A Helmholtz–Lie Type Characterization of Ellipsoids II

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Abstract. A closed convex surface S in $\mathbb{E}^d$ is an ellipsoid if and only if for any $x, y \in S$ there is an affinity mapping x onto y and a neighbourhood of x in S onto a neighbourhood of y in S.

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1 Introduction and Statement of Result

1.1 The present articles deal with the problem of characterizing ellipsoids among all closed convex surfaces in Euclidean d-space $\mathbb{E}^d$ by local transformation properties. Using topological tools the answer for odd d was obtained in the first article. Here the answer for all d will be given. More precisely, the following result will be proved where the *closed convex surface* is the boundary of a compact convex subset of $\mathbb{E}^d$ with non-empty interior.

Theorem. *Let S be a closed convex surface in $\mathbb{E}^d$ with the following property $\mathcal{A}$: to any pair of points $x, y \in S$ there corresponds an affine transformation A_{xy} of $\mathbb{E}^d$ which maps x onto y and a suitable neighbourhood N_{xy} of x in S onto a neighbourhood of y in S. Then S is an ellipsoid.*

For related results we refer to the introduction of the first article. To the convexity results cited there we include Mäurer [11]. Related results in the context of differential geometry and affine differential geometry are due to Szabó [15], and Liu and Wang [9].

1.2 The proof relies heavily on results of Leichtweiss on floating bodies and of Blaschke and Petty in affine differential geometry. In particular, a characterization of ellipsoids due to Petty is needed utilizing the notions of affine distance and Santaló point. Moreover, use is made of a characterization of spheres by the property that their Gauss curvature is constant, which goes back to Liebmann, and of a characterization of convex sets due to Tietze. In addition, we use tools from the first article and in case $d = 2$ the solution of a functional equation is needed.

2 General Tools and Preliminaries

For concepts not explained below and results for which no reference is given we refer to [14]. Let $|\cdot|$, conv, relint, and det stand for volume, convex hull, relative interior with respect to a given surface, and determinant of the linear part of an affinity, respectively. $\|\cdot\|$ and S^{d-1} denote the Euclidean norm and the unit sphere in $\mathbb{E}^d$.

Let S be a closed convex surface in $\mathbb{E}^d$.

2.1 It is well-known that

(1) a convex function on an interval in $\mathbb{R}$ is almost everywhere twice differentiable.

2.2 The supporting function $h(S,\cdot)$ of S is defined on S^{d-1}. Assume now that S is smooth, i.e. of class $\mathcal{C}^1$. Then for $x \in S$ we denote by $n(S,x)$ the exterior normal unit vector of C at x. Given $\delta > 0$, let $C(S,x,\delta)$ be the *cap* of S with centre x and height δ, that is the part of S between the supporting hyperplane of S at x and its translate by the vector $-\delta n(S,x)$. The *base* $B(S,x,\delta)$ of $C(S,x,\delta)$ is the convex hull of the intersection of S with the translated supporting hyperplane. If $\mu = |\operatorname{conv} C(S,x,\delta)|$ then μ is called the *volume* of $C(S,x,\delta)$ and instead of $B(S,x,\delta)$ and $C(S,x,\delta)$ we also write $B(S,x,\mu)$ and $C(S,x,\mu)$, respectively. For S of class $\mathcal{C}^2$ let $\kappa(S,u)$ be the Gauss curvature of S at the point(s) of S with exterior normal unit vector u. Thus $\kappa(S,\cdot)$ is defined on S^{d-1}.

2.3 Let S be of class $\mathcal{C}^1$. For $\nu > 0$ a closed convex surface $S_{[\nu]}$ in the interior of S is the *floating surface* of S corresponding to ν if any supporting hyperplane of $S_{[\nu]}$ cuts off from S a cap of volume ν. The following result is well-known.

(2) Let $S_{[\nu]}$ be a floating surface of S. Then the base of each cap of S of volume ν touches $S_{[\nu]}$ at a unique point and this point is the centroid of the base.

In order to state the next result we introduce the following notion where $\varepsilon > 0$. The surface S is *ε-smooth* if for any $x \in S$ there is a (solid Euclidean) ball of radius ε contained in $\operatorname{conv} S$ which touches S at x. If for each $x \in S$ there is a ball of radius $\frac{1}{\varepsilon}$ containing S and such that its boundary touches S at x, then S will be called *ε-strictly convex.*

From Leichtweiss [6] we take the following proposition.

(3) Let S be ε-smooth for a given $\varepsilon > 0$ and strictly convex. Then there is a $\lambda > 0$ such that for $0 < \nu < \lambda$ the floating surface $S_{[\nu]}$ of S exists, is of class $\mathcal{C}^2$ and the inequality $\kappa(S_{[\nu]},\cdot) > 0$ holds.

2.4 Elementary calculations yield the next result, compare [7].

(4) Let S be of class $\mathcal{C}^2$ and let L be a volume-preserving linear transformation of $\mathbb{E}^d$. Then, for each $x \in S$ and $u = n(S,x)$,

$$n(L(S), L(x)) = \frac{L^{-t}(u)}{\|L^{-t}(u)\|} = v, \text{ say,}$$

and

$$\kappa(L(S), v) = \frac{\kappa(S,u)}{\|L^{-t}(u)\|^{d+1}}.$$

Here L^{-t} is the inverse of the adjoint transformation of L.

2.5 A closed set U in $\mathbb{E}^d$ is said to have the *local supporting property* if for each of its boundary points x there is a hyperplane through x such that all points of U in a suitable neighbourhood of x are on the same side of the hyperplane, possibly on the hyperplane. A result of Tietze [16] implies the following.

(5) Let U be a closed compact surface in $\mathbb{E}^d$ which is starshaped with respect to the origin o, that is, each open ray starting at o meets U in precisely one point. If U has the local supporting property, then it is a closed convex surface.

2.6 Let S be of class $\mathcal{C}^2$ with $\kappa(S,\cdot) > 0$ and assume that o is in its interior. Generalizing a result of Blaschke [3], Petty [12] proved for the *affine distance*,

(6) $a(S,u) = h(S,u)\kappa(S,u)^{-1/(d+1)}$

from o to the (unique) point $x \in S$ where $u = n(S,x)$, the formula

(7) $$a(S,u)^{d+1} = \lim_{\delta\to+0} \frac{d^{d+1}|\operatorname{conv}(\{o\}\cup B(S,x,\delta))|}{(d+1)^{d-1}\kappa_{d-1}^2|\operatorname{conv} C(S,x,\delta)|}.$$

Here κ_{d-1} is the volume of the unit ball in $\mathbb{E}^{d-1}$.

The *Santaló point* of S is the point

(8) $$\operatorname{san} S = \int_{S^{d-1}} u h(S,u)^{-(d+1)} d\sigma(u),$$

where σ is the ordinary surface area measure on S^{d-1} and the integral is to be understood componentwise.

The following characterization of ellipsoids is due to Petty [13].

(9) Let S be of class $\mathcal{C}^2$ with $\kappa(S,\cdot) > 0$ where $o = \operatorname{san} S$. If $a(S,\cdot)$ is constant on S^{d-1}, then S is an ellipsoid with centre o.

2.7 It is well-known that,

(10) if S is of class $\mathcal{C}^2$ with $\kappa(S,\cdot) > 0$, then

$$\int_{S^{d-1}} \frac{u}{\kappa(S,u)} d\sigma(u) = o.$$

Also well-known is the next result of which a first version was given by Liebmann [8].

(11) Let S be of class $\mathcal{C}^2$. If $\kappa(S,\cdot)$ is constant, then S is a sphere.

3 Proof of the Theorem

Let S be as stated in the Theorem. The proof that S is an ellipsoid is split into three parts: in Section 3.1 some needed tools from the first article are cited and a lemma on ε-smoothness is proved. In Sections 3.2 and 3.3 we distinguish the cases where there are, resp. are not points $p, q \in S$ such that for corresponding affinities A_{pq} and B_{pq} the non-equality $|\det A_{pq}| \neq |\det B_{pq}|$ holds. In the former case we treat the cases $d = 2$ and $d \geq 3$ separately.

3.1 Tools from the first article and a lemma.

3.1.1 From [5], Sections 3.2, 3.3, 3.4.2 and 3.4.3, respectively, we take the following propositions.

(12) S is of class $\mathcal{C}^1$ and strictly convex.

(13) $\det A_{xy} \neq 0$ for all $x, y \in S$.

(14) Assume that for each pair $x, y \in S$ the value of $|\det A_{xy}|$ is the same for all affinities A_{xy}. Then, given $p \in S$, the function

$$x \to |\det A_{px}| : x \in S$$

is continuous and thus bounded between positive constants by (13).

(15) Assume that for each pair $x, y \in S$ the value of $|\det A_{xy}|$ is the same for all affinities A_{xy}. Then, given $p \in S$, there is a $\mu > 0$ such that

$$C(S, p, \mu) \subset N_{px},$$

$$A_{px}(C(S, p, \mu)) = C(S, x, |\det A_{px}|\mu)$$

for each $x \in S$ and suitable A_{px}.

(16) Let $d \geq 3$ and assume that there are points $p, q \in S$ and corresponding affinities A_{pq} and B_{pq} with $|\det A_{pq}| \neq |\det B_{pq}|$. Then S is an ellipsoid.

3.1.2 This subsection contains the proof of the following lemma.

(17) There is an $\varepsilon > 0$ such that S is ε-smooth and ε-strictly convex.

Since the proofs for ε-smoothness and for ε-strict convexity are very similar, only the former will be given. The first step is to show the following.

(18) For each $x \in S$ there is a ball B_x, with $x \in B_x \subset \operatorname{conv} S$.

Obviously, there are a point $p \in S$ and a ball B such that $p \in B \subset \operatorname{conv} S$. Now, for any $x \in S$, by replacing B by a suitable smaller ball which then is also denoted by B, if necessary, we may assume that $B \subset \operatorname{conv} N_{px}$ where N_{px} is a neighbourhood corresponding to p, x. Then

$$x = A_{px}(p) \in A_{px}(B) \subset A_{px}(\operatorname{conv} N_{px}) = \operatorname{conv} A_{px}(N_{px}) \subset \operatorname{conv} S.$$

Now choose a ball B_x contained in the ellipsoid $A_{px}(B)$ with $x \in B_x$. The proof of (18) is complete.

Clearly, we may assume that for each $x \in S$ the ball B_x has maximum radius, say ε_x. A simple compactness argument then yields,

(19) $S_n = \{x \in S : \varepsilon_x \geq \frac{1}{n}\}$ is closed in S for $n = 1, 2, \ldots$

By (18),

(20) $S = \bigcup_{n=1}^{\infty} S_n$.

Since S is (with the induced metric) a complete metric space, a version of the Baire category theorem together with (19) and (20) implies that

$$\operatorname{relint} S_n \neq \emptyset \text{ for a suitable index } n.$$

Let $p \in \operatorname{relint} S_n$. Obviously,

$$A_{px}(N_{px} \cap \operatorname{relint} S_n),\ x \in S,\ \text{is an open covering of } S.$$

By the compactness of S there are open neighbourhoods $N_1, \ldots, N_k$ of p in S_n and non-singular affinities $A_1, \ldots, A_k$ such that

$$A_1(N_1), \ldots, A_k(N_k) \text{ is an open covering of } S.$$

Again, the compactness of S in conjunction with Lebesgue's covering lemma then shows that there are sets

(21) $M_1 \subset N_1, \ldots, M_k \subset N_k$, compact,

while still

(22) $A_1(M_1), \ldots, A_k(M_k)$ is a covering of S.

Since $M_i \subset N_i \subset S_n$ for $i = 1, \ldots, k$, the definition of S_n in (19) implies,

(23) for each $q \in M_i, i = 1, \ldots, k$, there is a ball of radius $\varepsilon = \frac{1}{n}$ in $\operatorname{conv} S$ which touches S at q.

Taking into account (12) and (21), by decreasing ε if necessary, we may replace (23) by the following proposition.

(24) There is an $\varepsilon > 0$ such that for each $q \in M_i, i = 1, \ldots, k$, there is a ball B_q of radius ε with $q \in B_q \subset \operatorname{conv} N_i$.

Now (22), (24) and an argument which is slightly more complicated than the one that led to (18) give (17). (Here we have to deal with k ellipsoids.)

3.2 Case 1. We assume here that

(25) there are points $p, q \in S$ and corresponding affinities A_{pq} and B_{pq} with $|\det A_{pq}| \neq |\det B_{pq}|$.

Our aim is to show that then

(26) S is an ellipsoid.

Remark: Since the affinities which map neighbourhoods of points on an ellipsoid onto neighbourhoods on the ellipsoid are volume-preserving, this shows that Case 1 actually cannot hold.

3.2.1 $d = 2$. At first it will be shown that a convex arc which properly contains an affine image of itself must contain an arc of a conic. The proof of this lemma in essence consists of the solution of a functional equation. The lemma together with the assumption (25) then easily leads to (26).

We first show the following lemma.

(27) Let $p \in S$ and let A be an arc of S starting at p. Assume that $C_{pp} \neq id$ is an affinity with $\det C_{pp} > 0$ such that $C_{pp}(A)$ is also an arc in S which starts at p in the same direction as A. Then a sub-arc of A is an arc of a conic.

At first A and $C_{pp}(A)$ will be represented in a suitable Cartesian coordinate system: choose p as the origin, let the supporting line of S at p (which is unique by (12)) be the first coordinate axis such that A starts in the direction of the positive axis and let the positive second axis point into the halfplane containing S. The coordinates in this system will be denoted s, t.

Clearly, we may represent a sub-arc of A which starts at p in the form

$$t = f(s) \text{ or } s = g(t)$$

with suitable functions f, g. It follows from (1) and the property $\mathcal{A}$ of S that S is everywhere twice differentiable. Hence

(28) $f(s) \sim \alpha s^2$ as $s \to +0$ and thus $g(t) \sim \beta t^{1/2}$ as $t \to +0$,

where $\alpha > 0$ by (17) and $\beta = \alpha^{-1/2}$.

Since the arcs A and $C_{pp}(A)$ on S both start at (the origin) p in the direction of the positive first coordinate axis, the affinity C_{pp} maps the positive axis onto itself. Hence we may represent C_{pp} in the form

$$\begin{pmatrix} s \\ t \end{pmatrix} \to \begin{pmatrix} as + bt \\ ct \end{pmatrix} \text{ for } \begin{pmatrix} s \\ t \end{pmatrix} \in \mathbb{R}^2$$

with suitable $a > 0, b, c$. Clearly, $\det C_{pp} = ac > 0$ implies that $c > 0$. By replacing C_{pp} by C_{pp}^{-1} and writing C_{pp} for C_{pp}^{-1}, if necessary, we may assume that

(29) $0 < a \leq 1,\ c > 0.$

C_{pp} maps $(g(t), t)^t$ onto $(ag(t) + bt, ct)^t$. For small $t > 0$ the latter point is also on the sub-arc of A which is represented by $g(\cdot)$. Thus

(30) $ag(t) + bt = g(ct)$ for small $t > 0$,

and therefore

$$a\beta t^{1/2} + bt \sim c^{1/2}\beta t^{1/2} \text{ as } t \to +0$$

by (28). Hence $a = c^{1/2}$ or $c = a^2$. Thus, if $a = 1$ we have $c = 1$ and (30) yields $b = 0$, that is, $C_{pp} = id$, a contradiction. Hence $a \neq 1$ and instead of (29) the sharper statement

(31) $0 < a < 1,\ c = a^2$

holds.

The final step in the proof of (27) is to show that

(32) $g(t) = \beta t^{1/2} - \dfrac{b}{a(1-a)} t$ for small $t > 0$.

(30), (31), and (28) show that for small $t > 0$

$$\begin{aligned}
g(t) &= \frac{1}{a} g(a^2 t) - \frac{b}{a} t \\
&= \frac{1}{a}\{\frac{1}{a} g(a^4 t) - \frac{b}{a} a^2 t\} - \frac{b}{a} t = \frac{1}{a^2} g(a^4 t) - \frac{b}{a}(1+a)t \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&= \frac{1}{a^n} g(a^{2n} t) - \frac{b}{a}(1 + a + \ldots + a^{n-1})t = \frac{1}{a^n} g(a^{2n} t) - \frac{b}{a}\frac{1-a^n}{1-a} t \\
&\to \beta t^{1/2} - \frac{b}{a(1-a)} t \text{ as } n \to \infty,
\end{aligned}$$

which concludes the proof of (32).

Since the arc of S defined by $g(\cdot)$ where $t > 0$ is small is a conic arc by (32), the proof of (27) is complete.

Having proved (27), the proof of (26) is simple. Note the assumption (25) and let $C_{pp} = A_{pq}^{-1} B_{pq}$. By replacing C_{pp} by C_{pp}^2 and writing C_{pp} for C_{pp}^2, if necessary, we may suppose that

(33) $0 < \det C_{pp} \neq 1$ and thus, in particular, $C_{pp} \neq id$.

Clearly, C_{pp} maps p onto p and for a suitable arc A in S starting at p also $C_{pp}(A)$ is an arc of S starting at p. The strict convexity of S (see (12)) and $\det C_{pp} > 0$ (see (33)) imply that A and $C_{pp}(A)$ both start in the same direction. Thus an application of (27) shows that S contains an arc of a conic. The transformation property $\mathcal{A}$, the compactness of S, and the fact that two overlapping arcs of conics actually are on the same conic then implies that S is a conic itself. Being bounded, it is an ellipse, concluding the proof of (26).

3.2.2 $d \geq 3$. The assumption (25) together with (16) immediately yield (26).

3.3 Case 2. We now assume that

(34) for each pair $x, y \in S$ the value of $|\det A_{xy}|$ is the same for all corresponding affinities A_{xy}.

Again, our aim is to prove that

(35) S is an ellipsoid.

As a first step it will be shown that $|\det A_{xy}| = 1$ for all $x, y \in S$. Then we will prove that the floating surfaces $T = S_{[\nu]}$ also have property $\mathcal{A}$. Next, to each T we assign a closed starshaped surface U. (It is perhaps worth noting that T is a dilatation of the curvature image of U, compare Lutwak [10].) It then turns out that U has also property $\mathcal{A}$ but with linear affinities. This yields in particular, using Tietze's theorem (5), that U is a closed convex surface. Hence one may consider the floating surfaces $V = U_{[\nu]}$ of U. They, again, have property $\mathcal{A}$ where the affinities are linear. A further property of the V's deals with the affine distance. This permits the application of Petty's characterization (9) of ellipsoids. Hence each V is an ellipsoid. Then, going back from V to U, from U to T, and then to S, we see that also S is an ellipsoid.

3.3.1 This subsection contains the proof that

(36) $|\det A_{px}| = 1$ for all $p, x \in S$.

Property $\mathcal{A}$ and the assumption (34) yield the following proposition.

(37) Let $p, z, y \in S$ and let A_{pz}, A_{zy}, A_{py} be corresponding affinities. Then $|\det A_{py}| = |\det A_{pz}|\,|\det A_{zy}|$.

Let $p \in S$. The assumption (34) together with (14) implies that there is a $q \in S$ such that

$$|\det A_{pq}| = \max\{|\det A_{pz}| : z \in S\}.$$

Thus

(38) $|\det A_{pz}| \leq |\det A_{pq}|$ for all $z \in S$.

Let $x \in S$ and consider A_{qx}. The affinity A_{qx} maps a neighbourhood N_{qx} of q in S onto a neighbourhood N_x of x in S. For any $y \in N_x$ there is a point $z \in N_{qx}$ with $A_{qx}(z) = y$. By (34) we thus have

$$|\det A_{zy}| = |\det A_{qx}|.$$

This, (37) and (38) together then show

$$\begin{aligned} |\det A_{py}| &= |\det A_{pz}|\,|\det A_{zy}| \\ &\leq |\det A_{pq}|\,|\det A_{qx}| = |\det A_{px}| \text{ for all } y \in N_x. \end{aligned}$$

The function $x \to |\det A_{px}|$ thus has a local maximum at any point $x \in S$. Since this function is continuous by (34) and (14), it is a constant. Taking $x = p$, it follows that this constant is 1, concluding the proof of (36).

3.3.2 By (3) and (17),

(39) there is a $\lambda > 0$ such that $T = S_{[\nu]}$ exists, is of class $\mathcal{C}^2$ and $\kappa(S_{[\nu]}, \cdot) > 0$ for $0 < \nu < \lambda$.

Since $|\det A_{xy}| = 1$ for all $x, y \in S$ by (36), the following is a simple consequence of (15), where for A_{xy} we take $A_{py}A_{px}^{-1}$:

(40) there is a $\mu > 0$ such that

$$A_{xy}(C(S, x, \mu)) = C(S, y, \mu)$$

for all $x, y \in S$ and suitable A_{xy}.

In the following when we write A_{xy} it is to be understood that A_{xy} is a volume-preserving affinity which satisfies (40).

Our aim in this subsection is to show that

(41) $T = S_{[\nu]}$ has property $\mathcal{A}$ for $0 < \nu < \min\{\lambda, \mu\}$ where the affinities are volume-preserving.

For the proof of (41) it is sufficient to verify the following.

(42) Let $0 < \nu < \min\{\lambda, \mu\}$ and $u, v \in T = S_{[\nu]}$ be chosen. Since by (12) and (39) the surfaces S and T are smooth and strictly convex, there are unique $x, y \in S$ with $n(S, x) = n(T, u)$ and $n(S, y) = n(T, v)$. Then there are a neighbourhood M of u in T and a volume-preserving affinity $B_{uv}(= A_{xy})$ which maps u onto v and M into T.

The supporting hyperplane of T at u cuts off from S the cap $C(S, x, \nu)$ of volume ν. Since $\nu < \mu$, this cap is contained in $\operatorname{relint} C(S, x, \mu)$. Thus, since T is smooth and strictly convex (see (39)), there is a neighbourhood M of u in T such that

(43) for each $w \in M$ the following hold: if $z \in S$ is chosen (uniquely) such that $n(S, z) = n(T, w)$, then

- (a) the supporting hyperplane of T at w cuts off from S the cap $C(S, z, \nu)$ of volume ν, where
- (b) $C(S, z, \nu) \subset C(S, x, \mu)$, and
- (c) w is the centroid of the base $B(S, z, \nu)$.

(a), (b) are clear and (c) is implied by (2).

The proof that $B_{uv} = A_{xy}$ maps u onto v in essence is a special case of the proof that B_{uv} maps M into T. Hence only the latter will be given: let $w \in M$ and choose $z \in S$ such that $n(S,z) = n(T,w)$. By (43) w is the centroid of the base $B(S,z,\nu)$ of the cap $C(S,z,\nu)$. Since A_{xy} is non-singular, $A_{xy}(w)$ is the centroid of the base of the cap

$$A_{xy}(C(S,z,\nu)) = C(S, A_{xy}(z), \nu) \subset S,$$

see the definition of M, (43), and (40). This implies that $B_{uv}(w) = A_{xy}(w)$ is the point where the basis of the cap $C(S, A_{xy}(z), \nu)$ touches T. Thus $B_{uv}(w) \in T$, concluding the proof of (42) and thus of (41).

3.3.3 The next aim is to establish the following proposition.

(44) Let $T(= S_{[\nu]})$ be a closed convex surface of class $\mathcal{C}^2$ with $\kappa(T,\cdot) > 0$ which has property $\mathcal{A}$. Define a closed surface U, starshaped with respect to o, by

$$U = \{\kappa(T,t)^{-1/(d+1)} t : t \in S^{d-1}\}.$$

Then U has property $\mathcal{A}$ where the affinities are volume-preserving and linear and U is convex with o in its interior.

For the proof that U has property $\mathcal{A}$ it is sufficient to prove the following:

(45) let $r, s \in S^{d-1}$ and choose (unique) $u, v \in T$ such that $r = n(T,u)$ and $s = n(T,v)$. Then there are a neighbourhood N of $\kappa(T,r)^{-1/(d+1)} r$ in U and a volume-preserving linear transformation L_{rs}^{-t} (L_{rs} is the linear part of B_{uv}) which maps $\kappa(T,r)^{-1(d+1)} r$ onto $\kappa(T,r)^{-1/(d+1)} s$ and N into U.

By the assumptions in (44) there is a neighbourhood M of u in T and a volume-preserving affinity B_{uv} mapping u onto v and M into T. The assumptions in (44) also show that T is smooth and strictly convex. Hence the exterior normal unit vectors of T at the points of M form a neighbourhood of $r = n(T,u)$ in S^{d-1}. As t ranges over this neighbourhood, the vectors $\kappa(T,t)^{-1/(d+1)} t$ form a neighbourhood N of $\kappa(T,r)^{-1/(d+1)} r$ in U. In order to show that L_{rs}^{-t} maps N into U, let $\kappa(T,t)^{-1/(d+1)} t \in N$. By the definitions of N and U there is a $w \in M$ with $t = n(T,w)$. Now apply (4):

$$\begin{aligned}
L_{rs}^{-t}(\kappa(T,t)^{-1/(d+1)} t) &= \kappa(T,t)^{-1/(d+1)} L_{rs}^{-t}(t) \\
&= \Big(\frac{\kappa(T,t)}{\|L_{rs}^{-t}(t)\|^{d+1}}\Big)^{-1/(d+1)} \frac{L_{rs}^{-t}(t)}{\|L_{rs}^{-t}(t)\|} \\
&= \kappa\Big(L_{rs}(T), \frac{L_{rs}^{-t}(t)}{\|L_{rs}^{-t}(t)\|}\Big)^{-1/(d+1)} n(L_{rs}(T), L_{rs}(w)) \\
&= \kappa(B_{uv}(T), n(B_{uv}(T), B_{uv}(w)))^{-1/(d+1)} n(B_{uv}(T), B_{uv}(w)) \\
&= \kappa(T, n(T, B_{uv}(w)))^{-1/(d+1)} n(T, B_{uv}(w)) \in U,
\end{aligned}$$

since $B_{uv}(w) \in T$. Thus $L_{rs}^{-t}(N) \subset U$. For $w = u$ we have $t = n(T, w = u) = r$ and for $v = B_{uv}(u)$ we have $n(T, v) = s$. Hence

$$L_{rs}^{-t}(\kappa(T, r)^{-1/(d+1)} r) = \kappa(T, s)^{-1/(d+1)} s.$$

Since L_{rs}^{-t} is volume-preserving, the proof of (45) is finished.

Now we show that

(46) U is convex with o in its interior.

Choose $p \in U$ having maximum distance from o. The hyperplane through p orthogonal to the vector p supports U locally at p (even globally, but we do not need this). The transformation property $\mathcal{A}$ of U (see (45)) then implies that U is supported locally at each of its points. Thus (5) shows that U is convex. By definition of U (see (44)), o is in the interior of U, concluding the proof of (46).

Having proved (45) and (46), proposition (44) follows.

Taking into account (44), the argument of 3.3.2 shows that

(47) for each sufficiently small $\nu > 0$ the floating surface $V = U_{[\nu]}$ exists, is of class $\mathcal{C}^2$ with $\kappa(V, \cdot) > 0$, has property $\mathcal{A}$ where the affinities are (volume-preserving and) linear, and o is in the interior of V.

3.3.4 This subsection is devoted to the proof of the following proposition.

(48) Let $V(= U_{[\nu]})$ be a closed convex surface of class $\mathcal{C}^2$ with $\kappa(V, \cdot) > 0$ and o in its interior which has property $\mathcal{A}$ where the affinities are (volume-preserving and) linear. Then V is an ellipsoid with centre o.

In the proof of (48) we first show that the affine distance from o,

(49) $a(V, \cdot)$ is constant on V.

Choose $x, y \in V$ and let $u = n(V, x), v = n(V, y)$. For all sufficiently small $\delta > 0$ the cap $C(V, x, \delta)$ is mapped by a suitable volume-preserving linear transformation L_{xy} onto a cap of the form $C(V, y, \varepsilon)$ of the same volume. The assumptions in (48) imply that V is smooth and strictly convex. Hence $\varepsilon = \varepsilon(\delta) \to 0$ as $\delta \to +0$. Clearly, L_{xy} maps the cone $\operatorname{conv}(\{o\} \cup B(V, x, \delta))$ onto the cone $\operatorname{conv}(\{o\} \cup B(V, y, \varepsilon))$. Therefore both cones have the same volume. This holding for all sufficiently small $\delta > 0$ together with (7) yield $a(V, u) = a(V, v)$, concluding the proof of (49).

From (8), (6), (49) and (10) it follows for the Santaló point of V,

$$\begin{aligned} \operatorname{san} V &= \int_{S^{d-1}} u h(V, u)^{-(d+1)} d\sigma(u) = \int_{S^{d-1}} u a(V, u)^{-(d+1)} \kappa(V, u)^{-1} d\sigma(u) \\ &= \text{const} \int_{S^{d-1}} \frac{u}{\kappa(V, u)} d\sigma(u) = o. \end{aligned}$$

Combining this with (49), we see that the affine distance from the Santaló point o of V is constant on V. Note that V is of class $\mathcal{C}^2$, see (47). Hence an application of Petty's characterization (9) shows that V is an ellipsoid with centre o. This concludes the proof of (48).

3.3.5 Now we make use of what has been proved in earlier subsections to show (35), i. e., S is an ellipsoid.

For sufficiently small $\nu > 0$ the floating surfaces $V = U_{[\nu]}$ exist and are ellipsoids with centre o, see (47) and (48). Clearly, they approximate U arbitrarily closely from the interior as $\nu \to +0$. Hence

(50) U is an ellipsoid with centre o.

Next it will be shown that

(51) $T (= S_{[\nu]})$ is an ellipsoid for all sufficiently small $\nu > 0$.

The floating surface $T = S_{[\nu]}$ exists for all sufficiently small $\nu > 0$, compare (39). For such ν the corresponding closed convex surface U is also an ellipsoid and its centre is o by (44) and (50). Let L^{-t} be a volume-preserving linear transformation which transforms U into a sphere with centre o. Using (4) and (44) the following relation obtains:

$$\begin{aligned} L^{-t}(U) &= \{L^{-t}(\kappa(T,t)^{-1/(d+1)}t) : t \in S^{d-1}\} \\ &= \{\kappa(T,t)^{-1/(d+1)}L^{-t}(t) : t \in S^{d-1}\} \\ &= \{(\frac{\kappa(T,t)}{\|L^{-t}(t)\|^{d+1}})^{-1/(d+1)} \frac{L^{-t}(t)}{\|L^{-t}(t)\|} : t \in S^{d-1}\} \\ &= \{\kappa(L(T),s)^{-1/(d+1)}s : s \in S^{d-1}\}, \end{aligned}$$

see [10]. Since $L^{-t}(U)$ is a sphere with centre o, this can hold only if $\kappa(L(T),\cdot)$ is constant on S^{d-1}. Now, noting (39), proposition (11) implies that $L(T)$ is a sphere. This concludes the proof of (53).

The floating surfaces $T = S_{[\nu]}$ approximate S arbitrarily closely as $\nu \to +0$. This together with (51) shows that S is an ellipsoid. Thus the proof of (35) is complete.

3.4 Having shown that S is an ellipsoid, the proof of the Theorem is complete.

Acknowledgement

In a first version of this article we proved the Theorem for $d = 2$ and for $d \geq 3$ under the additional assumption that $|\det A_{xy}| = 1$ for all $x, y \in S$. Then Dr. Daniel Hug pointed out that this assumption may easily be eliminated; see Subsection 3.3.1 for an argument similar to his. We are obliged to Dr. Hug for these most helpful comments. Many thanks are also due to Professor John Chalk for his numerous hints.

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