P, x) + b(s + t)\, h(m(P), x) & \text{for } q = 1 \end{cases}$$

for every $I = [-s\,e_n, t\,e_n]$ and $P \in \mathcal{P}_0(e_n^\perp)$.

Proof. For $I \in \mathcal{P}_0(e_n)$, define $\mathrm{Z}_I : \mathcal{P}_0^{n-1} \to \langle \mathcal{K}^{n-1}, + \rangle$ by setting $h(\mathrm{Z}_I P, x) = h(\mathrm{Z}[P, I], x)$ for $x \in \mathbb{R}^{n-1}$ and $P \in \mathcal{P}_0^{n-1}$, where we identify $e_n^\perp$ and $\mathbb{R}^{n-1}$. Note that Z_I is a valuation on $\mathcal{P}_0^{n-1}$ that is $\mathrm{GL}(n-1)$ covariant of weight q. By Theorem 4.3 for $n = 3$ and by Proposition 5.2 in dimension $(n-1)$ for $n > 3$, we obtain that there are $a(I), b(I) \in \mathbb{R}$ such that for $x \in e_n^\perp$,

$$h(\mathrm{Z}[P, I], x) = \begin{cases} a(I)\, h(P, x) + b(I)\, h(-P, x) & \text{for } q = 0. \\ a(I)\, h(\mathrm{M}\, P, x) + b(I)\, h(m(P), x) & \text{for } q = 1. \end{cases}$$

If $\phi_r \in \mathrm{GL}(n)$ is a transformation that leaves $e_n^\perp$ fixed and multiplies e_n by $r > 0$, then (2) implies $h(\mathrm{Z}[P, r\, I], x) = r^q h(\mathrm{Z}[P, I], x)$ for $x \in e_n^\perp$. Thus a and b are valuations on $\mathcal{P}_0^1$ which are homogeneous of degree q. By (2), we see that $a(I) = a(-I)$ and $b(I) = b(-I)$ for $I \in \mathcal{P}_0^1$. Thus (10) and (11) imply the statement of the lemma. q.e.d.

Lemma 5.11. *If $q = 0$ or $q = 1$, then the operator $P \mapsto h(\mathrm{Z}\, P, \cdot)$ for $P \in \mathcal{P}_0^n$ has the Cauchy property for $\mathcal{P}_0(e_n^\perp)$.*

Proof. Let B be a centered ball and $s, t > 0$. First, we show that for all $P \in \mathcal{P}_0(e_n^\perp)$ with $P \subset B$ and for all $u = (u', -1)$ and $v = (v', 1)$ with $u', v' \in \mathbb{R}^{n-1}$, we have for $x' \in e_n^\perp \cap B$,

$$(44) \quad h(\mathrm{Z}[P, s\, u, t\, v], x') = \begin{cases} a\, h(P, x') + b\, h(-P, x') + o(1) & \text{for } q = 0 \\ o(1) & \text{for } q = 1 \end{cases}$$

as $s, t \to 0$ uniformly whenever $[s\, u, t\, v]$ intersects P.

Let $I = [-s\, e_n, t\, e_n]$ and define $\phi_v, \phi_r \in \mathrm{GL}(n)$ as in Lemma 4.5. Since $[P, -s\, v, t\, v] = \phi_v[P, I]$, we obtain from (2) that

$$h(\mathrm{Z}[P, -s\, v, t\, v], x') = h(\mathrm{Z}(\phi_v[P, I]), x') = h(\mathrm{Z}[P, I], x' + (x' \cdot v)\, e_n).$$

By Lemmas 5.6 and 5.10 it follows from (4) that

$$h(\mathrm{Z}[P, -s\, v, t\, v], x') = \begin{cases} a\, h(P, x') + b\, h(-P, x') + o(1) & \text{for } q = 0 \\ o(1) & \text{for } q = 1 \end{cases}$$

as $s, t \to 0$ uniformly. In particular, $\lim_{s,t \to 0} h(\mathrm{Z}[P, I], x')$ exists uniformly. It follows from (2) that $h(\mathrm{Z}(\phi_r[P, I], x') = h(\mathrm{Z}[P, I], x')$. Thus we apply Lemma 4.5 for $z = h(\mathrm{Z}[P, u, v], x')$ and obtain by Lemma 5.10 that (44) holds.

Let $x = (x', x_n) \in S^{n-1}$, where $x' \in e_n^\perp$ and $x_n \in \mathbb{R}$. From (4) and Lemma 5.6, we obtain that

$$h(\mathrm{Z}[P, s\, u, t\, v], x) = h(\mathrm{Z}[P, s\, u, t\, v], x') + o(1)$$

as $s, t \to 0$. Combined with (44), this completes the proof of the lemma. q.e.d.

Lemma 5.12. *If $q = 0$, then there are constants $c_0, c_1 \geq 0$ such that*

$$\mathrm{Z}\, P = c_0\, P + c_1(-P)$$

for all $P \in \mathcal{P}_0^n$. If $q = 1$, then there are constants $c_1 \geq 0$ and $c_2 \in \mathbb{R}$ such that

$$\mathrm{Z}\, P = c_1\, \mathrm{M}\, P + c_2\, m(P)$$

for all $P \in \mathcal{P}_0^n$.

Proof. For $P \in \mathcal{P}_0^n$, set $\mathrm{Y}(P) = h(\mathrm{Z}\, P, \cdot)$. By Lemmas 5.9 and 5.11 we can apply Lemma 3.1 and extend Y and therefore Z to $\overline{\mathcal{P}}_0^n$. Hence Theorem 5.4 implies the statement of the lemma. q.e.d.

5.2. Proof of Proposition 5.3. Let $\mathrm{Z} : \mathcal{P}_0^n \to \langle \mathcal{K}^n, + \rangle$, where $n \geq 3$, be a measurable valuation which is $\mathrm{GL}(n)$ contravariant of weight q. For $n > 3$, assume that Proposition 5.3 holds in dimension $(n-1)$.

Lemma 5.13. *For $q \geq 1$, there exist $a, b \in \mathbb{R}$ such that*

$$h(\mathrm{Z}[P, I], e_n) \quad = \quad \begin{cases} (a \log(t/s) + b) V_{n-1}(P) & \text{for } q = 1 \\ 0 & \text{otherwise} \end{cases}$$

for every $I = [-s\, e_n, t\, e_n]$ and $P \in \mathcal{P}_0(e_n^\perp)$.

Proof. Set $z([P, I]) = h(\mathrm{Z}[P, I], e_n)$. If $\phi \in \mathrm{GL}(n)$ leaves $e_n^\perp$ fixed and multiplies e_n by $r > 0$, then (12) implies $z([P, r\, I]) = r^{q-1} z([P, I])$. Thus $z([P, \cdot])$ is a valuation which is homogeneous of degree $q - 1$. If $\phi \in \mathrm{GL}(n)$ is a transformation that leaves e_n fixed, then (12) implies $z(\phi[P, I]) = |\det \phi|^q z([P, I])$. Thus $z([\cdot, I])$ is a valuation for which (17) holds in dimension $(n-1)$. The statements follow from Theorem 1.1 and (10). q.e.d.

Lemma 5.14. *For $q > 1$, we have $\mathrm{Z}[P, I] = \{0\}$ for every $I \in \mathcal{P}_0(e_n)$ and $P \in \mathcal{P}_0(e_n^\perp)$.*

Proof. By Lemma 5.13 and the $\mathrm{GL}(n)$ contravariance of Z,

$$h(\mathrm{Z}[P, I], e_n) = h(\mathrm{Z}[P, I], -e_n) = 0. \tag{45}$$

For $I \in \mathcal{P}_0(e_n)$, define $\mathrm{Z}_I : \mathcal{P}_0^{n-1} \to \langle \mathcal{K}^{n-1}, + \rangle$ by setting $h(\mathrm{Z}_I\, P, x) = h(\mathrm{Z}[P, I], x)$ for $x \in \mathbb{R}^{n-1}$ and $P \in \mathcal{P}_0^{n-1}$, where we identify $e_n^\perp$ and $\mathbb{R}^{n-1}$. Note that Z_I is a $\mathrm{GL}(n-1)$ contravariant valuation of weight q on $\mathcal{P}_0^{n-1}$.

Let $n = 3$. Define $\mathrm{Z}_I^s : \mathcal{P}_0^2 \to \langle \mathcal{K}_c^2, + \rangle$ by $\mathrm{Z}_I^s\, P = \mathrm{Z}_I(P) + \mathrm{Z}_I(-P)$. Since Z_I^s is $\mathrm{GL}(2)$ contravariant of weight q, (16) implies that $\rho_{\pi/2}\, \mathrm{Z}_I$ is $\mathrm{GL}(2)$ covariant of weight $q - 1$. It follows from Lemma 4.6 that there is $a(I) \in \mathbb{R}$ such that

$$\begin{aligned} h(\rho_{\pi/2}\, \mathrm{Z}_I^s[I_1, I_2], e_1) &= a(I)(s_1^q + t_1^q)(s_2^{q-1} + t_2^{q-1}) \\ h(\rho_{\pi/2}\, \mathrm{Z}_I^s[I_1, I_2], e_2) &= a(I)(s_1^{q-1} + t_1^{q-1})(s_2^q + t_2^q) \end{aligned}$$

where $I_i = [-s_i\, e_i, t_i\, e_i]$. Define $Q \in \mathcal{P}_0^3$ as the convex hull of I_1, I_2 and I. Let ψ be the linear transformation so that $\psi\, e_1 = e_3$ and $\psi\, e_2 = e_2$ and $\psi\, e_3 = e_1$. Since Z is $\mathrm{GL}(3)$ contravariant, it follows from (45) that

$$h(\mathrm{Z}(\psi Q), e_1) = h(\mathrm{Z}\, Q, e_3) = 0.$$

Let $I = [-s\, e_3, t\, e_3]$ with $s, t > 0$ and set $s_i = s$ and $t_i = t$ for $i = 1, 2$. We conclude that

$$h(\mathrm{Z}_I^s[I_1, I_2], e_1) = 0.$$

Hence $a(I) = 0$. Consequently, Lemma 3.2 shows that we can extend Z_I^s to $\overline{\mathcal{P}}_0^2$. By Theorem 4.4, we have $\rho_{-\pi/2}\, \mathrm{Z}_I^s\, P = \{0\}$ for all $P \in \mathcal{P}_0^2$. Since $\mathrm{Z}_I(P) + \mathrm{Z}_I(-P) = \{0\}$ for all $P \in \mathcal{P}_0^2$, Z_I is vector-valued. Theorem 1.2

implies that $\rho_{\pi/2}\,\mathrm{Z}_I\,P = \{0\}$ for all $P \in \mathcal{P}_0^2$. Hence the lemma holds true for $n = 3$.

Let $n > 3$. Since Proposition 5.3 holds in dimension $(n-1)$, we obtain that $\mathrm{Z}_I\,P = \{0\}$ for all $P \in \mathcal{P}_0(e_n^\perp)$. Hence the statement of the lemma holds true for every $n > 3$. q.e.d.

Lemma 5.15. *If $q > 1$, then $\mathrm{Z}\,P = \{0\}$ for all $P \in \mathcal{P}_0^n$.*

Proof. Set $\mathrm{Z}_s\,P = \mathrm{Z}\,P + \mathrm{Z}(-P)$ and $\mathrm{Y}\,P = h(\mathrm{Z}_s\,P,\cdot)$. Note that $\mathrm{Y} : \mathcal{P}_0 \to C_1^+(\mathbb{R}^n)$ is a $\mathrm{GL}(n)$ contravariant valuation. By Lemma 5.14 and Lemma 3.2 we can extend Y and thus Z_s to $\overline{\mathcal{P}}_0^n$. Theorem 5.5 implies that $\mathrm{Z}_s\,P = \{0\}$ for all $P \in \mathcal{P}_0$. Hence Z is vector-valued and Theorem 1.3 implies the statement of the lemma. q.e.d.

Lemma 5.16. *For $q = 1$, there exist $a, b \in \mathbb{R}$ such that for $x \in e_n^\perp$*

$$h(\mathrm{Z}[P,I],x) = \begin{cases} a(s+t)\,h(\rho_{\pi/2}P,x) + b(s+t)\,h(-\rho_{\pi/2}P,x) & \textit{for } n = 3 \\ a(s+t)\,h(\Pi\,P,x) & \textit{for } n > 3 \end{cases}$$

for every $I = [-s\,e_n, t\,e_n]$ and $P \in \mathcal{P}_0(e_n^\perp)$.

Proof. For $I \in \mathcal{P}_0(e_n)$, define $\mathrm{Z}_I : \mathcal{P}_0^{n-1} \to \langle \mathcal{K}^{n-1}, +\rangle$ by setting $h(\mathrm{Z}_I\,P,x) = h(\mathrm{Z}[P,I],x)$ for $x \in \mathbb{R}^{n-1}$ and $P \in \mathcal{P}^{n-1}$, where we identify $e_n^\perp$ and $\mathbb{R}^{n-1}$. Note that Z_I is a valuation on $\mathcal{P}_0^{n-1}$ that is $\mathrm{GL}(n-1)$ contravariant of weight $q = 1$.

By Proposition 4.2, we obtain that there are $a(I), b(I) \in \mathbb{R}$ such that for $x \in e_n^\perp$

$$h(\mathrm{Z}[P,I],x) = \begin{cases} a(I)\,h(\rho_{\pi/2}P,x) + b(I)\,h(-\rho_{\pi/2}P,x) & \text{for } n = 3. \\ a(I)\,h(\Pi\,P,x) & \text{for } n > 3. \end{cases}$$

If $\phi_r \in \mathrm{GL}(n)$ is a transformation that leaves $e_n^\perp$ fixed and multiplies e_n by $r > 0$, then (12) implies $h(\mathrm{Z}[P,r\,I],x) = r\,h(\mathrm{Z}[P,I],x)$. Thus a and b are valuations on $\mathcal{P}_0^1$ which are homogeneous of degree 1. By (12), we see that $a(I) = a(-I)$ and $b(I) = b(-I)$ for $I \in \mathcal{P}_0^1$. Thus (11) implies the statement of the lemma. q.e.d.

Lemma 5.17. *For $q = 1$, there exists $b \in \mathbb{R}$ such that*

$$h(\mathrm{Z}[P,I],e_n) = b\,V_{n-1}(P)$$

for every $I = [-s\,e_n, t\,e_n]$ and $P \in \mathcal{P}_0(e_n^\perp)$.

Proof. By Lemma 5.13, there are $a, b \in \mathbb{R}$ such that

$$h(\mathrm{Z}[P,I],e_n) = (a\log(\frac{t}{s}) + b)V_{n-1}(P). \tag{46}$$

Let $P = [I_1,\ldots,I_{n-1}]$ and $I_i = [-s_i\,e_i, t_i\,e_i]$. Define $\psi \in \mathrm{GL}(n)$ by $\psi e_1 = e_n$ and $\psi e_n = e_1$ and $\psi e_i = e_i$ for $i = 2,\ldots,n-1$. Since Z is $\mathrm{GL}(n)$ contravariant, we have

$$h(\mathrm{Z}[P,I],e_n) = h(\mathrm{Z}(\psi[P,I]),e_1).$$

It follows from Lemma 5.16 that there are $c, d \in \mathbb{R}$ such that

$$h(\mathrm{Z}(\psi[P, I]), e_1) = c(s_1+t_1)h(\rho_{\pi/2}[\hat{I}, I_2], e_1)+d(s_1+t_1)h(-\rho_{\pi/2}[\hat{I}, I_2], e_1)$$

for $n = 3$ and

$$h(\mathrm{Z}(\psi[P, I]), e_1) = c(s_1 + t_1)\, h(\Pi[\hat{I}, I_2, \dots, I_{n-1}], e_1)$$

for $n > 3$, where $\hat{I} = [-s\, e_1, t\, e_1]$. Comparing coefficients in the above equations and in (46) gives that $a = 0$ and completes the proof of the lemma. q.e.d.

Lemma 5.18. *For $q = 1$, there exist constants $a, b \in \mathbb{R}$ so that for all $P \in \mathcal{P}_0(e_n^\perp)$, for all $u = (u', -1)$ and $v = (v', 1)$ with $u', v' \in \mathbb{R}^{n-1}$, and for all $s, t > 0$, we have for $x \in e_n^\perp$,*

$$h(\mathrm{Z}[P, s\, u, t\, v], x) = a(s + t)\, h(\rho_{\pi/2}P, x) + b(s + t)\, h(-\rho_{\pi/2}P, x)$$

for $n = 3$ and

$$h(\mathrm{Z}[P, s\, u, t\, v], x) = a(s + t)\, h(\Pi\, P, x)$$

for $n > 3$, whenever $[s\, u, t\, v]$ intersects P.

Proof. Let $I = [-s\, e_n, t\, e_n]$ with $s, t > 0$ and $x \in e_n^\perp$. Lemma 5.16 implies that $\lim_{s,t\to 0} h(\mathrm{Z}[P, I], x)$ exists uniformly for $P \in \mathcal{P}_0(e_n^\perp)$ contained in a centered ball. Let $\phi_v, \phi_r \in \mathrm{GL}(n)$ be defined as in Lemma 4.5. Since $[P, -s\, v, t\, v] = \phi_v[P, I]$, by (12)

$$h(\mathrm{Z}[P, -s\, v, t\, v], x) = h(\mathrm{Z}[P, I], \phi_v^{-1}x) = h(\mathrm{Z}[P, I], x).$$

By (12), we get

$$h(\mathrm{Z}(\phi_r[P, u, v], x) = r^{q-1}h(\mathrm{Z}[P, u, v], x).$$

Thus we apply Lemma 4.5 and obtain by Lemma 5.16 that

$$\lim_{s,t\to 0} h(\mathrm{Z}[P, s\, u, t\, v], x) = 0$$

for $x \in e_n^\perp$.

Since Z is a valuation, we have for $s', t' > 0$ sufficiently small,

$$\begin{aligned} h(\mathrm{Z}[P, s\, u, t\, v], x) &= h(\mathrm{Z}[P, -t'\, v, t\, v], x) - h(\mathrm{Z}[P, -t'\, v, t'\, v], x) \\ &\quad + h(\mathrm{Z}[P, -s'\, u, s\, u], x) + h(\mathrm{Z}[P, s'\, u, t'\, v], x) \\ &\quad - h(\mathrm{Z}[P, -s'\, u, s'\, u], x). \end{aligned}$$

Taking the limit as $s', t' \to 0$ and using Lemma 5.16 gives the statement of the lemma. q.e.d.

Lemma 5.19. *For $q = 1$, the operator $P \mapsto h(\mathrm{Z}\, P, \cdot)$ for $P \in \mathcal{P}_0^n$ has the Cauchy property for $\mathcal{P}_0(e_n^\perp)$.*

Proof. For $I = [-s\,e_n, t\,e_n]$ with $s, t > 0$, it follows from Lemma 5.17 that $\lim_{s,t\to 0} h(\mathrm{Z}[P, I], e_n)$ exists uniformly for $P \in \mathcal{P}_0(e_n^\perp)$ contained in a centered ball. Let $u = (u', -1)$ and $v = (v', 1)$ with $u', v' \in \mathbb{R}^{n-1}$ and let $\phi_v, \phi_r \in \mathrm{GL}(n)$ be defined as in Lemma 4.5. Since we have $[P, -s\,v, t\,v] = \phi_v[P, I]$, we obtain from (12) that

$$h(\mathrm{Z}[P, -s\,v, t\,v], e_n) = h(\mathrm{Z}(\phi_v[P, I]), e_n) = h(\mathrm{Z}[P, I], -v' + e_n).$$

Let B be a centered ball. By Lemmas 5.17 and 5.16 it follows from (4) that

$$h(\mathrm{Z}[P, -s\,v, t\,v], e_n) = b\,V_{n-1}(P) + o(1)$$

as $s, t \to 0$ uniformly for $P \subset B$. By (12), we have

$$h(\mathrm{Z}(\phi_r[P, u, v], e_n) = h(\mathrm{Z}[P, u, v], e_n).$$

Thus we apply Lemma 4.5 and obtain that

$$\lim_{s,t\to 0} h(\mathrm{Z}[P, s\,u, t\,v], e_n) = b\,V_{n-1}(P)$$

uniformly for $P \subset B$.

Let $x = (x', x_n) \in S^{n-1}$. From (4) and Lemma 5.16, we obtain that

$$h(\mathrm{Z}[P, s\,u, t\,v], x) = h(\mathrm{Z}[P, s\,u, t\,v], x') + o(1)$$

as $s, t \to 0$ uniformly for $x \in S^{n-1}$ and $P \subset B$. Combined with Lemma 5.16, this completes the proof of the lemma. q.e.d.

Lemma 5.20. *If $q = 1$, then there is a constant $c \geq 0$ such that*

$$\mathrm{Z}\,P = c\,\Pi\,P$$

for all $P \in \mathcal{P}_0^n$.

Proof. For $P \in \mathcal{P}_0^n$, set $\mathrm{Y}(P) = h(\mathrm{Z}\,P, \cdot)$. By Lemmas 5.18 and 5.19, we can apply Lemma 3.1 and extend Y and therefore Z to $\overline{\mathcal{P}}_0^n$. Hence Theorem 5.5 implies the statement of the lemma. q.e.d.

6. Proof of Theorem 1

Let H be a hyperplane containing the origin and let u be a unit normal vector to H. By Property (iv) in the definition of invariant area in Section 2, there exists $c_B(u) > 0$ such that

$$z(K, B) = c_B(u)\,V_{n-1}(K) \tag{47}$$

for all $K \in \mathcal{K}(H)$. Define $c_B(x)$ for $x \in \mathbb{R}^n$ by setting $c_B(t\,u) = t\,c_B(u)$ for $t > 0$. Note that $c_B : \mathbb{R}^n \to (0, \infty)$ is sublinear by Property (v) and hence a support function. Define the convex body $\mathbb{I}\,B$ by setting $h(\mathbb{I}\,B, x) = c_B(x)$ for $x \in \mathbb{R}^n$. Note that $\mathbb{I}\,B \in \mathcal{K}_0^n$ and that $\mathbb{I}\,B$ is origin symmetric.

By Groemer's Extension Theorem (cf. [**31**]), $z(\cdot, B)$ can be extended to a valuation on finite unions of convex polytopes in $\mathbb{R}^n$. Hence, for a polytope $P \in \mathcal{K}^n$ with facets $F_1, \ldots, F_m$ lying in hyperplanes with outer normal unit vectors $u_1, \dots, u_m$, we obtain by (47) and the definition of $\mathbb{I}\, B$ that

$$z(P, B) = \sum_{i=1}^{m} h(\mathbb{I}\, B, u_i) V_{n-1}(F_i) = n\, V_1(P, \mathbb{I}\, B),$$

where we used (8). By continuity,

$$z(K, B) = n\, V_1(K, \mathbb{I}\, B) \quad \text{for every } \; K \in \mathcal{K}^n.$$

The convex body $\mathbb{I}\, B$ is called *isoperimetrix* since it turns out to be the solution to the isoperimetric problem.

It follows from Property (i), (5) and (6) that

$$\begin{aligned} V_1(K, \mathbb{I}\, B) &= V_1(\phi K, \mathbb{I}(\phi B)) \\ &= |\det \phi|\, V_1(K, \phi^{-1}\, \mathbb{I}(\phi B)) \\ &= V_1(K, |\det \phi|\, \phi^{-1}\, \mathbb{I}(\phi B)) \end{aligned}$$

for all $K \in \mathcal{K}^n$ and $B \in \mathcal{K}_0^n$. Thus it follows from (9) that

$$\mathbb{I}(\phi B) = |\det \phi|^{-1}\, \phi\, \mathbb{I}\, B$$

for all $\phi \in \mathrm{GL}(n)$ and $B \in \mathcal{K}_0^n$. This shows that $\mathbb{I} : \mathcal{K}_0^n \to \mathcal{K}_0^n$ is $\mathrm{GL}(n)$ covariant of weight $q = -1$. Since we assume that z is a bivaluation,

$$z(K, B_1) + z(K, B_2) = z(K, B_1 \cup B_2) + z(K, B_1 \cap B_2)$$

and therefore by (7)

$$V_1(K, \mathbb{I}\, B_1 + \mathbb{I}\, B_2) = V_1(K, \mathbb{I}(B_1 \cup B_2) + \mathbb{I}(B_1 \cap B_2))$$

for $K \in \mathcal{K}^n$ and $B_1, B_2, B_1 \cup B_2 \in \mathcal{K}_0^n$. Thus it follows from (9) that $\mathbb{I}$ is a Minkowski valuation on $\mathcal{K}_0^n$. Since z is continuous, also $\mathbb{I}$ is continuous. Hence Theorem 4.3 and Theorem 5.1 imply that there is a constant $c \geq 0$ such that $\mathbb{I}\, B = c\, \Pi\, B^*$ for all $B \in \mathcal{K}_0^n$.

7. An open problem

Theorem 1 shows that the Holmes-Thompson area is the only bivaluation on $\mathcal{K}^n \times \mathcal{K}_0^n$ that is an invariant area. Is it also possible to obtain a complete classifications of bivaluations on $\mathcal{K}^n \times \mathcal{K}_c^n$ that are invariant areas? Is the Holmes-Thompson area again the unique such area?

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