HESSIAN VALUATIONS

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ABSTRACT. A new class of continuous valuations on the space of convex functions on $\mathbb{R}^n$ is introduced. On smooth convex functions, they are defined for $i = 0, \ldots, n$ by

$$u \mapsto \int_{\mathbb{R}^n} \zeta(u(x), x, \nabla u(x)) \, [\mathrm{D}^2 u(x)]_i \, \mathrm{d}x$$

where $\zeta \in C(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n)$ and $[\mathrm{D}^2 u]_i$ is the ith elementary symmetric function of the eigenvalues of the Hessian matrix, $\mathrm{D}^2 u$, of u. Under suitable assumptions on ζ, these valuations are shown to be invariant under translations and rotations on convex and coercive functions.

1. INTRODUCTION

The purpose of this paper is to introduce a new type of valuations on convex functions. On smooth convex functions, they are defined for $i = 0, \ldots, n$ by

$$(1.1) \qquad u \mapsto \int_{\mathbb{R}^n} \zeta(u(x), x, \nabla u(x)) \, [\mathrm{D}^2 u(x)]_i \, \mathrm{d}x,$$

where $\zeta : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is continuous and $[\mathrm{D}^2 u]_i$ is the ith elementary symmetric function of the eigenvalues of the Hessian matrix of u, with the usual convention $[\mathrm{D}^2 u]_0 = 1$ (alternatively, $[\mathrm{D}^2 u]_i$ is the sum of the $i \times i$ principal minors of $\mathrm{D}^2 u$). We show that these functionals can be extended to a rather ample class of convex functions, providing a family of continuous valuations that we call *Hessian valuations*. Here continuity is with respect to epi-convergence (see Section 2). Under suitable assumptions on ζ, Hessian valuations are invariant under rotations on convex functions and under translations and rotations on convex and coercive functions.

The theory of valuations on function spaces has been rapidly growing in recent years. Several spaces of functions have been investigated already, including Lebesgue and Sobolev spaces, functions of bounded variations, and quasi-concave functions (see [3–5, 9, 12, 13, 22–27, 31, 37–39]). Needless to say, the major impulse to this area comes from the rich and beautiful theory of valuations defined on the family $\mathcal{K}^n$ of convex bodies (that is, compact convex subsets) in $\mathbb{R}^n$, which is one of the most active branches of convex geometry (see [35, Chapter 6] for a recent survey on the theory of valuations on convex bodies).

As a general framework, we start from the following family of convex functions:

$$\mathrm{Conv}(\mathbb{R}^n) = \{u \colon \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \colon u \text{ convex and l.s.c., } u \not\equiv +\infty\},$$

where l.s.c. stands for lower semicontinuous. A (real-valued) valuation on $\mathrm{Conv}(\mathbb{R}^n)$ is a functional $\mathrm{Z} \colon \mathrm{Conv}(\mathbb{R}^n) \to \mathbb{R}$ which has the additivity property

$$\mathrm{Z}(u \vee v) + \mathrm{Z}(u \wedge v) = \mathrm{Z}(u) + \mathrm{Z}(v),$$

for every $u, v \in \mathrm{Conv}(\mathbb{R}^n)$ such that $u \wedge v \in \mathrm{Conv}(\mathbb{R}^n)$. Here $\vee$ and $\wedge$ denote the pointwise maximum and minimum, respectively.

The basic tool to extend a functional of the form (1.1) to $\mathrm{Conv}(\mathbb{R}^n)$ are *Hessian measures* of convex functions (see Section 7), that will be denoted by $\Theta_i(u, \cdot)$ for $i \in \{0, \dots, n\}$ and $u \in \mathrm{Conv}(\mathbb{R}^n)$. Like support measures of convex bodies, Hessian measures can be defined as coefficients of a local Steiner formula,

$$(1.2) \qquad \mathcal{H}^n(P_s(u, \eta)) = \sum_{i=0}^{n} \binom{n}{i} s^i \, \Theta_{n-i}(u, \eta),$$

where $\mathcal{H}^n$ is the n-dimensional Hausdorff measure and $s \geq 0$ while for a Borel subset η of $\mathbb{R}^n \times \mathbb{R}^n$,

$$P_s(u, \eta) = \{x + sy \colon (x, y) \in \eta, \ y \in \partial u(x)\}.$$

Here $\partial u(x)$ denotes the subdifferential of u at the point x (see Section 2). Hessian measures permit to extend to non-smooth convex functions integrals of the elementary symmetric functions of the eigenvalues of the Hessian matrix. Indeed, if $u \in \mathrm{Conv}(\mathbb{R}^n) \cap C^2(\mathbb{R}^n)$ and $i \in \{0, \dots, n\}$, then

$$\Theta_i(u, \beta \times \mathbb{R}^n) = \int_{\beta} [\mathrm{D}^2 u(x)]_{n-i} \, \mathrm{d}x$$

for every Borel subset β of $\mathbb{R}^n$. Hessian measures have been considered in connection with convex (and more general) functions in [8,10,11,18] and are related to non-linear elliptic partial differential equations (see, *e.g.*, [6, 16, 36]).

To define the integral (1.1) for an arbitrary $u \in \mathrm{Conv}(\mathbb{R}^n)$, we integrate $\zeta(u(x), x, y)$ with respect to the Hessian measure of u, where the variable y plays the role of ∇u. To guarantee integrability, we assume that $\zeta(t, x, y)$ has compact support with respect to the second and third variables, that is, there exists $r > 0$ such that ζ vanishes in the complement set of the cylinder $\{(t, x, y) \colon |x| \leq r, |y| \leq r\}$, where $|\cdot|$ is the Euclidean norm. Our first result is the following.

Theorem 1.1. *Let* $\zeta \in C(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n)$ *have compact support with respect to the second and third variables. For every* $i \in \{0, 1, \dots, n\}$, *the functional* $\mathrm{Z}_{\zeta,i} \colon \mathrm{Conv}(\mathbb{R}^n) \to \mathbb{R}$, *defined by*

$$(1.3) \qquad \mathrm{Z}_{\zeta,i}(u) = \int_{\mathbb{R}^{2n}} \zeta(u(x), x, y) \, \mathrm{d}\Theta_i(u, (x, y)),$$

is a continuous valuation on $\mathrm{Conv}(\mathbb{R}^n)$. *If* $u \in \mathrm{Conv}(\mathbb{R}^n) \cap C^2(\mathbb{R}^n)$, *then* $\mathrm{Z}_{\zeta,i}(u)$ *takes the form* (1.1).

For ζ and i given as above, two additional properties of $\mathrm{Z}_{\zeta,i}$ will be established.

(i) The functional $\mathrm{Z}_{\zeta,i}$ is i-simple, that is, we have $\mathrm{Z}_{\zeta,i}(u) = 0$ for all $u \in \mathrm{Conv}(\mathbb{R}^n)$ such that $\dim(\mathrm{dom}(u)) < i$, where $\mathrm{dom}(u) = \{x \in \mathbb{R}^n \colon u(x) < +\infty\}$ is the domain of u and $\dim$ stands for dimension.

(ii) If ζ is of the form $\zeta(t, x, y) = \xi(t, |x|, |y|)$ for some $\xi \in C(\mathbb{R} \times [0, +\infty)^2)$, then (1.3) is invariant under rotations, that is, $\mathrm{Z}_{\zeta,i}(u) = \mathrm{Z}_{\zeta,i}(u \circ \phi^{-1})$ for every $\phi \in \mathrm{O}(n)$.

Remark. In [3], Alesker considers the space of convex functions on an open and convex set $U \subset \mathbb{R}^n$. Using Monge-Ampère measures, he introduces a class of valuations which extend to general convex functions functionals of the form

$$u \mapsto \int_U \xi(x) \det(\mathrm{D}^2 u(x), \dots, \mathrm{D}^2 u(x), A_{k+1}(x), \dots, A_n(x)) \, \mathrm{d}x,$$

where the function $\xi \in C(\mathbb{R}^n)$ has compact support, $\det$ is the mixed discriminant operator of n matrices, the Hessian $\mathrm{D}^2 u(x)$ appears k times with $k \in \{0, \dots, n\}$, and A_i is a symmetric $n \times n$ matrix having as coefficients continuous and compactly supported functions for every $i \in \{k+1, \dots, n\}$. There is an overlap between Alesker's valuations and the ones introduced here. Indeed, for $U = \mathbb{R}^n$, if $\zeta = \xi$

depends on the x-variable only and we choose $A_i \equiv I_n$ for $i = k + 1, \ldots, n$, where I_n is the identity matrix of order n, then the notions coincide for every $u \in \mathrm{Conv}(\mathbb{R}^n) \cap C^2(\mathbb{R}^n)$.

In [7, 14, 15, 30], valuations defined on the space of convex and coercive functions,

$$\mathrm{Conv}_{\mathrm{coe}}(\mathbb{R}^n) = \{u \colon \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \colon u \text{ convex, l.s.c., and coercive, } u \not\equiv +\infty\}$$

were studied and classified. Here a function $u : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is coercive if $\lim_{|x| \to +\infty} u(x) = +\infty$. In [7, 15], the following type of valuations on $\mathrm{Conv}_{\mathrm{coe}}(\mathbb{R}^n)$ were considered for suitable $\zeta : \mathbb{R} \to \mathbb{R}$:

$$(1.4) \qquad\qquad u \mapsto \int_{\mathbb{R}^n} \zeta(u(x)) \, \mathrm{d}x.$$

In [15], it is proved that (1.4) defines a continuous and rigid motion invariant valuation on $\mathrm{Conv}_{\mathrm{coe}}(\mathbb{R}^n)$ if and only if ζ is continuous on $\mathbb{R}$ and has finite $(n-1)$th moment. Moreover, it is proved that every continuous, non-negative, $\mathrm{SL}(n)$ and translation invariant valuation on $\mathrm{Conv}_{\mathrm{coe}}(\mathbb{R}^n)$ can be written as a functional of the form (1.4) plus a function only depending on $\min_{\mathbb{R}^n} u$.

We obtain new valuations on $\mathrm{Conv}_{\mathrm{coe}}(\mathbb{R}^n)$. The coercivity guarantees that integrability is preserved under less restrictive conditions on ζ. In particular, we may remove the dependence on the space variable x, gaining translation invariance.

Theorem 1.2. *Let $\zeta \in C(\mathbb{R} \times \mathbb{R}^n)$ have compact support. For every $i \in \{0, \ldots, n\}$, the functional $Z_{\zeta,i} \colon \mathrm{Conv}_{coe}(\mathbb{R}^n) \to \mathbb{R}$, defined by*

$$(1.5) \qquad\qquad Z_{\zeta,i}(u) = \int_{\mathbb{R}^{2n}} \zeta(u(x), y) \, \mathrm{d}\Theta_i(u, (x, y)),$$

is a continuous, translation invariant, i-simple valuation. Moreover, if there exists $\xi \in C(\mathbb{R} \times [0, +\infty))$ such that $\zeta(t, y) = \xi(t, |y|)$ on $\mathbb{R} \times \mathbb{R}^n$, then $Z_{\zeta,i}$ is also rotation invariant.

Hence, for every $\zeta \in C(\mathbb{R} \times [0, +\infty))$ with compact support, the valuation

$$u \mapsto \int_{\mathbb{R}^{2n}} \zeta(u(x), |y|) \, \mathrm{d}\Theta_i(u, (x, y))$$

is continuous and rigid motion invariant for $i \in \{0, \ldots, n\}$.

We mention that in [7] the construction of a class of valuations on $\mathrm{Conv}_{\mathrm{coe}}(\mathbb{R}^n)$ is presented, based on quermassintegrals (or intrinsic volumes) of level sets. Similar ideas can be found in [5, 28, 29]. A comparison between those and Hessian valuations is carried out in Section 11.1, where in particular we show that there are Hessian valuations that are not included in the class described in [7].

In the following, we collect results needed for the preparation of the proofs of the main results. In Section 3, we recall some basic facts on convex functions. In Section 4, the Lipschitz regularization of convex functions is described. Sections 5 and 6 are devoted to inclusion-exclusion type properties of the subdifferential, the subdifferential graph and local parallel sets of convex functions. These properties will be critical to prove the valuation property of Hessian measures, which is discussed in Sections 7 and 8. In particular, in Section 7 we extend to $\mathrm{Conv}(\mathbb{R}^n)$ the results of [11] concerning existence and integral representations of Hessian measures, while in the next section we discuss several (known and new) facts about them.

Subsequently, after giving the main definitions for valuations on $\mathrm{Conv}(\mathbb{R}^n)$ in Section 9, we prove Theorems 1.1 and 1.2 in Sections 10 and 11. In Section 12, we briefly analyze what happens if we restrict Hessian valuations to the subset $\mathrm{Conv}_{\mathrm{hom}}(\mathbb{R}^n)$ of $\mathrm{Conv}(\mathbb{R}^n)$, formed by finite convex functions homogeneous of degree 1, that is, by support functions of convex bodies. Note that continuous valuations on $\mathrm{Conv}_{\mathrm{hom}}(\mathbb{R}^n)$ correspond to continuous valuations on $\mathcal{K}^n$.

2. PRELIMINARIES

We work in n-dimensional Euclidean space, $\mathbb{R}^n$, for $n \geq 1$, endowed with the usual Euclidean norm $|\cdot|$ and scalar product $\langle \cdot, \cdot \rangle$. We set $\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n \colon |x| = 1\}$ and $B^n = \{x \in \mathbb{R}^n : |x| \leq 1\}$. For $r > 0$, let B_r^n be the closed ball of $\mathbb{R}^n$, centered at the origin and with radius r. Given a subset A of $\mathbb{R}^n$, its interior and boundary will be denoted by $\mathrm{int}(A)$ and $\mathrm{bd}(A)$, respectively. For $k \in [0, n]$, the k-dimensional Hausdorff measure in $\mathbb{R}^n$ is denoted by $\mathcal{H}^k$. In particular, $\mathcal{H}^n$ is the Lebesgue measure in $\mathbb{R}^n$. We write π_1 and π_2 for the canonical projections of $\mathbb{R}^n \times \mathbb{R}^n$ onto the first and the second component, respectively; that is,

$$\pi_1(x, y) = x, \quad \pi_2(x, y) = y$$

for $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$.

A subset C of $\mathbb{R}^n$ is convex if for every $x_0, x_1 \in C$ and $t \in [0, 1]$, we have $(1 - t)x_0 + tx_1 \in C$. If $C \subset \mathbb{R}^n$ is convex, we define its dimension, $\dim(C)$, as the minimum integer $k \in \{0, 1, \ldots, n\}$ such that there exists an affine subspace of dimension k containing C. The relative interior of a convex set C of dimension k is the subset of those points x of C for which there exists a k-dimensional ball centered at x and contained in C. The relative interior will be denoted by $\mathrm{relint}(C)$. Note that $\dim(C) = n$ if and only if $\mathrm{relint}(C) = \mathrm{int}(C)$. The group of rotations of $\mathbb{R}^n$ will be denoted by $\mathrm{O}(n)$. By a *rigid motion* we mean the composition of a translation and an element of $\mathrm{O}(n)$.

A *convex body* is a compact convex subset of $\mathbb{R}^n$. We will denote by $\mathcal{K}^n$ the family of convex bodies in $\mathbb{R}^n$. Our main reference for the theory of convex body is the monograph [35]. We will sometimes need separation theorems, especially for convex sets. For the notion of separation, strict separation and strong separation of two subsets of $\mathbb{R}^n$ by a hyperplane we refer to [35, Section 1.3].

If u and v are functions defined in $\mathbb{R}^n$, taking values in $\mathbb{R}$ or in $\mathbb{R} \cup \{+\infty\}$, we denote by $u \vee v$ and $u \wedge v$ the pointwise maximum function and the pointwise minimum function of u and v, respectively. In other words, for $x \in \mathbb{R}^n$,

$$(u \vee v)(x) = \max\{u(x), v(x)\}, \quad (u \wedge v)(x) = \min\{u(x), v(x)\}.$$

For $u \colon \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, we define its epigraph by

$$\mathrm{epi}(u) = \{(x, t) \in \mathbb{R}^n \times \mathbb{R} \colon t \geq u(x)\}.$$

We say that a function $w \colon \mathbb{R}^n \to \mathbb{R}$ is affine if there exist $y \in \mathbb{R}^n$ and $c \in \mathbb{R}$ such that $w(x) = \langle x, y \rangle + c$ for $x \in \mathbb{R}^n$. We denote by $\mathrm{Aff}(\mathbb{R}^n)$ the family of affine functions on $\mathbb{R}^n$.

3. CONVEX FUNCTIONS

A function $u \colon \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is convex if for every $x_0, x_1 \in \mathbb{R}^n$ and for every $t \in [0, 1]$, we have

$$u((1 - t)x_0 + tx_1) \leq (1 - t)u(x_0) + tu(x_1).$$

Recall that a function u is convex if and only if the epigraph of u is a convex subset of $\mathbb{R}^n \times \mathbb{R}$. Note that the lower semicontinuity of u is equivalent to $\mathrm{epi}(u)$ being closed. Such functions are also called *closed*. Our main reference texts on convex analysis – the theory of convex functions – are [33] and [34].

For $u \in \mathrm{Conv}(\mathbb{R}^n)$, we define the *domain* of u as

$$\mathrm{dom}(u) := \{x \in \mathbb{R}^n \colon u(x) < +\infty\}.$$

By the convexity of u, its domain is a convex set. Every convex function u is continuous in the interior of $\operatorname{dom}(u)$.

Example 3.1. Let K be a convex body in $\mathbb{R}^n$. The *convex indicator function*, $I_K \colon \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, of K is defined by

$$I_K(x) = \begin{cases} 0 & \text{if } x \in K, \\ +\infty & \text{if } x \notin K. \end{cases}$$

Hence $I_K \in \operatorname{Conv}_{\mathrm{coe}}(\mathbb{R}^n)$ for every $K \in \mathcal{K}^n$.

Example 3.2. Another function associated to a convex body K is its *support function*, denoted by h_K and defined on $\mathbb{R}^n$ by

$$h_K(y) = \sup_{x \in K} \langle x, y \rangle$$

(see [35, Chapter 1]). The support function h_K is a 1-homogeneous convex function for every $K \in \mathcal{K}^n$, where $f \colon \mathbb{R}^n \to \mathbb{R}$ is called 1-homogeneous, if $f(t\,x) = t\,u(x)$ for all $t > 0$ and $x \in \mathbb{R}^n$. In particular, the support function is everywhere finite by the compactness of K.

The pointwise maximum of two convex functions is again a convex function, but this does not guarantee that $\operatorname{Conv}(\mathbb{R}^n)$ is closed with respect to $\vee$, as $u \vee v$ may be identically $+\infty$ for some $u, v \in \operatorname{Conv}(\mathbb{R}^n)$. On the other hand, it is easy to see that if $u, v \in \operatorname{Conv}(\mathbb{R}^n)$ are such that $u \wedge v \in \operatorname{Conv}(\mathbb{R}^n)$, then $u \vee v \in \operatorname{Conv}(\mathbb{R}^n)$.

3.1. **The conjugate of a convex function.** We recall the notion of conjugate of a convex function (see [33]). For $u \in \operatorname{Conv}(\mathbb{R}^n)$ and $y \in \mathbb{R}^n$, we set

$$u^*(y) := \sup_{x \in \mathbb{R}^n} \big(\langle x, y \rangle - u(x) \big).$$

As every function $u \in \operatorname{Conv}(\mathbb{R}^n)$ is closed with $u \not\equiv +\infty$, the following result is a consequence of Theorem 12.2 and Corollary 12.2.1 in [33].

Proposition 3.3. *If $u \in \operatorname{Conv}(\mathbb{R}^n)$, then $u^* \in \operatorname{Conv}(\mathbb{R}^n)$ and $u^{**} := (u^*)^* = u$.*

We require the following result.

Lemma 3.4. *Let $u, v \in \operatorname{Conv}(\mathbb{R}^n)$. If $u \wedge v \in \operatorname{Conv}(\mathbb{R}^n)$, then so is $u^* \wedge v^*$.*

Proof. For $y \in \mathbb{R}^n$ and $s \in \mathbb{R}$, define the affine function $\ell_{y,s}$ by $\ell_{y,s}(x) = \langle y, x \rangle - s$. It follows immediately from the definition of conjugate functions that $(y, s) \in \operatorname{epi}(u^*)$ if and only if $\ell_{y,s} \leq u$.

It is clear that $u^* \wedge v^*$ is closed and proper. Suppose that $u^* \wedge v^*$ is not convex. Then there are $(y_1, s_1) \in \operatorname{epi}(u^*)$, $(y_2, s_2) \in \operatorname{epi}(v^*)$ and $\lambda \in (0, 1)$ such that

$$(z, t) = (1 - \lambda)(y_1, s_1) + \lambda(y_2, s_2) \notin \operatorname{epi}(u^* \wedge v^*).$$

Hence there exist $x_1, x_2 \in \mathbb{R}^n$ such that $\ell_{z,t}(x_1) > u(x_1)$ and $\ell_{z,t}(x_2) > v(x_2)$. Since $\operatorname{epi}(u \wedge v)$ is convex, the segment connecting $(x_1, \ell_{z,t}(x_1) - \varepsilon)$ and $(x_2, \ell_{z,t}(x_2) - \varepsilon)$ intersects $\operatorname{epi}(u \vee v)$ for $\varepsilon > 0$ sufficiently small. Hence there is a point x_0 such that

$$(3.1) \qquad\qquad \ell_{z,t}(x_0) > (u \vee v)(x_0).$$

But, since $(y_1, s_1) \in \operatorname{epi}(u^*)$ and $(y_2, s_2) \in \operatorname{epi}(v^*)$, we have

$$\ell_{z,t}(x) \leq (1 - \lambda)u(x) + \lambda\, v(x) \leq (u \vee v)(x).$$

This is a contradiction to (3.1). $\qquad\square$

Proposition 3.5. *If $u, v \in \mathrm{Conv}(\mathbb{R}^n)$ are such that $u \wedge v \in \mathrm{Conv}(\mathbb{R}^n)$, then*

$$(3.2) \qquad\qquad (u \wedge v)^* = u^* \vee v^*,$$

and

$$(3.3) \qquad\qquad (u \vee v)^* = u^* \wedge v^*.$$

Proof. For every $y \in \mathbb{R}^n$, we have

$$
\begin{aligned}
(u \wedge v)^*(y) &= \sup_{x \in \mathbb{R}^n} \left(\langle x, y \rangle - (u \wedge v)(x) \right) \\
&= \sup_{x \in \mathbb{R}^n} \left(\max \left\{ \langle x, y \rangle - u(x), \langle x, y \rangle - v(x) \right\} \right) \\
&= \max \left\{ \sup_{x \in \mathbb{R}^n} \left(\langle x, y \rangle - u(x), \sup_{x \in \mathbb{R}^n} \left(\langle x, y \rangle - v(x) \right) \right) \right\} \\
&= (u^* \vee v^*)(y).
\end{aligned}
$$

This proves (3.2). Together with Proposition 3.3 and Lemma 3.4 this gives

$$u^* \wedge v^* = ((u^* \wedge v^*)^*)^* = ((u^*)^* \vee (v^*)^*)^* = (u \vee v)^*,$$

which shows (3.3). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

3.2. **Topology in** $\mathrm{Conv}(\mathbb{R}^n)$**.** As in [15] and [14], we adopt the topology induced by *epi-convergence*. A sequence u_k of elements of $\mathrm{Conv}(\mathbb{R}^n)$ is epi-convergent to $u \in \mathrm{Conv}(\mathbb{R}^n)$ if for every $x \in \mathbb{R}^n$ the following conditions hold.

(i) For every sequence x_k that converges to x,

$$u(x) \le \liminf_{k \to +\infty} u_k(x_k).$$

(ii) There exists a sequence x_k that converges to x such that

$$u(x) = \lim_{k \to +\infty} u_k(x_k).$$

An exhaustive source for epi-convergence of convex functions is [34], where also the following result can be found.

Proposition 3.6. *A sequence u_k of functions from $\mathrm{Conv}(\mathbb{R}^n)$ epi-converges to $u \in \mathrm{Conv}(\mathbb{R}^n)$ if and only if the sequence u_k^* epi-converges to u^*.*

Remark. If u_k is a sequence of *finite* functions in $\mathrm{Conv}(\mathbb{R}^n)$, that is, $\mathrm{dom}(u_k) = \mathbb{R}^n$ for every k, and $u \in \mathrm{Conv}(\mathbb{R}^n)$ is also finite, then u_k epi-converges to u if and only if it converges to u pointwise in $\mathbb{R}^n$ and uniformly on compact sets (see for instance [34, Theorem 7.17]).

3.3. **Subdifferentials.** Let $u \in \mathrm{Conv}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$. A vector $y \in \mathbb{R}^n$ is said to be a *subgradient* of u at x if

$$u(z) \ge u(x) + \langle z - x, y \rangle$$

for all $z \in \mathbb{R}^n$. The (possibly empty) set of all vectors with this property will be denoted by $\partial u(x)$ and called the *subdifferential* of u at x. In particular, $\partial u(x) = \emptyset$ for every $x \notin \mathrm{dom}(u)$. Also note that if $x \in \mathbb{R}^n$ is such that $\partial u(x) \ne \emptyset$, then $\partial u(x)$ is closed and convex.

Remark. If $u \in \mathrm{Conv}(\mathbb{R}^n)$, then there exists at least one point $x \in \mathbb{R}^n$ such that $\partial u(x) \ne \emptyset$, since $\mathrm{dom}(u) \ne \emptyset$ implies that $\mathrm{relint}(\mathrm{dom}(u)) \ne \emptyset$ and by [33, p. 227].

The subdifferential is strongly connected to directional derivatives. Let $u \in \mathrm{Conv}(\mathbb{R}^n)$. Following the notation used in [33], for $x \in \mathrm{dom}(u)$ and a vector $y \in \mathbb{R}^n$, we set

$$u'(x; y) = \lim_{t \to 0^+} \frac{u(x + ty) - u(x)}{t}.$$

This limit always exists, finite or not, that is, $u'(x; y) \in \mathbb{R} \cup \{\pm\infty\}$ (see [33, Theorem 23.1]). The following result is [33, Theorem 23.2].

Lemma 3.7. *Let $u \in \mathrm{Conv}(\mathbb{R}^n)$ and $x \in \mathrm{dom}(u)$. For $y \in \mathbb{R}^n$, we have $y \in \partial u(x)$ if and only if $\langle z, y \rangle \leq u'(x; z)$ for all $z \in \mathbb{R}^n$.*

We also need the following result, which can be found in [33, Theorem 23.5].

Lemma 3.8. *For $u \in \mathrm{Conv}(\mathbb{R}^n)$ and $x, y \in \mathbb{R}^n$, the following are equivalent:*

(i) $y \in \partial u(x)$,
(ii) $x \in \partial u^*(y)$,
(iii) $\langle x, y \rangle = u(x) + u^*(y)$,
(iv) $x \in \mathrm{argmax}_{z \in \mathbb{R}^n} (\langle y, z \rangle - u(z))$,
(v) $y \in \mathrm{argmax}_{z \in \mathbb{R}^n} (\langle x, z \rangle - u^*(z))$.

Here $\mathrm{argmax}_{z \in V} v(z)$ denotes the set of points in the set V at which the function values of v are maximized on V.

4. LIPSCHITZ REGULARIZATION

For a function $u \in \mathrm{Conv}(\mathbb{R}^n)$ and $r > 0$, we consider its *Pasch-Hausdorff envelope* or *Lipschitz regularization*

$$\mathrm{reg}_r u = (u^* + I_{B^n_{1/r}})^*$$

(see [20, 21] and [34, Example 9.11]).

Remark. Given $u \in \mathrm{Conv}(\mathbb{R}^n)$ and $r > 0$, the Lipschitz regularization $\mathrm{reg}_r u$ admits the following equivalent definition

$$\mathrm{reg}_r u(x) = \sup\{w(x) \colon w \in \mathrm{Aff}(\mathbb{R}^n), w \leq u \text{ in } \mathbb{R}^n, |\nabla w| \leq 1/r\}.$$

In other words, the graph of $\mathrm{reg}_r u$ is the envelope of all supporting hyperplanes to the graph of u, with slope bounded by $1/r$. This can be also rephrased as follows: $\mathrm{reg}_r u$ is the largest convex function which is smaller than u and has Lipschitz constant bounded by $1/r$ (for sufficiently large r). For brevity, we omit the proof of the equivalence of these definitions.

Proposition 4.1. *For $u \in \mathrm{Conv}(\mathbb{R}^n)$ and $r > 0$, the Lipschitz regularization has the following properties.*

(i) For $r > 0$, the function $\mathrm{reg}_r u$ is convex in $\mathbb{R}^n$. There exists $r_0 > 0$ such that $\mathrm{reg}_r u(x) > -\infty$ for every $0 < r \leq r_0$ and $x \in \mathbb{R}^n$.
(ii) For $r > 0$, we have $\mathrm{reg}_r u < +\infty$ on $\mathbb{R}^n$.
(iii) For $0 < r < t$, we have $\mathrm{reg}_t u(x) \leq \mathrm{reg}_r u(x) \leq u(x)$ for every $x \in \mathbb{R}^n$.
(iv) As $r \to 0$, the functions $\mathrm{reg}_r u$ epi-converge to u.
(v) For every $x \in \mathbb{R}^n$, we have $\lim_{r \to 0} \mathrm{reg}_r u(x) = u(x)$.
(vi) If $x \in \mathrm{dom}(u)$ and $y \in \partial u(x)$, then $\mathrm{reg}_r u(x) = u(x)$ and $\partial \mathrm{reg}_r u(x) = \partial u(x) \cap B^n_{1/r}$ for $|y| < \frac{1}{r}$.

Proof. (i) Since $\mathrm{reg}_r u$ is defined as the conjugate of a convex function, it is convex itself. Furthermore, since $u^* \not\equiv +\infty$, there exists $r_0 > 0$ such that $\mathrm{dom}(u^*) \cap B^n_{1/r} \neq \emptyset$ for every $0 < r \leq r_0$ and therefore $u^* + I_{B^n_{1/r}} \in \mathrm{Conv}(\mathbb{R}^n)$. Hence, Proposition 3.3 implies that $\mathrm{reg}_r u \in \mathrm{Conv}(\mathbb{R}^n)$ for every $0 < r \leq r_0$.

(ii) It follows from the definition of the convex conjugate, that

$$\operatorname{reg}_r u(x) = \sup_{y \in \mathbb{R}^n} \left(\langle x, y \rangle - (u^*(y) + I_{B^n_{1/r}}(y)) \right) = \sup_{y \in B^n_{1/r}} \left(\langle x, y \rangle - u^*(y) \right) < +\infty.$$

(iii) This follows from the fact that convex conjugation is order reversing, that is, if $u \le v$ pointwise, then $u^* \ge v^*$ pointwise.

(iv) By Proposition 3.6, it is enough to show that $u^* + I_{B^n_{1/r}}$ is epi-convergent to u^*, which follows from the properties of epi-convergence (see also [34, Theorem 7.46]).

(v) By (iii), the function $\operatorname{reg}_r u(x)$ is bounded above by $u(x)$ and decreasing in r. Hence, $\lim_{r \to 0} \operatorname{reg}_r u(x)$ exists in $(-\infty, +\infty]$. Together with (iv) and the definition of epi-convergence, this gives

$$u(x) \le \lim_{s \to 0} \operatorname{reg}_r u(x) \le u(x).$$

(vi) For $x \in \operatorname{dom}(u)$, we have by Lemma 3.8

$$y \in \partial u(x), \quad |y| \le \frac{1}{r} \quad \Longleftrightarrow \quad y \in \operatorname{argmax}_{z \in B^n_{1/r}} \left(\langle x, z \rangle - u^*(z) \right)$$

$$\Longleftrightarrow \quad y \in \operatorname{argmax}_{z \in \mathbb{R}^n} \left(\langle x, z \rangle - (u^*(z) + I_{B^n_{1/r}}(z)) \right)$$

$$\Longleftrightarrow \quad y \in \operatorname{argmax}_{z \in \mathbb{R}^n} \left(\langle x, z \rangle - (\operatorname{reg}_r u)^*(z) \right)$$

$$\Longleftrightarrow \quad y \in \partial \operatorname{reg}_r u(x).$$

Hence, for $y \in \partial u(x)$ with $|y| < \frac{1}{r}$,

$$u(x) = \sup_{z \in B^n_{1/r}} \left(\langle x, z \rangle - u^*(z) \right) = \sup_{z \in \mathbb{R}^n} \left(\langle x, z \rangle - u^*(z) - I_{B^n_{1/r}}(z) \right) = \operatorname{reg}_r u(x),$$

which concludes the proof. $\square$

Proposition 4.2. *Let $u_j, u \in \operatorname{Conv}(\mathbb{R}^n)$. If u_j epi-converges to u, then $\operatorname{reg}_r u_j$ epi-converges to $\operatorname{reg}_r u$ as $j \to +\infty$ for sufficiently small $r > 0$.*

Proof. By Proposition 3.6, it is enough to prove that given a sequence $u_j \in \operatorname{Conv}(\mathbb{R}^n)$ that is epi-convergent to $u \in \operatorname{Conv}(\mathbb{R}^n)$, then, for $s > 0$ sufficiently large, the sequence $u_j + I_{B^n_s}$ epi-converges to $u + I_{B^n_s}$.

Let us fix $s > 0$ such that there exists $\bar{x} \in \operatorname{dom}(u) \cap B^n_{s/4}$. By epi-convergence, we may assume that for every j there exists $\bar{x}_j \in B^n_{s/2}$ such that $u_j(\bar{x}_j) \le 2u(\bar{x})$. Let $x \in \mathbb{R}^n$ and x_j be a sequence converging to x. The inequality

$$\liminf_{j \to +\infty} (u_j + I_{B^n_s})(x_j) \ge (u + I_{B^n_s})(x)$$

follows easily from the epi-convergence of the sequence u_j to u. To conclude, we have to show that there exists a specific sequence x_j converging to x and such that

$$(4.1) \qquad \lim_{j \to +\infty} (u_j + I_{B^n_s})(x_j) = (u + I_{B^n_s})(x).$$

If either $|x| < s$ or $|x| > s$, this is again a straightforward consequence of epi-convergence. So assume that $|x| = s$ and let x_j be a sequence such that

$$\lim_{j \to +\infty} u_j(x_j) = u(x).$$

If all but a finite number of the elements of x_j belong to B^n_s, then (4.1) follows. Hence we may assume that $|x_j| > s$ for every j. Let $\hat{x}_j$ be the intersection of the segment joining x_j and $\bar{x}_j$ with $\operatorname{bd}(B^n_s)$. Then

$$\hat{x}_j = (1 - t_j)\bar{x}_j + t_j x_j$$

for a suitable $t_j \in [0, 1]$. As $|x| = |x_j| = s$ and $|\bar{x}_j| \leq s/2$ for every j, up to extracting a subsequence, we may assume that $t_j \to 1$ as $j \to +\infty$, so that, in particular

$$\lim_{j \to +\infty} \hat{x}_j = x.$$

Hence

$$\liminf_{j \to +\infty}(u_j + I_{B_s^n})(\hat{x}_j) = \liminf_{j \to +\infty} u_j(\hat{x}_j) \geq u(x) = (u + I_{B_s^n})(x).$$

On the other hand, by convexity

$$(u_j + I_{B_r^n})(\hat{x}_j) = u_j(\hat{x}_j) \leq (1 - t_j)u_j(\bar{x}_j) + t_j u_j(x_j)$$

for $j \in \mathbb{N}$. Passing to the limit as $j \to +\infty$ (and recalling that the sequence $u_j(\bar{x}_j)$ is bounded from above), we obtain

$$\limsup_{j \to +\infty}(u_j + I_{B_s^n})(\hat{x}_j) \leq u(x) = (u + I_{B_s^n})(x)$$

which concludes the proof. $\qquad\square$

Proposition 4.3. *Let* $u, v \in \mathrm{Conv}(\mathbb{R}^n)$. *If* $u \wedge v \in \mathrm{Conv}(\mathbb{R}^n)$, *then*

$$\mathrm{reg}_r(u \wedge v) = \mathrm{reg}_r u \wedge \mathrm{reg}_r v, \quad \mathrm{reg}_r(u \vee v) = \mathrm{reg}_r u \vee \mathrm{reg}_r v$$

for sufficiently small $r > 0$.

Proof. By Proposition 3.5, the function $u^* \wedge v^*$ is convex. Hence,

$$(\mathrm{reg}_r u)^* \wedge (\mathrm{reg}_r v)^* = (u^* + I_{B_{1/r}^n}) \wedge (v^* + I_{B_{1/r}^n}) = (u^* \wedge v^*) + I_{B_{1/r}^n}$$

is convex as well, and this implies that $\mathrm{reg}_r u \wedge \mathrm{reg}_r v$ is convex. Consequently

$$
\begin{aligned}
\mathrm{reg}_r(u \wedge v) &= ((u \wedge v)^* + I_{B_{1/r}^n})^* \\
&= (u^* \vee v^* + I_{B_{1/r}^n})^* \\
&= ((u^* + I_{B_{1/r}^n}) \vee (v^* + I_{B_{1/r}^n}))^* \\
&= ((\mathrm{reg}_r u)^* \vee (\mathrm{reg}_r v)^*)^* \\
&= \mathrm{reg}_r u \wedge \mathrm{reg}_r u.
\end{aligned}
$$

The second equation is proved analogously. $\qquad\square$

5. INCLUSION-EXCLUSION PROPERTIES OF THE SUBGRADIENT MAP

In this section we will prove the following result.

Theorem 5.1. *Let* $u, v \in \mathrm{Conv}(\mathbb{R}^n)$. *If* $u \wedge v \in \mathrm{Conv}(\mathbb{R}^n)$, *then*

$$
\begin{aligned}
\partial(u \vee v)(x) \cup \partial(u \wedge v)(x) &= \partial u(x) \cup \partial v(x), \\
\partial(u \vee v)(x) \cap \partial(u \wedge v)(x) &= \partial u(x) \cap \partial v(x)
\end{aligned}
$$

for every $x \in \mathbb{R}^n$.

The proof will require some preliminary steps.

Lemma 5.2. *Let* C_1 *and* C_2 *be non-empty, closed and convex subsets of* $\mathbb{R}^n$. *If* $C_1 \cup C_2$ *is convex, then* $C_1 \cap C_2 \neq \emptyset$.

Proof. Let $x_1 \in C_1$; $x_2 \in C_2$. Then the closed segment $[x_1; x_2]$ is contained in $C_1 \cup C_2$ by convexity. Since $[x_1; x_2]$ is connected and C_1 and C_2 are closed, there exists $y \in C_1 \cap C_2 \cap [x_1; x_2]$. $\qquad\square$

Note that the claim of the previous lemma fails to be true without assuming the sets to be closed.

Lemma 5.3. *Let $u, v \in \mathrm{Conv}(\mathbb{R}^n)$. If $u \wedge v \in \mathrm{Conv}(\mathbb{R}^n)$ and $u \vee v \geq 0$ in $\mathbb{R}^n$, then $u \geq 0$ or $v \geq 0$ in $\mathbb{R}^n$.*

Proof. By contradiction, assume that there exist $x_1, x_2 \in \mathbb{R}^n$ such that $c := u(x_1) \vee v(x_2) < 0$. Let

$$C_1 = \{x \in \mathbb{R}^n \colon u(x) \leq c\}, \quad C_2 = \{x \in \mathbb{R}^n \colon v(x) \leq c\}.$$

These are non-empty, closed convex sets. Moreover

$$\{x \in \mathbb{R}^n \colon (u \wedge v)(x) \leq c\} = C_1 \cup C_2,$$

that is, their union is convex. By the previous lemma, there exists $x \in C_1 \cap C_2$; on the other hand this leads to

$$(u \vee v)(x) \leq c < 0,$$

which is a contradiction. $\square$

We proceed with a one-dimensional result.

Lemma 5.4. *Let $u, v \colon \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ be convex and l.s.c. and assume that $u \wedge v$ is convex. If $\bar{x} \in \mathbb{R}$ is such that $u(\bar{x}) < v(\bar{x})$, then there exists $\delta > 0$ such that $u(x) \leq v(x)$ for all $x \in [\bar{x} - \delta, \bar{x} + \delta]$.*

Proof. Clearly $\bar{x} \in \mathrm{dom}(u)$. If $\bar{x}$ is in the interior of $\mathrm{dom}(u)$, then u is continuous at $\bar{x}$ and the statement follows. If $\bar{x}$ is a boundary point of $\mathrm{dom}(u)$ and $u(x) = +\infty$ for $x > \bar{x}$, say, then the convexity of $u \wedge v$ implies that $v(x) = +\infty$ for $x > \bar{x}$, too. $\square$

Lemma 5.5. *Let $u, v \in \mathrm{Conv}(\mathbb{R}^n)$ and $u \wedge v \in \mathrm{Conv}(\mathbb{R}^n)$. If $x \in \mathbb{R}^n$ is such that $u(x) < v(x)$, then $(u \wedge v)'(x; y) = u'(x; y)$ for every $y \in \mathbb{R}^n$. Moreover, if $v(x)$ is finite, then $(u \vee v)'(x; y) = v'(x; y)$ for every $y \in \mathbb{R}^n$.*

Proof. The statement is trivially true for $y = 0$; so we assume $y \neq 0$. Consider the line L passing through x and parallel to y. The restrictions of u and v to L are l.s.c. convex functions of one variable, such that their minimum is convex. By the previous lemma, for $|t|$ sufficiently small,

$$(u \wedge v)(x + ty) = u(x + ty), \quad (u \vee v)(x + ty) = v(x + ty).$$

The conclusion follows immediately. $\square$

Proof of Theorem 5.1. First, if $u(x) < v(x)$, then Lemma 3.7 and Lemma 5.5 imply that

(5.1) $$\partial(u \wedge v)(x) = \partial u(x) \quad \text{and} \quad \partial(u \vee v)(x) = \partial v(x).$$

Second, if $u(x) = v(x)$, we show that

(5.2) $$\begin{aligned} \partial(u \wedge v)(x) &= \partial u(x) \cap \partial v(x), \\ \partial(u \vee v)(x) &= \partial u(x) \cup \partial v(x). \end{aligned}$$

Indeed, if $u(x) = v(x) = +\infty$, then all subdifferentials are empty. So assume that $x \in \mathrm{dom}(u) \cap \mathrm{dom}(v)$. The first equation in (5.2) is a straightforward consequence of the definition of subdifferentials and the equality $u(x) = v(x)$. Concerning the second equation, let

$$C = \partial(u \vee v)(x), \quad D = \partial u(x) \cup \partial v(x).$$

If $y \in D$, then we may assume that $y \in \partial u(x)$, which yields, for every $z \in \mathbb{R}^n$,

$$(u \vee v)(z) \geq u(z) \geq u(x) + \langle z - x, y \rangle = (u \vee v)(x) + \langle z - x, y \rangle,$$

that is, $y \in C$. Now assume that $y \in C$. Define $w \in \mathrm{Conv}(\mathbb{R}^n)$ as

$$w(z) = u(x) + \langle z - x, y \rangle,$$

and set $\bar{u} = u - w$ and $\bar{v} = v - w$. Then $\bar{u}, \bar{v} \in \mathrm{Conv}(\mathbb{R}^n)$ and

$$\bar{u} \wedge \bar{v} = (u \wedge v) - w, \quad \bar{u} \vee \bar{v} = (u \vee v) - w.$$

In particular, $\bar{u} \wedge \bar{v} \in \mathrm{Conv}(\mathbb{R}^n)$. Now, $y \in \partial(u \vee v)(x)$ implies $0 \in \partial(\bar{u} \vee \bar{v})(x)$; moreover $(\bar{u} \vee \bar{v})(x) = 0$. Hence $\bar{u} \vee \bar{v} \geq 0$ in $\mathbb{R}^n$. By Lemma 5.3, one of the two functions $\bar{u}$ and $\bar{v}$ is non-negative in $\mathbb{R}^n$. Assuming for instance $\bar{u} \geq 0$, we obtain, as $\bar{u}$ vanishes at x, that $0 \in \partial\bar{u}(x)$ which implies $y \in \partial u(x)$, that is, $y \in D$.

Finally, note that (5.1) and (5.2) imply the statement of the theorem. $\square$

6. THE GRAPH OF THE SUBGRADIENT MAP AND PARALLEL SETS OF FUNCTIONS

We start by recalling two important definitions.

Definition 6.1. For $u \in \mathrm{Conv}(\mathbb{R}^n)$, the graph of the subdifferential map of u is defined as

$$\Gamma_u := \{(x, y) \colon x \in \mathbb{R}^n, \ y \in \partial u(x)\}.$$

Next we define parallel sets of a convex function.

Definition 6.2. Let $u \in \mathrm{Conv}(\mathbb{R}^n)$ and $\eta \subset \mathbb{R}^n \times \mathbb{R}^n$. For $s \geq 0$, we set

$$P_s(u, \eta) = \{x + sy \colon (x, y) \in \eta \cap \Gamma_u\}.$$

Note that for $s = 0$ we have $P_0(u, \eta) = \pi_1(\eta \cap \Gamma_u)$.

6.1. **Inclusion-exclusion results.** The following result is an immediate consequence of Theorem 5.1.

Proposition 6.3. *Let $u, v \in \mathrm{Conv}(\mathbb{R}^n)$. If $u \wedge v \in \mathrm{Conv}(\mathbb{R}^n)$, then*

$$\Gamma_{u \vee v} \cap \Gamma_{u \wedge v} = \Gamma_u \cap \Gamma_v \quad \text{and} \quad \Gamma_{u \vee v} \cup \Gamma_{u \wedge v} = \Gamma_u \cup \Gamma_v.$$

Next, we establish a corresponding result for the parallel sets of the subdifferential graph.

Proposition 6.4. *Let $u, v \in \mathrm{Conv}(\mathbb{R}^n)$. If $u \wedge v \in \mathrm{Conv}(\mathbb{R}^n)$, then*

(6.1) $$P_s(u \vee v, \eta) \cup P_s(u \wedge v, \eta) = P_s(u, \eta) \cup P_s(v, \eta),$$
(6.2) $$P_s(u \vee v, \eta) \cap P_s(u \wedge v, \eta) = P_s(u, \eta) \cap P_s(v, \eta)$$

for every $\eta \subset \mathbb{R}^n \times \mathbb{R}^n$ and $s \geq 0$.

Proof. The case $s = 0$ follows easily from the previous proposition. For $s > 0$, we have

$$\begin{aligned}
P_s(u \vee v, \eta) \cup P_s(u \wedge v, \eta) &= \{x + sy \colon (x, y) \in \eta \cap \Gamma_{u \vee v}\} \cup \{x + sy \colon (x, y) \in \eta \cap \Gamma_{u \wedge v}\} \\
&= \{x + sy \colon (x, y) \in \eta \cap (\Gamma_{u \vee v} \cup \Gamma_{u \wedge v})\} \\
&= \{x + sy \colon (x, y) \in \eta \cap (\Gamma_u \cup \Gamma_v)\} \\
&= \{x + sy \colon (x, y) \in \eta \cap \Gamma_u\} \cup \{x + sy \colon (x, y) \in \eta \cap \Gamma_v\} \\
&= P_s(u, \eta) \cup P_s(v, \eta).
\end{aligned}$$

This proves (6.1). The proof of (6.2) is analogous. $\square$

6.2. **An auxiliary proposition.** Given $u \in \mathrm{Conv}(\mathbb{R}^n)$ and $r > 0$, we set

$$\Gamma_u^r := \Gamma_u \cap \{(x, y) \in \mathbb{R}^{2n} : |y| \leq r\}.$$

We require the following properties of the Lipschitz regularization of a function $u \in \mathrm{Conv}(\mathbb{R}^n)$ (in addition to those presented in Section 4).

Proposition 6.5. *If $u \in \mathrm{Conv}(\mathbb{R}^n)$ and $r > 0$, then $u_r = \mathrm{reg}_r u$ has the following properties.*

(i) $\Gamma_u^r \subset \Gamma_{u_r}$.
(ii) For every $\eta \subset \Gamma_u^r$ and $s \geq 0$, we have $P_s(u, \eta) = P_s(u_r, \eta)$.

Proof. By (vi) from Proposition 4.1, it follows from $(x, y) \in \Gamma_u^r$ that $u_r(x) = u(x)$ and $\partial u_r(x) = \partial u(x) \cap \{y \in \mathbb{R}^n : |y| \leq r\}$, which implies (i) and therefore (ii). $\square$

7. HESSIAN MEASURES

We recall and extend the definition of Hessian measures. First, we establish the following result.

Theorem 7.1. *For $u \in \mathrm{Conv}(\mathbb{R}^n)$, there are non-negative Borel measures $\Theta_0(u, \cdot), \ldots, \Theta_n(u, \cdot)$ on $\mathbb{R}^{2n}$ such that*

$$(7.1) \qquad \mathcal{H}^n(P_s(u, \eta)) = \sum_{i=0}^{n} \binom{n}{i} s^i \, \Theta_{n-i}(u, \eta)$$

for every $\eta \in \mathcal{B}(\mathbb{R}^{2n})$ and $s \geq 0$,

We will call the measures $\Theta_i(u, \cdot)$ for $i = 0, \ldots, n$ the *Hessian measures* of u. The proof makes use of the following result from [10] (see Theorem 3.1; also see [11, Section 5]). For an open subset U of $\mathbb{R}^n$, we denote by $\mathcal{B}(U)$ the family of Borel subsets of U.

Theorem 7.2. *If $u \in \mathrm{Conv}(\mathbb{R}^n)$ and $U := \mathrm{int}(\mathrm{dom}(u))$ is not empty, then there are non-negative Borel measures $\Theta_0(u, \cdot), \ldots, \Theta_n(u, \cdot)$ on $U \times \mathbb{R}^n$ such that (7.1) holds for every $\eta \in \mathcal{B}(U \times \mathbb{R}^n)$ and $s \geq 0$.*

7.1. **Proof of Theorem 7.1.** Let $r > 0$ and set $u_r = \mathrm{reg}_r u$. For $\eta \in \mathcal{B}(\mathbb{R}^{2n})$ with $\eta \subset \mathbb{R}^n \times B_r^n$, it follows from (ii) of Proposition 6.5 that $P_s(u, \eta) = P_s(u_r, \eta \cap \Gamma_u)$. Hence, by Theorem 7.2,

$$\mathcal{H}^n(P_s(u, \eta)) = \sum_{i=0}^{n} \binom{n}{i} s^i \, \Theta_{n-i}(u_r, \eta \cap \Gamma_u).$$

In this way we have proved the following fact: for every $r > 0$, there exists a set of $(n + 1)$ Borel measures on $\mathcal{B}(\mathbb{R}^n \times B_r^n)$, namely $\Theta_{i,r}(u, \cdot \cap \Gamma_u) := \Theta_i(u_r, \cdot \cap \Gamma_u)$, such that (7.1) holds for every $\eta \in \mathcal{B}(\mathbb{R}^n \times B_r^n)$ and $s \geq 0$. If $r' \geq r \geq 0$, as $\mathcal{B}(\mathbb{R}^n \times B_{r'}^n) \supset \mathcal{B}(\mathbb{R}^n \times B_r^n)$, we easily get

$$\Theta_{i,r}(u, \eta) = \Theta_{i,r'}(u, \eta)$$

for all $\eta \in \mathcal{B}(\mathbb{R}^n \times B_r^n)$ and $i = 0, \ldots, n$. Such measures can be extended to $\mathcal{B}(\mathbb{R}^{2n})$ by a standard procedure, as they are non-negative:

$$\Theta_i(u, \eta) = \lim_{r \to +\infty} \Theta_{i,r}(u, \eta \cap (\mathbb{R}^n \times B_r^n))$$

for $\eta \in \mathcal{B}(\mathbb{R}^{2n})$ and $i = 0, \ldots, n$.

The validity of the Steiner formula (7.1) is preserved as

$$P_s(u, \eta) = \bigcup_{r \geq 0} P_s(u, \eta \cap \Gamma_u^r)$$

so that

$$\mathcal{H}^n(P_s(u, \eta)) = \lim_{r \to +\infty} \mathcal{H}^n(P_s(u, \eta \cap (\mathbb{R}^n \times B_r^n)))$$

for every $\eta \in \mathcal{B}(\mathbb{R}^{2n})$ and $s \geq 0$. $\hfill\square$

Remark. The notion of Hessian measure is clearly of local nature. Let u and v be real-valued convex functions defined in an open convex set $U \subset \mathbb{R}^n$. Assume that $\partial u(x) = \partial v(x)$ for every $x \in \beta \in \mathcal{B}(U)$. If $\eta \in \mathcal{B}(U \times \mathbb{R}^n)$ is such that $\pi_1(\eta) \subset \beta$, then

$$P_s(u, \eta) = P_s(v, \eta).$$

for every $s \geq 0$. Hence

$$\Theta_i(u, \eta) = \Theta_i(v, \eta)$$

for $i = 0, \ldots, n$.

Remark. For $u \in \mathrm{Conv}(\mathbb{R}^n)$, the support of the Hessian measures of u is contained in Γ_u. Indeed, if $\eta \subset \mathbb{R}^n \setminus \Gamma_u$, then for every $(x, y) \in \eta$, we have $\partial u(x) = \emptyset$. Hence, for $s \geq 0$, we have $P_s(u, \eta) = \emptyset$, which implies that

$$0 = \mathcal{H}^n(P_s(u, \eta)) = \sum_{i=0}^{n} \binom{n}{i} s^i \, \Theta_{n-i}(u, \eta).$$

As Hessian measures are non-negative, we get $\Theta_i(u, \eta) = 0$ for every $i = 0, \ldots, n$.

Remark. Hessian measures are *locally finite*. Indeed, assume that $|(x, y)| \leq r$ for every $(x, y) \in \eta$, for some $r > 0$. As each $\Theta_i(u, \cdot)$ is non-negative, by the Steiner formula (for $s = 1$) we get

$$\binom{n}{i} \Theta_i(u, \eta) \leq \mathcal{H}^n(P_1(u, \eta)).$$

On the other hand, we have $P_1(u, \eta) \subset \{z \in \mathbb{R}^n \colon |z| \leq 2r\}$. Hence, for $i = 0, \ldots, n$,

$$(7.2) \qquad \Theta_i(u, \eta) \leq c(n) \, \mathrm{diam}(\eta)^n$$

for every $u \in \mathrm{Conv}(\mathbb{R}^n)$ and $\eta \in \mathcal{B}(\mathbb{R}^{2n})$, where $c(n)$ only depends on the dimension and diam stands for diameter.

7.2. Continuity.

We show that Hessian measures are weakly continuous with respect to epi-convergence.

Theorem 7.3. *Let u_k be a sequence in $\mathrm{Conv}(\mathbb{R}^n)$. If u_k epi-converges to $u \in \mathrm{Conv}(\mathbb{R}^n)$, then the sequence of measures $\Theta_i(u_k, \cdot)$ converges weakly to $\Theta_i(u, \cdot)$ as $k \to +\infty$ for every $i = 0, \ldots, n$.*

Proof. First, using the Steiner formula (7.1), it is easy to see that it is sufficient to prove for every $s > 0$ that the sequence of measures $\eta \mapsto \mathcal{H}^n(P_s(u_k, \eta))$ converges weakly to the measure $\eta \mapsto \mathcal{H}^n(P_s(u, \eta))$ as $k \to +\infty$.

Second, to establish the previous claim we prove that if $\eta \subset \mathbb{R}^{2n}$ is compact, then

$$(7.3) \qquad \limsup_{k \to +\infty} \mathcal{H}^n(P_s(u_k, \eta)) \leq \mathcal{H}^n(P_s(u, \eta)),$$

and, if $\eta \subset \mathbb{R}^{2n}$ is open, then

$$(7.4) \qquad \liminf_{k \to +\infty} \mathcal{H}^n(P_s(u_k, \eta)) \geq \mathcal{H}^n(P_s(u, \eta)).$$

Third, we prove (7.3). To simplify notation, for fixed $\eta \subset \mathbb{R}^{2n}$ and $s > 0$, we set $A = P_s(u, \eta)$ and $A_k = P_s(u_k, \eta)$ for $k \in \mathbb{N}$. As η is compact, A is compact as well. Indeed, it is obviously bounded. To prove that it is closed, let z_j be a sequence in A, converging to some $\bar{z} \in \mathbb{R}^n$. Then there exists a sequence x_j in $\mathbb{R}^n$ and a sequence y_j such that $y_j \in \partial u(x_j)$ and $(x_j, y_j) \in \eta$ for every j and $z_j = x_j + s y_j$. By

compactness, we may assume that x_j and y_j converge to some $\bar{x} \in \mathbb{R}^n$ and $\bar{y} \in \mathbb{R}^n$, respectively, with $(\bar{x}, \bar{y}) \in \eta$. For every $x \in \mathbb{R}^n$, we have

$$u(x) \geq u(x_j) + \langle x - x_j, y_j \rangle.$$

Passing to the limit and using the lower semicontinuity of u, we obtain

$$u(x) \geq u(\bar{x}) + \langle x - \bar{x}, \bar{y} \rangle$$

for all $x \in \mathbb{R}^n$. Hence $\bar{y} \in \partial u(\bar{x})$, so that $\bar{z} \in A$.

Next, we prove that for every $\varepsilon > 0$ there exists $\bar{k} \in \mathbb{N}$ such that

(7.5) $$A_k \subset (A)_\varepsilon$$

for all $k \geq \bar{k}$, where $A_\varepsilon := \{x \in \mathbb{R}^n \colon \operatorname{dist}(x, A) \leq \varepsilon\}$ and $\operatorname{dist}(x, A) = \inf\{|x - y| : y \in A\}$. Note that (7.3) follows from (7.5), as $A = \bigcap_{\varepsilon>0} A_\varepsilon$ implies that $\mathcal{H}^n(A) = \lim_{\varepsilon \to 0^+} \mathcal{H}^n(A_\varepsilon)$, where the first equality holds because A is closed. We argue by contradiction. Assume that there exists $\varepsilon > 0$, a sequence $x_k \in \mathbb{R}^n$ and a sequence $y_k \in \mathbb{R}^n$, with $y_k \in \partial u_k(x_k)$ and $(x_k, y_k) \in \eta$ for every $k \in \mathbb{N}$, such that

(7.6) $$\operatorname{dist}(x_k + s y_k, A) \geq \varepsilon$$

for all $k \in \mathbb{N}$. By compactness, we may assume that the sequences x_k and y_k converge to $\bar{x}$ and $\bar{y} \in \mathbb{R}^n$, respectively, with $(\bar{x}, \bar{y}) \in \eta$. Let $x \in \mathbb{R}^n$; by the definition of epi-convergence, there exists a sequence $\hat{x}_k$ such that

$$\lim_{k \to +\infty} u_k(\hat{x}_k) = u(x).$$

Moreover

$$\liminf_{k \to +\infty} u_k(x_k) \geq u(\bar{x}),$$

again by epi-convergence. From the inequality

$$u_k(\hat{x}_k) \geq u_k(x_k) + \langle \hat{x}_k - x_k, y_k \rangle$$

for all $k \in \mathbb{N}$, passing to the limit in k, we deduce

$$u(x) \geq u(\bar{x}) + \langle x - \bar{x}, \bar{y} \rangle$$

for $x \in \mathbb{R}^n$, that is, $\bar{y} \in \partial u(\bar{x})$ so that $x_k + s y_k$ converges to a point of A, which contradicts (7.6).

Finally, we prove (7.4). Define $w, w_k \in \operatorname{Conv}(\mathbb{R}^n)$ as

$$w(x) = s\, u(x) + \frac{|x|^2}{2}, \quad w_k(x) = s\, u_k(x) + \frac{|x|^2}{2},$$

and note that $\partial w(x) = x + s\, \partial u(x)$ while $\partial w_k(x) = x + s\, \partial u_k(x)$. Let us fix $\bar{x} \in \mathbb{R}^n$ and $\bar{z} \in \partial w(x)$. The function $\hat{w} \in \operatorname{Conv}(\mathbb{R}^n)$ defined as

$$\hat{w}(x) = w(x) - \langle x - \bar{x}, \bar{z} \rangle$$

has an absolute minimum at $x = \bar{x}$. Consider now, for $k \in \mathbb{N}$, the function $\hat{w}_k \in \operatorname{Conv}(\mathbb{R}^n)$ defined as

$$\hat{w}_k(x) = w_k(x) - \langle x - \bar{x}, \bar{z} \rangle.$$

This is a strictly convex and lower semicontinuous function on $\mathbb{R}^n$, which verifies

$$\lim_{|x| \to +\infty} \hat{w}_k(x) = +\infty.$$

This follows from the fact that u_k is bounded from below by an affine function, since its subdifferential is non-empty at least at one point. Hence w_k admits a unique (by strict convexity) absolute minimum point x_k. This implies $0 \in \partial \hat{w}_k(x_k)$ which in turn implies that $\bar{z} \in \partial w_k(x_k)$.

As u_k epi-converges to u, the sequence $\hat{w}_k$ epi-converges to $\hat{w}$, so that (see for instance Theorem 7.33 in [34])

$$\lim_{k \to +\infty} x_k = \bar{x}.$$

We have proved that given $(\bar{x}, \bar{y}) \in \eta \cap \Gamma_u$, there exists a sequence (x_k, y_k) with $(x_k, y_k) \in \Gamma_{u_k}$ for every k, such that $\bar{x} + s\bar{y} = x_k + sy_k$ for every $k \in \mathbb{N}$; moreover, x_k tends to $\bar{x}$ as $k \to +\infty$. These conditions imply that y_k tends to $\bar{y}$. Hence, as η is open, (x_k, y_k) is definitively contained in η. Thus every $z \in P_s(u, \eta)$ is contained in $P_s(u_k, \eta)$. Consequently,

$$P_s(u, \eta) \subset \bigcup_{i \in \mathbb{N}} \left(\bigcap_{k \geq i} P_s(u_k, \eta) \right).$$

This easily implies (7.4). $\qquad \square$

Remark. In the special case of finite functions, the previous theorem was proved in [11, Theorem 1.1].

7.3. **Integral representations of Hessian measures.** Our starting point is the following result, which follows in a rather direct way from results proved in [10]. We recall that a subset A of $\mathbb{R}^{2n}$ is *countably n-rectifiable* if there exist a countable family $\{U_i \colon i \in \mathbb{N}\}$ of open subsets of $\mathbb{R}^n$ and a countable family $\{f_i \colon i \in \mathbb{N}\}$ of Lipschitz maps, such that $f_i \colon U_i \to \mathbb{R}^{2n}$ and

$$\mathcal{H}^n \left(A \setminus \bigcup_{i \in \mathbb{N}} f_i(U_i) \right) = 0$$

(see *e.g.* [17]).

Proposition 7.4. *If $u \in \mathrm{Conv}(\mathbb{R}^n)$ is such that $\mathrm{dom}(u) = \mathbb{R}^n$, then Γ_u is a countably n-rectifiable set. Moreover, for $i \in \{0, \dots, n\}$, there exists a non-negative, $\mathcal{H}^n$-measurable function $\theta_i(u, \cdot) \colon \Gamma_u \to \mathbb{R}$ such that*

$$\Theta_i(u, \eta) = \int_{\eta \cap \Gamma_u} \theta_i(u, (x, y)) \, \mathrm{d}\mathcal{H}^n(x, y)$$

for every $\eta \in \mathcal{B}(\mathbb{R}^{2n})$.

Proof. The countable n-rectifiability of Γ_u was observed in [10, Section 3].

The set $\mathrm{epi}(u)$ is a non-empty, closed, convex subset of $\mathbb{R}^{n+1}$, which does not coincide with $\mathbb{R}^{n+1}$ itself. In particular, its *normal bundle*, $\mathrm{Nor}(\mathrm{epi}(u))$, is well defined:

$$\mathrm{Nor}(\mathrm{epi}(u)) = \{(p, q) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \colon p \in \mathrm{bd}(\mathrm{epi}(u)), \ q \in N(\mathrm{epi}(u), p)\},$$

where $N(\mathrm{epi}(u), p)$ denotes the outer normal cone to $\mathrm{epi}(u)$ at p. Finally, we set

$$\mathcal{F}_u = \mathrm{Nor}(\mathrm{epi}(u)) \cap \left(\{(x, u(x)) \colon x \in \mathbb{R}^n\} \times \mathbb{S}^n \right).$$

In [10, Section 3] the following facts are proved:

(i) $\mathcal{F}_u$ is a countably n-rectifiable Borel set;

(ii) there exists an homeomorphism $T \colon \Gamma_u \to \mathcal{F}_u$; moreover T and its inverse are locally Lipschitz.

By [10, Theorem 3.1], each Hessian measure of u admits an integral representation on $\mathcal{F}_u$: there exist $\mathcal{H}^n$-measurable, non-negative functions $\bar{\theta}_i(u, \cdot) \colon \mathcal{F}_u \to \mathbb{R}$, such that

$$(7.7) \qquad \Theta_i(u, \eta) = \int_{\bar{\eta} \cap \mathcal{F}_u} \bar{\theta}_i(u, (\bar{x}, \bar{y}) \, \mathrm{d}\mathcal{H}^n(\bar{x}, \bar{y}),$$

for every $\eta \in \mathcal{B}(\mathbb{R}^{2n})$ and $i \in \{0, \dots, n\}$, where $\bar{\eta} = T(\eta \cap \Gamma_u)$. Now (7.8) follows from performing the change of variables $(\bar{x}, \bar{y}) = T(x, y)$ in (7.7) and using the co-area formula (see for instance [17]). $\qquad \square$

Next we extend the previous result to the general case.

Theorem 7.5. *If $u \in \mathrm{Conv}(\mathbb{R}^n)$, then Γ_u is a countably n-rectifiable set. Moreover, for $i \in \{0, \ldots, n\}$, there exists a non-negative, $\mathcal{H}^n$-measurable function $\theta_i(u, \cdot) \colon \Gamma_u \to \mathbb{R}$ such that*

$$(7.8) \qquad \Theta_i(u, \eta) = \int_{\eta \cap \Gamma_u} \theta_i(u, (x, y)) \, \mathrm{d}\mathcal{H}^n(x, y).$$

for every $\eta \in \mathcal{B}(\mathbb{R}^{2n})$.

Proof. For $r > 0$, set $u_r = \mathrm{reg}_r u$ and note that $\mathrm{dom}(u_r) = \mathbb{R}^n$. By Proposition 6.5, $\Gamma_u^{1/r} \subset \Gamma_{u_r}$. By Proposition 7.4, the set Γ_{u_r} is countably n-rectifiable, and hence the same conclusion is valid for $\Gamma_u^{1/r}$. As

$$\Gamma_u = \bigcup_{r > 0} \Gamma_u^{1/r}$$

we deduce that Γ_u is countably n-rectifiable as well.

Let us fix $i \in \{0, \ldots, n\}$. As in the proof of Theorem 7.1, we have $P_s(u, \eta) = P_s(u_r, \eta)$ for every Borel set $\eta \subset \Gamma_u \cap (\mathbb{R}^n \times B_r^n)$. So, in particular

$$\Theta_i(u, \eta) = \Theta_i(u_r, \eta)$$

for every $\eta \in \mathcal{B}(\mathbb{R}^{2n})$ contained in $\Gamma_u \cap (\mathbb{R}^n \times B_r^n)$. By Proposition 7.4, there exists a non-negative $\mathcal{H}^n$-measurable function $\theta_i(u_r, \cdot)$ defined on Γ_{u_r} such that

$$\Theta_i(u, \eta) = \Theta_i(u_r, \eta) = \int_{\eta \cap \Gamma_{u_r}} \theta_i(u_r, (x, y)) \, \mathrm{d}\mathcal{H}^n(x, y) = \int_{\eta \cap \Gamma_u} \theta_i(u_r, (x, y)) \, \mathrm{d}\mathcal{H}^n(x, y).$$

Clearly for $r' \geq r$, the function $\theta_i(u_{r'}, \cdot)$ extends $\theta_i(u_r, \cdot)$. Hence there is a function $\theta_i(u, \cdot)$, defined on Γ_u, such that (7.8) holds for every $\eta \in \mathcal{B}(\mathbb{R}^{2n})$ contained in $\mathbb{R}^n \times B_r^n$. The conclusion follows from letting $r \to +\infty$. $\qquad\square$

7.4. **A second auxiliary result.** For future use, we state the following result that follows from the argument used in the proofs in the previous part of this section.

Proposition 7.6. *Let $u \in \mathrm{Conv}(\mathbb{R}^n)$. For $r > 0$, the function $u_r = \mathrm{reg}_r u$ has the following properties.*

(i) *For every $\eta \subset \Gamma_u^r$ and $i \in \{0, \ldots, n\}$,*

$$\Theta_i(u, \eta) = \Theta_i(u_r, \eta).$$

(ii) *For $\mathcal{H}^n$-a.e. $(x, y) \in \Gamma_u^r$ and $i \in \{0, \ldots, n\}$,*

$$\theta_i(u, (x, y)) = \theta_i(u_r, (x, y)).$$

8. Further Properties of Hessian Measures

8.1. **Invariance and covariance properties.** Let $u \in \mathrm{Conv}(\mathbb{R}^n)$ and $x_0 \in \mathbb{R}^n$; we denote by u_{x_0} the composition of u and the translation by x_0; that is,

$$u_{x_0}(x) = u(x - x_0)$$

for $x \in \mathbb{R}^n$. Clearly, $u_{x_0} \in \mathrm{Conv}(\mathbb{R}^n)$. Let ϕ be an element of $\mathrm{O}(n)$. Given $u \in \mathrm{Conv}(\mathbb{R}^n)$, we define $u_\phi \colon \mathbb{R}^n \to \mathbb{R}^n$ by

$$u_\phi(y) = u(\phi^{-1} y)$$

for $y \in \mathbb{R}^n$. As before, we have $u_\phi \in \mathrm{Conv}(\mathbb{R}^n)$. For $\phi \in \mathrm{O}(n)$, we also define $\tilde{\phi} \colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n$ by $\tilde{\phi}(x, y) = (\phi x, \phi y)$.

We establish the following covariance properties of Hessian measures.

Proposition 8.1. *Let $u \in \mathrm{Conv}(\mathbb{R}^n)$. For $x_0 \in \mathbb{R}^n$ and $\phi \in \mathrm{O}(n)$,*

$$\Theta_i(u_{x_0}, \eta) = \Theta_i(u, \eta + (x_0, 0)),$$
$$\Theta_i(u_\phi, \eta) = \Theta_i(u, \tilde{\phi}^{-1}\eta).$$

for every $\eta \in \mathcal{B}(\mathbb{R}^n \times \mathbb{R}^n)$ and $i \in \{0, \ldots, n\}$.

Proof. For every $x \in \mathbb{R}^n$, the definition of the subdifferential implies that $\partial u_{x_0}(x) = \partial u(x - x_0)$ and therefore $\Gamma_{u_{x_0}} = \Gamma_u + (x_0, 0)$. Consequently,

$$
\begin{aligned}
P_s(u_{x_0}, \eta) &= \{x + sy \colon (x, y) \in \eta \cap \Gamma_{u_{x_0}}\} \\
&= \{x + sy \colon (x, y) \in \eta \cap (\Gamma_u + (x_0, 0))\} \\
&= \{x + sy \colon (x, y) \in ((\eta - (x_0, 0)) \cap \Gamma_u) + (x_0, 0)\} \\
&= \{x + sy \colon (x, y) \in (\eta - (x_0, 0)) \cap \Gamma_u\} + x_0 \\
&= P_s(u, \eta - (x_0, 0)) + x_0.
\end{aligned}
$$

Similarly, $\phi^{-1}\partial u_\phi(x) = \partial u(\phi^{-1}x)$ and therefore $\Gamma_{u_\phi} = \tilde{\phi}(\Gamma_u)$, which implies that

$$
\begin{aligned}
P_s(u_\phi, \eta) &= \{x + sy \colon (x, y) \in \eta \cap \Gamma_{u_\phi}\} \\
&= \{x + sy \colon (x, y) \in \eta \cap \tilde{\phi}\,\Gamma_u\} \\
&= \{x + sy \colon (x, y) \in \tilde{\phi}(\tilde{\phi}^{-1}\eta \cap \Gamma_u)\} \\
&= \phi(P_s(u, \tilde{\phi}^{-1}\eta)).
\end{aligned}
$$

The conclusions follow from taking the Lebesgue measure of $P_s(u_{x_0}, \eta)$ and of $P_s(u_\phi, \eta)$, respectively, and from using the Steiner type formula (7.1) and the invariance of the Lebesgue measure under rotations and translations. $\square$

8.2. **Hessian measures of u and u^*.** Let $T \colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be defined by $T(x, y) = (y, x)$. Given a subset η of $\mathbb{R}^{2n}$, we set

$$\hat{\eta} = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n \colon (y, x) \in \eta\} = T(\eta).$$

If $u \in \mathrm{Conv}(\mathbb{R}^n)$, then its conjugate function u^* belongs to $\mathrm{Conv}(\mathbb{R}^n)$ as well. Moreover, by Lemma 3.8, for every $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$, we have $y \in \partial u(x)$ if and only if $x \in \partial u^*(y)$. Hence

$$\Gamma_u = T(\Gamma_{u^*}).$$

Similarly,

$$\Gamma_u \cap \eta = T(\Gamma_{u^*} \cap \hat{\eta})$$

for all $\eta \in \mathcal{B}(\mathbb{R}^{2n})$. We easily deduce the following result, already observed in [11, Theorem 5.8].

Theorem 8.2. *For $u \in \mathrm{Conv}(\mathbb{R}^n)$,*

$$\Theta_i(u, \eta) = \Theta_{n-i}(u^*, \hat{\eta})$$

for every $i = 0, \ldots, n$ and $\eta \in \mathcal{B}(\mathbb{R}^{2n})$.

8.3. **The measures Θ_0 and Θ_n.** We consider the extremal cases $i = 0$ and $i = n$ for Hessian measures.

Proposition 8.3. *For every $u \in \mathrm{Conv}(\mathbb{R}^n)$,*

$$\Theta_n(u, \eta) = \mathcal{H}^n(\pi_1(\eta \cap \Gamma_u)), \quad \Theta_0(u, \eta) = \mathcal{H}^n(\pi_2(\eta \cap \Gamma_u)),$$

for all $\eta \in \mathcal{B}(\mathbb{R}^{2n})$.

Proof. We have $P_0(u, \eta) = \pi_1(\eta \cap \Gamma_u)$. Hence the first equality follows from (7.1) with $s = 0$. The second is obtained by applying the first one to u^* and by Theorem 8.2. $\square$

A particular case of the previous result was already presented in [10, p. 3248].

8.4. **Lower dimensional domains.**

Proposition 8.4. *If* $u \in \mathrm{Conv}(\mathbb{R}^n)$*, then* $\Theta_i(u, \cdot) \equiv 0$ *for every* $i > \dim(\mathrm{dom}(u))$*.*

Proof. Let $\dim(\mathrm{dom}(u)) = k$. There exists an affine subspace E of $\mathbb{R}^n$ of dimension k, such that $\mathrm{dom}(u) \subset E$. Without loss of generality, we may assume that

$$E = \{(x', 0) \colon x' \in \mathbb{R}^k\},$$

where 0 stands for the zero element of $\mathbb{R}^{n-k}$. Let F be the orthogonal complement of E, that is,

$$F = \{(0, x'') \colon x'' \in \mathbb{R}^{n-k}\}.$$

Define $u_E \colon \mathbb{R}^k \to \mathbb{R} \cup \{+\infty\}$ as the restriction of u to E, that is, $u_E(x') = u(x', 0)$ for $x' \in \mathbb{R}^k$. It follows from $\mathrm{dom}(u) \subset E$ that $u_E \in \mathrm{Conv}(\mathbb{R}^k)$. It is not hard to check that, for every $x' \in \mathbb{R}^k$,

$$\partial u(x', 0) = \{(y', y'') \colon y' \in \partial u_E(x'), \ y'' \in \mathbb{R}^{n-k}\}.$$

Hence

$$\Gamma_u = \{((x', 0), (y', y'')) \colon x' \in \mathbb{R}^k, \ y' \in \partial u_E(x'), \ y'' \in \mathbb{R}^{n-k}\}.$$

Now let $\alpha', \beta' \in \mathcal{B}(\mathbb{R}^k)$ and $\alpha'', \beta'' \in \mathcal{B}(\mathbb{R}^{n-k})$ with $0 \in \alpha''$. Set

$$\eta = \alpha' \times \alpha'' \times \beta' \times \beta'' \in \mathcal{B}(\mathbb{R}^{2n}).$$

For $s \geq 0$, we obtain

$$\begin{aligned} P_s(u, \eta) &= \{(x' + sy', sy'') \colon (x', y') \in \Gamma_{u_E} \cap (\alpha' \times \beta'), \ y'' \in \beta''\} \\ &= P_s(u_E, \alpha' \times \beta') \times s\beta''. \end{aligned}$$

Taking Lebesgue measures on both sides, we get

$$\mathcal{H}^n(P_s(u, \eta)) = s^{n-k} \, \mathcal{H}^k(P_s(u_E, \alpha' \times \beta')).$$

This implies that the polynomial expansion of $\mathcal{H}^n(P_s(u, \eta))$ does not contain any term of order lower than $(n - k)$. Thus

$$\Theta_{k+1}(u, \eta) = \cdots = \Theta_n(u, \eta) = 0.$$

The conclusion now follows from the arbitrariness of $\alpha', \alpha'', \beta', \beta''$ and the non-negativity of Hessian measures. $\qquad\square$

8.5. **Smooth functions.** For $u \in \mathrm{Conv}(\mathbb{R}^n) \cap C^2(\mathbb{R}^n)$, we have

(8.1) $$\Gamma_u = \{(x, \nabla u(x)) \colon x \in \mathbb{R}^n\},$$

and

$$P_s(u, \beta \times \mathbb{R}^n) = \{x + s \, \nabla u(x) \colon x \in \beta \times \mathbb{R}^n\}$$

for every $\beta \in \mathcal{B}(\mathbb{R}^n)$ and $s \geq 0$. Hence

$$\mathcal{H}^n(P_s(u, \beta \times \mathbb{R}^n)) = \int_{P_s(u, \beta \times \mathbb{R}^n)} \mathrm{d}z = \int_\beta \det(I_n + s \, \mathrm{D}^2 u(x)) \, \mathrm{d}x,$$

by the change of variables $z = x + s \, \nabla u(x)$. On the other hand:

$$\det(I_n + s \, \mathrm{D}^2 u(x)) = \sum_{i=0}^n \binom{n}{i} s^i \, [\mathrm{D}^2 u(x)]_i.$$

By Theorem 7.1, we get

(8.2) $$\Theta_i(u, \beta \times \mathbb{R}^n) = \int_\beta [\mathrm{D}^2 u(x)]_{n-i} \, \mathrm{d}x$$

for every $i \in \{0, \ldots, n\}$ and $\beta \in \mathcal{B}(\mathbb{R}^n)$.

9. Valuations on Convex Functions

Let $\langle \mathcal{G}, + \rangle$ be an Abelian semi-group.

Definition 9.1. A function $Z \colon \mathrm{Conv}(\mathbb{R}^n) \to \mathcal{G}$ is called a valuation if

$$Z(u \vee v) + Z(u \wedge v) = Z(u) + Z(v)$$

for every $u, v \in \mathrm{Conv}(\mathbb{R}^n)$ such that $u \wedge v \in \mathrm{Conv}(\mathbb{R}^n)$.

We will only be interested in two cases: $\mathcal{G} = \mathbb{R}$ and $\mathcal{G} = \mathcal{M}(\mathbb{R}^{2n})$; the latter is the space of Borel measures on $\mathbb{R}^{2n}$. In both situations there exists a natural choice of topology, the Euclidean topology on $\mathbb{R}$ and the topology of weak convergence on $\mathcal{M}(\mathbb{R}^{2n})$. A valuation Z is called continuous if, for every sequence u_k in $\mathrm{Conv}(\mathbb{R}^n)$ epi-convergent to $u \in \mathrm{Conv}(\mathbb{R}^n)$, we have $\lim_{k \to +\infty} Z(u_k) = Z(u)$ in $\mathcal{G}$.

In the following definitions we use the notation introduced in the previous section for the composition of a function with a translation or a rotation. Let $Z \colon \mathrm{Conv}(\mathbb{R}^n) \to \mathbb{R}$ be a valuation. We say that

  (i) Z is translation invariant if $Z(u_{x_0}) = Z(u)$ for all $u \in \mathrm{Conv}(\mathbb{R}^n)$ and $x_0 \in \mathbb{R}^n$;
 (ii) Z is rotation invariant if $Z(u_\phi) = Z(u)$ for all $u \in \mathrm{Conv}(\mathbb{R}^n)$ and $\phi \in \mathrm{O}(n)$;
(iii) Z is rigid motion invariant if it is translation and rotation invariant.
(iv) Z is i-simple, for $i = 1, \ldots, n$, if $Z(u) = 0$ whenever $u \in \mathrm{Conv}(\mathbb{R}^n)$ and $\dim(\mathrm{dom}(u)) \leq i + 1$.

Clearly, all the previous definition can be repeated for functionals defined on subsets of $\mathrm{Conv}(\mathbb{R}^n)$.

9.1. Measure-valued valuations.

Theorem 9.2. *For $i \in \{0, \ldots, n\}$, the function $\Theta_i \colon \mathrm{Conv}(\mathbb{R}^n) \to \mathcal{M}(\mathbb{R}^{2n})$ is a continuous and i-simple valuation.*

Proof. Continuity and simplicity follow from Theorems 7.3 and 8.4, respectively. As for the valuation property, let $u, v \in \mathrm{Conv}(\mathbb{R}^n)$ be such that $u \wedge v \in \mathrm{Conv}(\mathbb{R}^n)$. We have to prove that

$$\Theta_i(u \wedge v, \eta) + \Theta_i(u \vee v, \eta) = \Theta_i(u, \eta) + \Theta_i(v, \eta)$$

for $\eta \in \mathcal{B}(\mathbb{R}^{2n})$. By Proposition 6.4 we have, for every $s \geq 0$,

$$\begin{aligned}
\mathcal{H}^n(P_s(u \wedge v, \eta)) &+ \mathcal{H}^n(P_s(u \vee v, \eta)) \\
&= \mathcal{H}^n(P_s(u \wedge v, \eta) \cup P_s(u \vee v, \eta)) + \mathcal{H}^n(P_s(u \wedge v, \eta) \cap P_s(u \vee v, \eta)) \\
&= \mathcal{H}^n(P_s(u, \eta) \cup P_s(v, \eta)) + \mathcal{H}^n(P_s(u, \eta) \cap P_s(v, \eta)) \\
&= \mathcal{H}^n(P_s(u, \eta)) + \mathcal{H}^n(P_s(v, \eta)).
\end{aligned}$$

The conclusion follows immediately from the Steiner type formula (7.1). $\qquad \square$

10. Hessian Valuations

Let $\zeta \colon \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a continuous function. We extend its definition to include the value $+\infty$ for its first variable, setting $\zeta(+\infty, x, y) = 0$ for all $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$. This extension will be always tacitly used in the rest of the paper.

We are now in a position to prove Theorem 1.1 and first establish the following result.

Theorem 10.1. *Let $\zeta \in C(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n)$ have compact support with respect to the second and third variables. For $i \in \{0, 1, \ldots, n\}$, the functional $Z_{\zeta,i} \colon \mathrm{Conv}(\mathbb{R}^n) \to \mathbb{R}$, defined by*

$$(10.1) \qquad Z_{\zeta,i}(u) = \int_{\Gamma_u} \zeta(u(x), x, y) \, \mathrm{d}\Theta_i(u, (x, y)),$$

is an i-simple and continuous valuation on $\mathrm{Conv}(\mathbb{R}^n)$.

Remark. If $u \in \mathrm{Conv}(\mathbb{R}^n) \cap C^2(\mathbb{R}^n)$, then, by (8.1) and (8.2),

$$Z_{\zeta,i}(u) = \int_{\Gamma_u} \zeta(u(x), x, y) \, \mathrm{d}\Theta_i(u, (x, y)) = \int_{\mathbb{R}^n} \zeta(u(x), x, \nabla u(x)) \, [\mathrm{D}^2 u(x))]_{n-i} \, \mathrm{d}x.$$

Hence, all statements of Theorem 1.1 follow from the above theorem.

The proof of Theorem 10.1 requires some preparatory steps.

Lemma 10.2. *Let $\zeta \in C(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n)$ and $u \in \mathrm{Conv}(\mathbb{R}^n)$. The function $\xi \colon \Gamma_u \to \mathbb{R}$, defined as $\xi(x, y) = \zeta(u(x), x, y)$, is continuous on Γ_u.*

Proof. For $r > 0$, by Proposition 4.1, the function $u_r = \mathrm{reg}_r u$ has $\mathrm{dom}(u_r) = \mathbb{R}^n$ and $u(x) = u_r(x)$ for x such that $|y| \le 1/r$ for every $y \in \partial u(x)$. Hence

$$\xi(x, y) = \zeta(u(x), x, y) = \zeta(u_r(x), x, y)$$

for $(x, y) \in \Gamma_u^r$. As u_r is continuous in $\mathbb{R}^n$, ξ is continuous on Γ_u^r. As $r > 0$ is arbitrary, the conclusion follows. $\qquad\square$

Proposition 10.3. *Let $\zeta \in C(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n)$ have compact support with respect to the second and third variables. For every $u \in \mathrm{Conv}(\mathbb{R}^n)$, the function $\xi \colon \Gamma_u \to \mathbb{R}$, defined as $\xi(x, y) = \zeta(u(x), x, y)$, is summable with respect to the measure $\Theta_i(u, \cdot)$ for $i \in \{0, \ldots, n\}$.*

Proof. We recall that the support of $\Theta_i(u, \cdot)$ is contained in Γ_u. Set $\zeta_+ = \max\{\zeta, 0\}$ and $\zeta_- = \min\{\zeta, 0\}$. We clearly have

$$\xi_+(x, y) := \max\{\xi(x, y), 0\} = \zeta_+(u(x), x, y), \quad \xi_-(x, y) := \min\{\xi(x, y), 0\} = \zeta_-(u(x), x, y).$$

By Lemma 10.2, the function ξ_+ is continuous and hence integrable with respect to the Borel measure $\Theta_i(u, \cdot)$ over Γ_u. Let us prove that ξ_+ is in fact summable (the proof for ξ_- is completely analogous). As ζ has compact support in (x, y), there exists $r > 0$ such that $\zeta(t, x, y) = 0$ whenever $|x| \ge r$ or $|y| \ge r$. Moreover, there exists $u_r \in \mathrm{Conv}(\mathbb{R}^n)$ such that $\mathrm{dom}(u_r) = \mathbb{R}^n$, and $u(x) = u_r(x)$ for all $x \in \pi_1(\Gamma_u^r)$. Hence

$$\xi(x, y) = \zeta(u_r(x), x, y)$$

for all $(x, y) \in \Gamma_u$. By continuity, u_r is bounded in the set $\{x \in \mathbb{R}^n \colon |x| \le r\}$ and there exists a constant $c > 0$ such that

$$\xi_+(x, y) \le c$$

for all $(x, y) \in \Gamma_u$. Consequently,

$$\int_{\mathbb{R}^{2n}} \zeta(u_r(x), x, y) \, \mathrm{d}\Theta_i(u_r, (x, y)) \le c \, \Theta_i(u_r, \eta)$$

for $\eta = \{(x, y) \in \mathbb{R}^{2n} \colon |x|, |y| \le r\}$. The conclusion follows from the fact that $\Theta_i(u_r, \cdot)$ is locally finite (see (7.2)). $\qquad\square$

10.1. **Proof of Theorem 10.1.** By Proposition 10.3, the integral (10.1) is finite.

For simplicity, for an arbitrary $u \in \mathrm{Conv}(\mathbb{R}^n)$ we write $\Theta_i(u, \cdot)$ as $\Theta(u, \cdot)$. As remarked above, the support of $\Theta(u, \cdot)$ is contained in Γ_u. Moreover, by Theorem 7.5, we have

$$(10.2) \qquad \int_{\Gamma_u} \zeta(u(x), x, y) \, \mathrm{d}\Theta(u, (x, y)) = \int_{\Gamma_u} \zeta(u(x), x, y) \theta(u, (x, y)) \, \mathrm{d}\mathcal{H}^n(x, y),$$

for a suitable non-negative, $\mathcal{H}^n$-measurable function $\theta(u, \cdot)$ defined on Γ_u. Let $r > 0$ be such that $\zeta(t, x, y) = 0$ if either $|x| > r$ or $|y| > r$. For $u \in \mathrm{Conv}(\mathbb{R}^n)$, we set $u_r = \mathrm{reg}_r u$ and use the properties established in Proposition 6.5 and Proposition 7.6.

1. The valuation property. Let $u, v \in \mathrm{Conv}(\mathbb{R}^n)$ be such that $u \wedge v \in \mathrm{Conv}(\mathbb{R}^n)$. Set

$$\Sigma = \Gamma_u \cup \Gamma_v = \Gamma_{u \vee v} \cup \Gamma_{u \wedge v}$$

(see Proposition 6.3). We will consider a generic point $(x, y) \in \Sigma$ such that the densities

$$\theta(u, (x, y)), \theta(v, (x, y)), \theta(u \vee v, (x, y)), \theta(u \wedge v, (x, y))$$

are well defined; this happens to be true for $\mathcal{H}^n$-a.e. $(x, y) \in \Sigma$. In view of (10.2), it is enough to prove that for $\mathcal{H}^n$-a.e. $(x, y) \in \Sigma$, we have

$$(10.3) \qquad \begin{aligned} &\zeta(u(x), x, y) \, \theta(u, (x, y)) + \zeta(v(x), x, y) \, \theta(v, (x, y)) \\ &\qquad = \zeta((u \vee v)(x), x, y) \, \theta(u \vee v, (x, y)) + \zeta((u \wedge v)(x), x, y) \, \theta(u \wedge v, (x, y)). \end{aligned}$$

Indeed, integrating over Σ we get

$$Z_{\zeta,i}(u) + Z_{\zeta,i}(v) = Z_{\zeta,i}(u \vee v) + Z_{\zeta,i}(u \wedge v).$$

It will suffice to prove (10.3) under the assumption $|y| \leq r$, as it is trivially true for $|y| > r$ since ζ vanishes at each point where it is computed.

By the valuation property of Hessian measures (Theorem 9.2), we deduce

$$(10.4) \qquad \theta(u \vee v, (x, y)) + \theta(u \wedge v, (x, y)) = \theta(u, (x, y)) + \theta(v, (x, y)) \quad \text{for } \mathcal{H}^n\text{-a.e. } (x, y) \in \Sigma.$$

This shows in particular that (10.3) holds true for every (x, y) such that $u(x) = v(x)$. Next assume that (x_0, y_0) is such that $u(x_0) \neq v(x_0)$ and without loss of generality, assume that $u(x_0) > v(x_0)$.

Case 1: Let $(x_0, y_0) \notin \Gamma_u \cap \Gamma_v$. Then we also have, by Proposition 6.3, $(x_0, y_0) \notin \Gamma_{u \vee v} \cap \Gamma_{u \wedge v}$. Assume that $(x_0, y_0) \in \Gamma_u \setminus \Gamma_v$. From (5.1), $y_0 \in \partial u(x_0) = \partial (u \vee v)(x_0)$; therefore

$$(x_0, y_0) \in \Gamma_{u \vee v} \setminus \Gamma_{u \wedge v}.$$

Hence

$$\theta(v, (x_0, y_0)) = \theta(u \wedge v, (x_0, y_0)) = 0.$$

In this case (10.4) reduces to

$$\theta(u \vee v, (x_0, y_0)) = \theta(u, (x_0, y_0))$$

and (10.3) follows from multiplying both sides of the previous equation by $(u \vee v)(x_0) = u(x_0)$. The case $(x_0, y_0) \in \Gamma_v \setminus \Gamma_u$ is completely analogous.

Case 2: Let $(x_0, y_0) \in \Gamma_u \cap \Gamma_v$. Then $u_r(x_0) = u(x_0) > v(x_0) = v_r(x_0)$. Let U be a neighborhood of x_0 such that $u_r > v_r$ in U (which exists by the continuity of u_r and v_r). Hence

$$u_r \wedge v_r \equiv v_r, \quad u_r \vee v_r \equiv u_r \quad \text{in } U.$$

Then we also have $\partial (u_r \wedge v_r)(x) = \partial v_r(x)$ and $\partial (u_r \vee v_r)(x) = \partial v_r(x)$ for every $x \in U$, and this implies that

$$P_s(u_r \wedge v_r, \eta) = P_s(v_r, \eta), \quad P_s(u_r \vee v_r, \eta) = P_s(u_r, \eta)$$

for every $s \geq 0$ and for every $\eta \subset \mathbb{R}^n \times \mathbb{R}^n$ such that $\pi_1(\eta) \subset U$. Consequently,

$$\Theta(u_r \wedge v_r, \eta) = \Theta(v_r, \eta), \quad \Theta(u_r \vee v_r, \eta) = \Theta(u_r, \eta),$$

for every such η. We deduce that

$$\theta(u_r \wedge v_r, x_0) = \theta(v_r, x_0), \quad \theta(u_r \vee v_r, x_0) = \theta(u_r, x_0).$$

By Proposition 4.3 and Proposition 7.6, we have

$$\theta(u \wedge v, x_0) = \theta(v, x_0), \quad \theta(u \vee v, x_0) = \theta(u, x_0).$$

These equations, combined with the obvious relations

$$(u \wedge v)(x_0) = v(x_0), \quad (u \vee v)(x_0) = u(x_0)$$

lead to (10.3).

2. Continuity. Let u_k be a sequence in $\mathrm{Conv}(\mathbb{R}^n)$ epi-converging to $u \in \mathrm{Conv}(\mathbb{R}^n)$. For simplicity, we set $w_k = (u_k)_r = \mathrm{reg}_r u_k$ and $w = u_r = \mathrm{reg}_r u$.

$$
\begin{aligned}
\left| Z_{\zeta,i}(u_k) - Z_{\zeta,i}(u) \right| &= \left| \int_{\mathbb{R}^{2n}} \zeta(u_k(x), x, y) \, \mathrm{d}\Theta(u_k, (x,y)) - \int_{\mathbb{R}^{2n}} \zeta(u(x), x, y) \, \mathrm{d}\Theta(u, (x,y)) \right| \\
&= \left| \int_{\mathbb{R}^{2n}} \zeta(w_k(x), x, y) \, \mathrm{d}\Theta(u_k, (x,y)) - \int_{\mathbb{R}^{2n}} \zeta(w(x), x, y) \, \mathrm{d}\Theta(u, (x,y)) \right| \\
&\leq \int_{\mathbb{R}^{2n}} \left| \zeta(w_k(x), x, y) - \zeta(w(x), x, y) \right| \, \mathrm{d}\Theta(u_k, (x,y)) + \\
&\quad + \left| \int_{\mathbb{R}^{2n}} \zeta(w(x), x, y) \, \mathrm{d}\Theta(u_k, (x,y)) - \int_{\mathbb{R}^{2n}} \zeta(w(x), x, y) \, \mathrm{d}\Theta(u, (x,y)) \right|.
\end{aligned}
$$

By Proposition 5.5, the sequence w_k epi-converges to w; as w_k and w are finite in $\mathbb{R}^n$, the convergence is uniform on compact sets. As ζ is continuous and with compact support with respect to x and y, the sequence $\zeta(w_k(x), x, y)$ converges uniformly to $\zeta(w(x), x, y)$ in $\mathbb{R}^{2n}$ as $k \to +\infty$. Hence for an arbitrary $\varepsilon > 0$, and for sufficiently large k, we have

$$\int_{\mathbb{R}^{2n}} |\zeta(w_k(x), x, y) - \zeta(w(x), x, y)| \, \mathrm{d}\Theta(u_k, (x,y)) \leq \varepsilon \, \Theta(u_k, C)$$

where C is a compact subset of $\mathbb{R}^{2n}$ such that $\zeta(t, x, y) = 0$ if $(x, y) \notin C$. On the other hand, as Hessian measures are locally finite, there exists a constant $c > 0$ depending only on C such that

$$\Theta(u_k, C) \leq c$$

for every $k \in \mathbb{N}$. This proves that

$$\lim_{k \to +\infty} \int_{\mathbb{R}^{2n}} |\zeta(w_k(x), x, y) - \zeta(w(x), x, y)| \, \mathrm{d}\Theta(u_k, (x,y)) = 0.$$

Set $\xi(x, y) = \zeta(w(x), x, y)$ for $(x, y) \in \mathbb{R}^n$. Recall that w is a convex function such that $\mathrm{dom}(w) = \mathbb{R}^n$; in particular it is continuous in $\mathbb{R}^n$, so that ξ is continuous. Moreover, ξ has compact support, as ζ has this property with respect to (x, y). As the sequence of measures $\Theta(u_k, \cdot)$ converges weakly to $\Theta(u, \cdot)$, we have

$$\lim_{k \to +\infty} \int_{\mathbb{R}^{2n}} \zeta(w(x), x, y) \, \mathrm{d}\Theta(u_k, (x,y)) - \int_{\mathbb{R}^{2n}} \zeta(w(x), x, y) \, \mathrm{d}\Theta(u, (x,y)) = 0,$$

which proves the weak continuity.

3. Simplicity. Proposition 8.4 implies simplicity. $\qquad \square$

10.2. **Invariance properties.**

Proposition 10.4. *Let $\zeta \in C(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n)$ have compact support with respect to the second and third variables. If there exists $\xi \in C(\mathbb{R} \times [0, +\infty)^2)$ such that $\zeta(t, x, y) = \xi(t, |x|, |y|)$ on $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$, then the valuation $Z_{\zeta,i}$ defined by (10.1) is rotation invariant.*

Proof. Let $\phi \in O(n)$, $u \in \mathrm{Conv}(\mathbb{R}^n)$ and u_ϕ be defined as in Section 8. We have

$$Z_{\zeta,i}(u_\phi) \;=\; \int_{\Gamma_{u_\phi}} \xi(u_\phi(x), |x|, |y|) \, \mathrm{d}\Theta_i(u_\phi, (x, y)).$$

As observed in the proof of Proposition 8.1, we have $\Gamma_{u_\phi} = \tilde{\phi}\,\Gamma_u$. The conclusion follows by the change of variables

$$(\bar{x}, \bar{y}) = (\phi x, \phi y) = \tilde{\phi}(x, y),$$

and Proposition 8.1. $\qquad\square$

10.3. **Composing a valuation with the conjugate function.** Let $\mathcal{G}$ be a topological Abelian semigroup and $Z\colon \mathrm{Conv}(\mathbb{R}^n) \to \mathcal{G}$ a continuous valuation. By Proposition 3.5 and Proposition 3.6, the functional $Z^*\colon \mathrm{Conv}(\mathbb{R}^n) \to \mathcal{G}$ defined by

$$Z^*(u) = Z(u^*)$$

is a continuous valuation as well.

Assume that $\mathcal{G} = \mathbb{R}$ and that $Z = Z_{\zeta,i}$ is of the form (10.1). Then for every $u \in \mathrm{Conv}(\mathbb{R}^n)$,

$$
\begin{aligned}
Z_{\zeta,i}^*(u) \;&=\; \int_{\mathbb{R}^{2n}} \zeta(u^*(x), x, y) \, \mathrm{d}\Theta_i(u^*, (x, y)) \\
&=\; \int_{\mathbb{R}^{2n}} \zeta(\langle x, y \rangle - u(y), x, y) \, \mathrm{d}\Theta_i(u^*, (x, y)) \\
&=\; \int_{\mathbb{R}^{2n}} \zeta(\langle x, y \rangle - u(x), y, x) \, \mathrm{d}\Theta_{n-i}(u, (y, x)) \\
&=\; Z_{\bar{\zeta}, n-i}(u),
\end{aligned}
$$

where we have used (iii) of Lemma 3.8 and Theorem 8.2, and we set

$$\bar{\zeta}(t, x, y) = \zeta(\langle x, y \rangle - t, y, x)$$

on $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$. This shows in particular that Z^* is still a Hessian valuation of the form (10.1).

10.4. **The cases $i = 0$ and $i = n$.** Let Z be of the form (10.1), with $i = n$:

$$Z_{\zeta,n}(u) = \int_{\Gamma_u} \zeta(u(x), x, y) \, \mathrm{d}\Theta_n(u, (x, y)),$$

for a suitable $\zeta \in C(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n)$, having compact support with respect to x and y.

Let $u \in \mathrm{Conv}(\mathbb{R}^n)$; if $\dim(\mathrm{dom}(u)) < n$, then $Z_{\zeta,n}(u) = 0$ by the simplicity property stated in Theorem 10.1. Assume that $\dim(\mathrm{dom}(u)) = n$, and let

$$D = \{x \in \mathrm{int}(\mathrm{dom}(u)) \colon u \text{ is differentiable at } x\}.$$

As u is differentiable $\mathcal{H}^n$-a.e. on $\mathrm{dom}(u)$, we have

$$\mathcal{H}^n(\mathrm{dom}(u) \setminus D) = 0.$$

Hence, setting

$$\Gamma_{u,D} = \{(x, y) \in \Gamma_u \colon x \in D\},$$

by Proposition 8.3,

$$
\begin{aligned}
\mathrm{Z}_{\zeta,n}(u) &= \int_{\Gamma_{u,D}} \zeta(x, u(x), \nabla u(x)) \, \mathrm{d}\Theta_n(u, (x, y)) \\
&= \int_D \zeta(x, u(x), \nabla u(x)) \, \mathrm{d}x.
\end{aligned}
$$

We conclude that

$$
\mathrm{Z}_{\zeta,n}(u) = \int_{\mathrm{dom}(u)} \zeta(x, u(x), \nabla u(x)) \, \mathrm{d}x
$$

for $u \in \mathrm{Conv}(\mathbb{R}^n)$. Using this and Theorem 8.2 we also get

$$
\mathrm{Z}_{\zeta,0}(u) = \int_{\mathrm{dom}(u^*)} \zeta(\langle \nabla u^*(y), y \rangle - u^*(y), \nabla u^*(y), y) \, \mathrm{d}y
$$

for $u \in \mathrm{Conv}(\mathbb{R}^n)$.

11. Valuations on Convex and Coercive Functions

We can now prove Theorem 1.2 and first establish the following result.

Theorem 11.1. *Let* $\zeta \in C(\mathbb{R} \times \mathbb{R}^n)$ *have compact support and let* $i \in \{0, \dots, n\}$. *The functional* $\mathrm{Z}_{\zeta,i} \colon \mathrm{Conv}_{coe}(\mathbb{R}^n) \to \mathbb{R}$, *defined by*

$$
(11.1) \qquad \qquad \mathrm{Z}_{\zeta,i}(u) = \int_{\mathbb{R}^{2n}} \zeta(u(x), y) \, \mathrm{d}\Theta_i(u, (x, y)),
$$

is a continuous, translation invariant, i-simple valuation. Moreover, if there exists $\xi \in C(\mathbb{R} \times [0, +\infty))$ such that $\zeta(t, y) = \xi(t, |y|)$ on $\mathbb{R} \times \mathbb{R}^n$, then $\mathrm{Z}_{\zeta,i}$ is also rotation invariant.

Proof. As ζ has compact support and u is coercive, the function $(x, y) \mapsto \zeta(u(x), y)$ has compact support as well. By Lemma 10.2 this function is also continuous on Γ_u; hence, as Hessian measures are locally finite, the integral in (11.1) is always finite. The proofs of the valuation property, continuity and rotation invariance are the same as in the case described in the previous section.

In order to prove translation invariance, let $u \in \mathrm{Conv}_{\mathrm{coe}}(\mathbb{R}^n)$ and $x_0 \in \mathbb{R}^n$. For the translated function u_{x_0}, we have

$$
\mathrm{Z}_{\zeta,i}(u_{x_0}) = \int_{\mathbb{R}^{2n}} \zeta(u(x - x_0), y) \, \mathrm{d}\Theta_i(u_{x_0}, (x, y)).
$$

The claim follows by the change of variables $(x - x_0, y) = (\bar{x}, \bar{y})$ and Proposition 8.1. $\qquad \square$

Remark. If ζ is as in the previous theorem and $u \in \mathrm{Conv}_{\mathrm{coe}}(\mathbb{R}^n) \cap C^2(\mathbb{R}^n)$, then

$$
\mathrm{Z}_{\zeta,i}(u) = \int_{\mathbb{R}^n} \zeta(u(x), \nabla u(x)) \, [\mathrm{D}^2 u(x)]_{n-i} \, \mathrm{d}x.
$$

Hence Theorem 11.1 implies Theorem 1.2.

Remark. We have seen that a sufficient condition on $\zeta \in C(\mathbb{R} \times \mathbb{R}^n)$ such that for $i \in \{0, \dots, n\}$, the integral

$$
(11.2) \qquad \qquad \mathrm{Z}_{\zeta,i}(u) = \int_{\mathbb{R}^{2n}} \zeta(u(x), y) \, \mathrm{d}\Theta_i(u, (x, y))
$$

is finite for every $u \in \mathrm{Conv}_{\mathrm{coe}}(\mathbb{R}^n)$, is that ζ has compact support. On the other hand, in some special cases necessary and sufficient conditions for integrability are known. For instance, in [15] is proved that if $i = n$ and $\zeta = \zeta(t)$ is non-negative, then (11.2) is finite for every $u \in \mathrm{Conv}_{\mathrm{coe}}(\mathbb{R}^n)$ if and only if

$$\int_0^{+\infty} t^{n-1} \zeta(t) \, \mathrm{d}t < +\infty.$$

It would be interesting to understand what the corresponding conditions are in the general case. In other words, it would be interesting to have for given i an explicit description of the space of functions

$$\mathrm{Int}_i(\mathbb{R} \times \mathbb{R}^n) = \{\zeta \in C(\mathbb{R} \times \mathbb{R}^n) \colon \int_{\mathbb{R}^{2n}} \zeta(u(x), y) \, \mathrm{d}\Theta_i(u, (x, y)) < \infty \text{ for every } u \in \mathrm{Conv}_{\mathrm{coe}}(\mathbb{R}^n)\}.$$

For the one-dimensional case, some progress in this direction was obtained in [32].

11.1. **Comparison with level-set based valuations.** Let $u \in \mathrm{Conv}_{\mathrm{coe}}(\mathbb{R}^n)$. For $t \in \mathbb{R}$, we consider the sublevel set $\{u \le t\} = \{x \in \mathbb{R}^n \colon u(x) \le t\}$. By the semi-continuity, convexity and coercivity of u, this is either a compact convex subset of $\mathbb{R}^n$, that is, a convex body, or the empty set. In particular, for every $k \in \{0, \ldots, n\}$, the kth quermassintegral $W_k(\{u \le t\})$ is well defined (with the convention $W_k(\emptyset) = 0$ for every k).

Let $\omega \in C(\mathbb{R})$ have compact support. For $k \in \{0, \ldots, n\}$, functionals of the form

$$(11.3) \qquad \qquad \mathrm{X}_k(u) = \int_{\mathbb{R}} \omega(t) \, W_k(\{u \le t\}) \, \mathrm{d}t$$

have been considered on $\mathrm{Conv}_{\mathrm{coe}}(\mathbb{R}^n)$ in [7] (with slightly different assumptions on ω), and in the context of log-concave or quasi-concave functions in [5, 12, 28, 29].

The functional in (11.3) defines a continuous, rigid motion invariant valuation on $\mathrm{Conv}_{\mathrm{coe}}(\mathbb{R}^n)$. Indeed, the assumption of compact support on ω and coercivity assure that it is finite. The valuation property was observed in [7], and rigid motion invariance is an immediate consequence of the rigid motion invariance of quermassintegrals. Continuity with respect to epi-convergence can be proved using Lemma 5 in [15], which concerns the relation between epi-convergence and convergence of level sets. We refer to functionals of the form (11.3) as to *level-set based valuations*.

When $k = n$, we have $W_n(K) = \kappa_n$ (the volume of the unit ball in $\mathbb{R}^n$) for every (non-empty) convex body. Hence

$$\mathrm{X}_n(u) = \xi(\min_{\mathbb{R}^n} u)$$

for $u \in \mathrm{Conv}_{\mathrm{coe}}(\mathbb{R}^n)$, where, for $t \in \mathbb{R}$, we set

$$\xi(t) = \kappa_n \int_t^\infty \omega(s) \, \mathrm{d}s.$$

For $k = 0$, as W_0 is the Lebesgue measure, by the Layer Cake Principle (see Section 6.3 in [7]), we have the equivalent representation

$$(11.4) \qquad \qquad \mathrm{X}_0(u) = \int_{\mathrm{dom}(u)} \xi(u(x)) \, \mathrm{d}x$$

for $u \in \mathrm{Conv}_{\mathrm{coe}}(\mathbb{R}^n)$, where, for $t \in \mathbb{R}$, we now set

$$\xi(t) = \int_t^{+\infty} \omega(s) \, \mathrm{d}s.$$

In particular, (11.4) is a valuation which can be written as a Hessian valuation and as a level-set based valuation.

The following examples show that the family of Hessian valuations is not included in that of level-set based valuations.

Example 11.2. Let $n = 1$. Note that there are only two types of level-set based valuations on $\mathrm{Conv}_{\mathrm{coe}}(\mathbb{R})$:

$$\mathrm{X}_0(u) = \xi_0(\min_{\mathbb{R}} u), \quad \mathrm{X}_1(u) = \int_{\mathrm{dom}(u)} \xi_1(u(x)) \, \mathrm{d}x,$$

where $\xi_0, \xi_1 \in C(\mathbb{R})$. Let $\zeta \in C(\mathbb{R}^2)$ be even with respect to second variable and have compact support. We consider the Hessian valuation Z defined on $\mathrm{Conv}_{\mathrm{coe}}(\mathbb{R}^n)$ as

$$\mathrm{Z}(u) = \int_{\Gamma_u} \zeta(u(x), y) \, \mathrm{d}\Theta_0(u, (x, y)).$$

Assume that Z can be written as linear combination of level-set based valuations, that is,

$$(11.5) \qquad \mathrm{Z}(u) = \xi_0(\min(u)) + \int_{\mathrm{dom}(u)} \xi_1(u(x)) \, \mathrm{d}x$$

for suitable ξ_0 and ξ_1. Evaluating (11.5) on functions of the form $u(x) = t + I_{[-r,r]}(x)$ with $t \in \mathbb{R}$ and $r > 0$ and using Proposition 8.3, we obtain

$$\int_{\mathbb{R}} \zeta(t, y) \, \mathrm{d}y = \xi_0(t) + 2r\xi_1(t).$$

As the last term is the only one depending on r, we get $\xi_1 \equiv 0$. Hence (11.5) becomes

$$\mathrm{Z}(u) = \xi_0(\min_{\mathbb{R}} u).$$

Plugging the function $u(x) = t + s|x|$ into the last equality with $t \in \mathbb{R}$ and $s > 0$, we deduce

$$\int_{-s}^{s} \zeta(t, y) \, \mathrm{d}y = 2 \int_0^s \zeta(t, y) \, \mathrm{d}y = \xi_0(t)$$

for $t \in \mathbb{R}$ and $s > 0$. Thus we obtain $\zeta \equiv 0$.

Example 11.3. Let $\zeta = \zeta(t, s)$ be a function in $C(\mathbb{R} \times [0, \infty))$ of compact support. Consider the Hessian valuation Z defined on $\mathrm{Conv}_{\mathrm{coe}}(\mathbb{R}^n)$ by

$$\mathrm{Z}(u) = \int_{\Gamma_u} \zeta(u(x), |y|) \, \mathrm{d}\Theta_n(u, (x, y)) = \int_{\mathrm{dom}(u)} \zeta(u(x), |\nabla u(x)|) \, \mathrm{d}x.$$

Assume that Z can be written as linear combination of level-set based valuations

$$\mathrm{Z}(u) = \sum_{k=0}^{n} \mathrm{X}_k(u).$$

Note that Z is n-simple, *i.e.*, it vanishes for all u such that $\dim(\mathrm{dom}(u)) < n$. The only level-set based valuations with this property are those for $k = 0$. It is not hard to see that this implies that the right hand-side of the previous inequality reduces to the term X_0, hence, on $\mathrm{Conv}_{\mathrm{coe}}(\mathbb{R}^n)$,

$$(11.6) \qquad \mathrm{Z}(u) = \int_{\mathrm{dom}(u)} \xi(u(x)) \, \mathrm{d}x$$

for some function $\xi \in C(\mathbb{R})$. We evaluate (11.6) at functions u of the form $u(x) = s|x| + I_{B_r^n}$, with $s, r > 0$, obtaining

$$\int_0^r \zeta(st, s) t^{n-1} \, \mathrm{d}t = \int_0^r \xi(st) t^{n-1} \, \mathrm{d}t.$$

As $r > 0$ is arbitrary, we deduce $\zeta(t, s) = \xi(t)$ for all $t \in \mathbb{R}$ and $s > 0$, which is possible only if ξ does not depend on the second variable.

12. The Space of 1-homogeneous Convex Functions

We analyze connections of the results on valuations on convex functions and valuations on convex bodies. Let

$$\mathrm{Conv}_{\mathrm{hom}}(\mathbb{R}^n) = \{u \in \mathrm{Conv}(\mathbb{R}^n) \colon \mathrm{dom}(u) = \mathbb{R}^n, \ u \text{ is 1-homogeneous}\}.$$

It is well known that $u \in \mathrm{Conv}_{\mathrm{hom}}(\mathbb{R}^n)$ if and only if there exists a convex body K in $\mathbb{R}^n$ such that $u = h_K$, the support function of K (see Example 3.2). We recall that $\mathcal{K}^n$ denotes the family of convex bodies in $\mathbb{R}^n$.

We want to discuss valuations on $\mathrm{Conv}_{\mathrm{hom}}(\mathbb{R}^n)$ and first look at Hessian measures on this space. Denote by u a generic element of $\mathrm{Conv}_{\mathrm{hom}}(\mathbb{R}^n)$ and by K the convex body such that $u = h_K$. For a point $y \in \mathrm{bd}(K)$, let $N(K, y)$ be the normal cone to K at y. For $x \neq 0$, we have $y \in \partial u(x)$ if and only if $y \in \mathrm{bd}(K)$ and $x \in N(K, y)$; moreover, $\partial u(0) = K$ (see [35, Theorem 1.7.4]). Hence

$$\Gamma_u = (\{0\} \times K) \cup \{(x, y) \in \mathbb{R}^{2n} \colon y \in \mathrm{bd}(K), \ x \in N(K, y)\}.$$

In particular,

$$\Gamma_u \subset \mathbb{R}^n \times K.$$

The 1-homogeneity of u implies that the subdifferential is 0-homogeneous, that is, $\partial u(tx) = \partial u(x)$ for $x \in \mathbb{R}^n$ and $t > 0$. Hence, for every $\eta \subset \mathbb{R}^{2n}$, $s \geq 0$ and $t > 0$,

$$
\begin{aligned}
P_s(u, t\eta) &= \{tx + sy \colon x \in \eta, \ y \in \partial u(x)\} \\
&= t\left\{x + \frac{s}{t}y \colon (x, y) \in \eta \cap \Gamma_u\right\} \\
&= t\, P_{s/t}(u, \eta).
\end{aligned}
$$

Taking Lebesgue measures and using the Steiner formula (7.1), we obtain the following homogeneity property for Hessian measures of support functions,

$$\Theta_i(u, t\eta) = t^i\, \Theta_i(u, \eta)$$

for $\eta \in \mathcal{B}(\mathbb{R}^{2n})$ and $i \in \{0, \ldots, n\}$.

Hessian measures of support functions are closely connected to *support measures* of the corresponding convex bodies. For the definition of the support measures $\Theta_0(K, \cdot), \ldots, \Theta_n(K, \cdot)$ of a convex body K, we refer to [35, Theorem 4.2.1]. The following relation was established in [11, Corollary 5.9]. For every $\alpha \in \mathcal{B}(\mathbb{S}^{n-1})$, $\beta \in \mathcal{B}(\mathbb{R}^n)$ and $i \in \{1, \ldots, n\}$:

$$\Theta_i(u, \hat{\alpha} \times \beta) = \tfrac{1}{n}\, \Theta_{n-i}(K, \beta \times \alpha),$$

where $\hat{\alpha}$ is the convex hull $\alpha \cup \{0\}$. In particular,

$$\Theta_i(u, B^n \times \mathbb{R}^n) = \Theta_i(u, \Gamma_u \cap (B^n \times \mathbb{R}^n)) = \tfrac{1}{n}\, \Theta_{n-i}(K, \mathbb{R}^n \times \mathbb{S}^{n-1}) = W_{n-i}(K)$$

where $W_j(K)$, for $j \in \{0, \ldots, n\}$, is the jth quermassintegral of K. The previous equality extends to the case $i = 0$ as

$$\Theta_0(u, B^n \times \mathbb{R}^n) = \Theta_0(u, \Gamma_u \cap (B^n \times \mathbb{R}^n)) = \mathcal{H}^n(\pi_2(\Gamma_u)) = \mathcal{H}^n(K) = W_0(K).$$

A real-valued valuation on $\mathcal{K}^n$ is a functional $\mathrm{Y} \colon \mathcal{K}^n \to \mathbb{R}$ which has the additivity property

$$\mathrm{Y}(K \cup L) + \mathrm{Y}(K \cap L) = \mathrm{Y}(K) + \mathrm{Y}(L)$$

for every $K, L \in \mathcal{K}^n$ such that $K \cup L \in \mathcal{K}^n$. The valuation Y is continuous, if it is continuous with respect to the Hausdorff metric.

Hadwiger's celebrated classification theorem is the following result (see [19]).

Theorem 12.1 (Hadwiger). *A functional* $\mathrm{Y}\colon \mathcal{K}^n \to \mathbb{R}$ *is a continuous valuation, invariant under rotations and translations if and only if there exist* $c_0, \dots, c_n \in \mathbb{R}$ *such that*

$$\mathrm{Y}(K) = \sum_{i=0}^{n} c_i \, W_i(K)$$

for every $K \in \mathcal{K}^n$.

To rewrite Hadwiger's theorem as a result on $\mathrm{Conv}_{\mathrm{hom}}(\mathbb{R}^n)$, we need the following facts. We associate with $\mathrm{Y}\colon \mathcal{K}^n \to \mathbb{R}$ the functional $\bar{\mathrm{Y}}\colon \mathrm{Conv}_{\mathrm{hom}}(\mathbb{R}^n) \to \mathbb{R}$, defined as $\bar{\mathrm{Y}}(h_K) = \mathrm{Y}(K)$. First, let K_j be a sequence of convex bodies in $\mathbb{R}^n$ and let $u_j = h_{K_j}$ for every $j \in \mathbb{N}$. The sequence K_j converges to a convex body K with respect to the Hausdorff metric if and only if the sequence of functions u_j converges pointwise to $u = h_K$ in $\mathbb{R}^n$ (and uniformly on compact sets). This is in turn equivalent to the fact that u_j epi-converges to u. Hence, $\mathrm{Y}\colon \mathcal{K}^n \to \mathbb{R}$ is continuous with respect to the Hausdorff metric if and only if $\bar{\mathrm{Y}}\colon \mathrm{Conv}_{\mathrm{hom}}(\mathbb{R}^n) \to \mathbb{R}$ is epi-continuous.

Second, if $K, L \in \mathcal{K}^n$ are such that $K \cup L \in \mathcal{K}^n$, then

$$h_{K\cup L} = h_K \vee h_L, \quad h_{K\cap L} = h_K \wedge h_L.$$

This follows from Example 3.2 and Proposition 3.5. Hence, if $\mathrm{Y}\colon \mathcal{K}^n \to \mathbb{R}$ is a valuation, then $\bar{\mathrm{Y}}\colon \mathrm{Conv}_{\mathrm{hom}}(\mathbb{R}^n) \to \mathbb{R}$ is also a valuation.

Third, for every $\phi \in \mathrm{O}(n)$, the support function of ϕK is given by $h_{\phi K}(x) = h_K(\phi^t x)$ for every $x \in \mathbb{R}^n$. Hence, $\mathrm{Y}\colon \mathcal{K}^n \to \mathbb{R}$ is rotation invariant if and only if $\bar{\mathrm{Y}}$ is also rotation invariant.

Finally, for every $x_0 \in \mathbb{R}^n$, the support function of the convex body $K + x_0$ is given by

$$h_{K+x_0}(x) = h_K(x) + \langle x, x_0 \rangle = u(x) + \langle x, x_0 \rangle$$

for every $x \in \mathbb{R}^n$. Hence, $\mathrm{Y}\colon \mathcal{K}^n \to \mathbb{R}$ is translation invariant if and only if $\bar{\mathrm{Y}}(u + w) = \bar{\mathrm{Y}}(w)$ for every linear function $w\colon \mathbb{R}^n \to \mathbb{R}$. In general, we say that $\mathrm{Z}\colon \mathrm{Conv}_{\mathrm{hom}}(\mathbb{R}^n) \to \mathbb{R}$ is unchanged by addition of linear functions if $\mathrm{Z}(u + w) = \mathrm{Z}(u)$ for every $u \in \mathrm{Conv}_{\mathrm{hom}}(\mathbb{R}^n)$ and linear function w.

As a consequence of the previous considerations, we obtain the following version of Hadwiger's theorem.

Theorem 12.2 (Hadwiger). *A functional* $\mathrm{Z}\colon \mathrm{Conv}_{\mathrm{hom}}(\mathbb{R}^n) \to \mathbb{R}$ *is a continuous and rotation invariant valuation that is unchanged by addition of linear functions if and only if there exists* $c_0, \dots, c_n \in \mathbb{R}$ *such that*

$$\mathrm{Z}(u) = \sum_{i=0}^{n} c_i \, \Theta_i(u, B^n \times \mathbb{R}^n)$$

for every $u \in \mathrm{Conv}_{\mathrm{hom}}(\mathbb{R}^n)$.

Without requiring that the functionals are unchanged by addition of linear functions, we obtain many further functionals.

Theorem 12.3. *Let* $\zeta \in C(\mathbb{R} \times [0, +\infty))$. *For* $i \in \{0, \dots, n\}$, *the functional* $\mathrm{Z}_{\zeta,i}\colon \mathrm{Conv}_{\mathrm{hom}}(\mathbb{R}^n) \to \mathbb{R}$, *defined by*

$$(12.1) \qquad\qquad \mathrm{Z}_{\zeta,i}(u) = \int_{B^n \times \mathbb{R}^n} \zeta(u(x), |y|) \, \mathrm{d}\Theta_i(u, (x, y)),$$

is a continuous and rotation invariant valuation on $\mathrm{Conv}_{\mathrm{hom}}(\mathbb{R}^n)$. *Consequently,* $K \mapsto \mathrm{Z}_{\zeta,i}(h_K)$ *is a continuous and rotation invariant valuation on* $\mathcal{K}^n$.

Proof. Finiteness of $Z_{\zeta,i}$ follows from the compactness of $\Gamma_u \cap (B^n \times \mathbb{R}^n)$, the continuity of u and ζ, and the local finiteness of Hessian measures. The proofs of the valuation property, continuity and rotation invariance are as in the proofs of Theorem 10.1. $\qquad\square$

Alesker [1] obtained a classification of continuous and rotation invariant valuations on convex bodies that are polynomial with respect to translations and showed that these are functionals of type (12.1).

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