GENERAL AFFINE SURFACE AREAS

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Abstract

Two families of general affine surface areas are introduced. Basic properties and affine isoperimetric inequalities for these new affine surface areas as well as for $L_\phi$ affine surface areas are established.

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Finding the right notion of affine surface area was one of the first questions asked within affine differential geometry. At the beginning of the last century, Blaschke [5] and his School studied this question and introduced equi-affine surface area – a notion of surface area that is equi-affine invariant, that is, $\text{SL}(n)$ and translation invariant. The first fundamental result regarding equi-affine surface area was the classical affine isoperimetric inequality of differential geometry [5]. Numerous important results regarding equi-affine surface area were obtained in recent years (see, for example, [1,2,45,48–51]). Using valuations on convex bodies, the author and Reitzner [27] were able to characterize a much richer family of affine surface areas (see Theorem 2). Classical equi-affine and centro-affine surface area as well as all $L_p$ affine surface areas for $p > 0$ belong to this family of $L_\phi$ affine surface areas.

The present paper has two aims. The first is to establish affine isoperimetric inequalities and basic duality relations for all $L_\phi$ affine surface areas. The second aim is to define new general notions of affine surface area that complement $L_\phi$ affine surface areas and include $L_p$ affine surface areas for $p < -n$ and $-n < p < 0$. Let $\mathcal{K}_0^n$ denote the space of convex bodies, that is, compact convex sets, in $\mathbb{R}^n$ that contain the origin in their interiors. Whereas $L_\phi$ affine surface areas are always finite and are upper semicontinuous functionals on $\mathcal{K}_0^n$, the affine surface areas of the new families are infinite for certain convex bodies including polytopes and are lower semicontinuous functionals on $\mathcal{K}_0^n$. Basic properties and affine isoperimetric inequalities for these new affine surface areas are established. In Section 6,

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it is conjectured that together with $L_\phi$ affine surface areas, these new affine surface areas constitute – in a certain sense – all affine surface areas.

For a smooth convex body $K \subset \mathbb{R}^n$, equi-affine surface area is defined by

$$\Omega(K) = \int_{\partial K} \kappa_0(K,x)^{1/(n+1)} \, d\mu_K(x).$$

(1)

Here $d\mu_K(x) = x \cdot u(K,x) \, d\mathcal{H}(x)$ is the cone measure on $\partial K$, $x \cdot u$ is the standard inner product of $x,u \in \mathbb{R}^n$, $u(K,x)$ is the exterior unit normal vector to $K$ at $x \in \partial K$, $\mathcal{H}$ is the $(n-1)$-dimensional Hausdorff measure,

$$\kappa_0(K,x) = \frac{\kappa(K,x)}{(x \cdot u(K,x))^{n+1}},$$

and $\kappa(K,x)$ is the Gaussian curvature of $K$ at $x$. Note that $\kappa_0(K,x)$ is (up to a constant) just a power of the volume of the origin-centered ellipsoid osculating $K$ at $x$ and thus is an SL$(n)$ covariant notion. Also $\mu_K$ is an SL$(n)$ covariant notion. Thus $\Omega$ is easily seen to be SL$(n)$ invariant and it is also easily seen to be translation invariant. The notion of equi-affine surface area is fundamental in affine differential and convex geometry. Since many basic problems in discrete and stochastic geometry are equi-affine invariant, equi-affine surface area has found numerous applications in these fields (see, for example, [3,4,12,40]).

The extension of the definition of equi-affine surface area to general convex bodies was obtained much more recently in a series of papers [21,29,43]. Since $\kappa_0(K,\cdot)$ exists $\mu_K$ a.e. on $\partial K$ by Aleksandrov’s differentiability theorem, definition (1) still can be used. The long conjectured upper semicontinuity of equi-affine surface area (for smooth surfaces as well as for general convex surfaces) was proved by Lutwak [29] in 1991, that is,

$$\limsup_{j \to \infty} \Omega(K_j) \leq \Omega(K)$$

for any sequence of convex bodies $K_j$ converging to $K$ (in the Hausdorff metric). Let $\mathcal{K}^n$ denote the space of convex bodies in $\mathbb{R}^n$. Schütt [42] showed that $\Omega$ is a valuation on $\mathcal{K}^n$, that is,

$$\Omega(K) + \Omega(L) = \Omega(K \cup L) + \Omega(K \cap L)$$

for all $K,L \in \mathcal{K}^n$ with $K \cup L \in \mathcal{K}^n$. An equi-affine version of Hadwiger’s celebrated classification theorem [18] was established in [26]: (up to multiplication with a positive constant) equi-affine surface area is the unique upper semicontinuous, SL$(n)$ and translation invariant valuation on $\mathcal{K}^n$ that vanishes on polytopes.
During the past decade and a half, there has been an explosive growth of an $L_p$ extension of the classical Brunn Minkowski theory (see, for example, [6–8,15–17,24,25,31,34–38,46,47]). Within this theory, $L_p$ affine surface area is the notion corresponding to equi-affine surface area in the classical Brunn Minkowski theory. For $p > 1$, $L_p$ affine surface area, $\Omega_p$, was introduced by Lutwak [32] and shown to be $\text{SL}(n)$ invariant, homogeneous of degree $q = p(n - p)/(n + p)$ (that is, $\Omega_p(tK) = t^q \Omega_p(K)$ for $t > 0$), and upper semicontinuous on $\mathcal{K}_0^n$. Hug [19] defined $L_p$ affine surface area for every $p > 0$ and obtained the following representation for $K \in \mathcal{K}_0^n$:

$$\Omega_p(K) = \int_{\partial K} \kappa_0(K,x)^{\frac{p}{n+p}} d\mu_K(x).$$

(2)

Note that $\Omega_1 = \Omega$ and that $\Omega_n$ is the classical (and $\text{GL}(n)$ invariant) centro-affine surface area. Geometric interpretations of $L_p$ affine surface areas were obtained in [11, 39, 44, 52], and an application of $L_p$ affine surface areas to partial differential equations is given in [33].

The $L_p$ affine surface areas for $p > 0$ are special cases of the following family of affine surface areas introduced in [27]. Let $\text{Conc}(0,\infty)$ be the set of functions $\phi : (0,\infty) \to (0,\infty)$ such that $\phi$ is concave, $\lim_{t \to 0} \phi(t) = 0$, and $\lim_{t \to \infty} \phi(t)/t = 0$. Set $\phi(0) = 0$. For $\phi \in \text{Conc}(0,\infty)$, we define the $L_\phi$ affine surface area of $K$ by

$$\Omega_{\phi}(K) = \int_{\partial K} \phi(\kappa_0(K,x)) d\mu_K(x).$$

(3)

The following basic properties of $L_\phi$ affine surface areas were established in [27]. Let $P_0^n$ denote the set of convex polytopes containing the origin in their interiors.

**Theorem 1** ([27]). If $\phi \in \text{Conc}(0,\infty)$, then $\Omega_{\phi}(K)$ is finite for every $K \in \mathcal{K}_0^n$ and $\Omega_{\phi}(P) = 0$ for every $P \in P_0^n$. In addition, $\Omega_{\phi} : \mathcal{K}_0^n \to [0,\infty)$ is both upper semicontinuous and an $\text{SL}(n)$ invariant valuation.

The family of $L_\phi$ affine surface areas for $\phi \in \text{Conc}(0,\infty)$ is distinguished by the following basic properties (see [23] and [27], for characterizations of functionals that do not necessarily vanish on polytopes).

**Theorem 2** ([27]). If $\Phi : \mathcal{K}_0^n \to \mathbb{R}$ is an upper semicontinuous and $\text{SL}(n)$ invariant valuation that vanishes on $P_0^n$, then there exists $\phi \in \text{Conc}(0,\infty)$ such that

$$\Phi(K) = \Omega_{\phi}(K)$$

for every $K \in \mathcal{K}_0^n$. 

3
One of the most important inequalities of affine geometry is the classical affine isoperimetric inequality. The following theorem establishes affine isoperimetric inequalities for all $L_\phi$ affine surface areas. Let $\mathcal{K}_c^n$ denote the space of $K \in \mathcal{K}_c^n$ that have their centroids at the origin and let $|K|$ denote the $n$-dimensional volume of $K$.

**Theorem 3.** Let $K \in \mathcal{K}_c^n$ and $B_K \in \mathcal{K}_c^n$ be the ball such that $|B_K| = |K|$. If $\phi \in \text{Conc}(0, \infty)$, then

$$\Omega_\phi(K) \leq \Omega_\phi(B_K)$$

and there is equality for strictly increasing $\phi$ if and only if $K$ is an ellipsoid.

For $\phi(t) = t^{1/(n+1)}$ and smooth convex bodies, Theorem 3 is the classical affine isoperimetric inequality of differential geometry. For general convex bodies, proofs of the classical affine isoperimetric inequality were given by Leichtweiß [21], Lutwak [29], and Hug [19]. For $L_p$ affine surface areas, the affine isoperimetric inequality was established by Lutwak [32] for $p > 1$ and by Werner and Ye [53] for $p > 0$.

Polarity on convex bodies induces the following duality on $L_\phi$ affine surface areas. Let $K^* = \{x \in \mathbb{R}^n : x \cdot y \leq 1 \text{ for } y \in K\}$ denote the polar body of $K \in \mathcal{K}_c^n$. For $\phi \in \text{Conc}(0, \infty)$, define $\phi^*: (0, \infty) \to (0, \infty)$ by $\phi^*(s) = s \phi(1/s)$.

**Theorem 4.** If $\phi \in \text{Conc}(0, \infty)$, then $\Omega_\phi(K^*) = \Omega_{\phi^*}(K)$ holds for every $K \in \mathcal{K}_c^n$.

For $L_p$ affine surface areas and $p > 0$, Theorem 4 is due to Hug [20]: $\Omega_p(K^*) = \Omega_{n^2/p}(K)$ for every $K \in \mathcal{K}_c^n$.

An alternative definition of $L_p$ affine surface area uses integrals of the curvature function $f(K, \cdot)$ over the unit sphere $S^{n-1}$ (see [32]). This approach can also be used for $L_\phi$ affine surface areas.

**Theorem 5.** If $\phi \in \text{Conc}(0, \infty)$, then

$$\Omega_\phi(K) = \int_{S^{n-1}} \phi^*(a_0(K, u)) \, d\nu_K(u)$$

for every $K \in \mathcal{K}_c^n$.

Here $a_0(K, u) = f_{-n}(K, u) = h(K, u)^{n+1} f(K, u)$ is the $L_p$ curvature function of $K$ (see [32]) for $p = -n$, while $h(K, u)$ is the support function of $K$, and $d\nu_K(u) = d\mathcal{H}(u)/h(K, u)^n$ (see Section 1 for precise definitions). For $L_p$ affine surface areas and $p > 0$, Theorem 5 is due to Hug [19].
The family of $L_φ$ affine surface areas for $φ \in \text{Conc}(0, \infty)$ includes all $\text{SL}(n)$ invariant and upper semicontinuous valuations on $\mathcal{K}_0^n$ that vanish on polytopes and, in particular, all $L_p$ affine surface areas for $p > 0$. However, $L_p$ affine surface areas for $p < 0$ do not belong to the family of $L_φ$ affine surface areas. Recent results by Meyer and Werner [39], Schütt and Werner [44], Werner [52], and Werner and Ye [53] underline the importance of $L_p$ affine surface area also for $p < 0$.

A new family of affine surface areas generalizes $L_p$ affine surface area for $-n < p < 0$. Let $\text{Conv}(0, \infty)$ be the set of functions $ψ : (0, \infty) \to (0, \infty)$ such that $ψ$ is convex, $\lim_{t \to 0} ψ(t) = \infty$, and $\lim_{t \to \infty} ψ(t) = 0$. Set $ψ(0) = \infty$. For $ψ \in \text{Conv}(0, \infty)$, we define the $L_ψ$ affine surface area of $K$ by

$$Ω_ψ(K) = \int_{\partial K} ψ(κ_0(K, x)) \, dμ_K(x). \quad (4)$$

The following theorem establishes basic properties of $L_ψ$ affine surface areas.

**Theorem 6.** If $ψ \in \text{Conv}(0, \infty)$, then $Ω_ψ(K)$ is positive for every $K \in \mathcal{K}_0^n$ and $Ω_ψ(P) = \infty$ for every $P \in \mathcal{P}_0^n$. In addition, $Ω_ψ : \mathcal{K}_0^n \to (0, \infty]$ is both lower semicontinuous and an $\text{SL}(n)$ invariant valuation.

An immediate consequence of Theorem 6 is the following result for $L_p$ affine surface area.

**Corollary 7.** If $-n < p < 0$, then $Ω_p(K)$ is positive for every $K \in \mathcal{K}_0^n$ and $Ω_p(P) = \infty$ for every $P \in \mathcal{P}_0^n$. In addition, $Ω_p : \mathcal{K}_0^n \to (0, \infty]$ is both lower semicontinuous and an $\text{SL}(n)$ invariant valuation.

Affine isoperimetric inequalities for $L_ψ$ affine surface areas are established in

**Theorem 8.** Let $K \in \mathcal{K}_c^n$ and $B_K \in \mathcal{K}_c^n$ be the ball such that $|B_K| = |K|$. If $ψ \in \text{Conv}(0, \infty)$, then

$$Ω_ψ(K) \geq Ω_ψ(B_K)$$

and there is equality for strictly decreasing $ψ$ if and only if $K$ is an ellipsoid.

For $ψ(t) = t^{p/(n+p)}$ and $-n < p < 0$, this result was proved (in a different way) by Werner and Ye [53].

For $ψ \in \text{Conv}(0, \infty)$, define $Ω_ψ^* : \mathcal{K}_0^n \to (0, \infty]$ by $Ω_ψ^*(K) := Ω_ψ(K^*)$. The following theorem establishes basic properties of these affine surface areas.
Theorem 9. If $\psi \in \text{Conv}(0, \infty)$, then $\Omega^*_\psi(K)$ is positive for every $K \in \mathcal{K}_0^n$ and $\Omega^*_\psi(P) = \infty$ for every $P \in \mathcal{P}_0^n$. In addition, $\Omega^*_\psi : \mathcal{K}_0^n \to (0, \infty]$ is both lower semicontinuous and an $\text{SL}(n)$ invariant valuation.

The family of affine surface areas $\Omega^*_\psi$ for $\psi \in \text{Conv}(0, \infty)$ complements $L_\phi$ affine surface areas for $\phi \in \text{Conc}(0, \infty)$ and $L_\psi$ affine surface areas for $\psi \in \text{Conv}(0, \infty)$. Whereas $L_\phi$ affine surface areas for $\phi \in \text{Conc}(0, \infty)$ include affine surface areas homogeneous of degree $q$ for all $|q| < n$ and $L_\psi$ affine surface areas for $\psi \in \text{Conv}(0, \infty)$ include affine surface areas homogeneous of degree $q$ for all $q > n$, the new family includes affine surface areas homogeneous of degree $q$ for all $q < -n$.

The next theorem gives a representation of $\Omega^*_\psi$ corresponding to that of Theorem 5.

Theorem 10. If $\psi \in \text{Conv}(0, \infty)$, then

$$\Omega^*_\psi(K) = \int_{S^{n-1}} \psi(a_0(K, u)) \, d\nu_K(u)$$

for every $K \in \mathcal{K}_0^n$.

For $p < -n$, $L_p$ affine surface area was defined by Schütz and Werner [44] using (2). Here a different approach is used and a different definition of $L_p$ affine surface areas for $p < -n$ is given:

$$\Omega_p(K) := \int_{S^{n-1}} a_0(K, u)^{n/p} \, d\nu_K(u). \quad (5)$$

By Theorem 10, $\Omega_p(K) = \Omega^*_n(K) = \Omega^*_\psi(K)$ with $\psi(t) = t^{n/(n+p)}$ and $p < -n$.

An immediate consequence of Theorem 9 is the following result for $L_p$ affine surface area as defined by (5).

Corollary 11. If $p < -n$, then $\Omega_p(K)$ is positive for every $K \in \mathcal{K}_0^n$ and $\Omega_p(P) = \infty$ for every $P \in \mathcal{P}_0^n$. In addition, $\Omega_p : \mathcal{K}_0^n \to (0, \infty]$ is both lower semicontinuous and an $\text{SL}(n)$ invariant valuation.

1 Tools

Basic notions on convex bodies and their curvature measures are collected. For detailed information, see [10,13,41]. Let $K \in \mathcal{K}_0^n$. The support function of $K$ is defined for $x \in \mathbb{R}^n$ by

$$h(K, x) = \max\{x \cdot y : y \in K\}.$$
The radial function of $K$ is defined for $x \in \mathbb{R}^n$ and $x \neq 0$ by
\[
\rho(K, x) = \max\{t > 0 : tx \in K\}.
\]
Note that these definitions immediately imply that
\[
\rho(K, x) = 1 \quad \text{for} \quad x \in \partial K,
\]
and
\[
\rho(K, tu) = \frac{1}{t} \rho(K, x) \quad \text{for} \quad t > 0,
\]
where $K^*$ is the polar body of $K$.

Let $B(\mathbb{R}^n)$ denote the family of Borel sets in $\mathbb{R}^n$ and $\sigma(K, \beta)$ the spherical image of $\beta \in B(\mathbb{R}^n)$, that is, the set of all exterior unit normal vectors of $K$ at points of $\beta$. Note that $\sigma(K, \beta)$ is Lebesgue measurable for each $\beta \in B(\mathbb{R}^n)$. For a sequence of convex bodies $K_j \in \mathcal{K}_0^n$ converging to $K \in \mathcal{K}_0^n$ and a closed set $\beta \subset \mathbb{R}^n$, we have
\[
\limsup_{j \to \infty} \sigma(K_j, \beta) \subset \sigma(K, \beta).
\]
For $\beta \in B(\mathbb{R}^n)$, set
\[
C(K, \beta) = \int_{\sigma(K,\beta)} \frac{d\mathcal{H}(u)}{h(K, u)^n},
\]
where $\mathcal{H}$ denotes the $(n-1)$-dimensional Hausdorff measure. Hence $C(K, \cdot)$ is a Borel measure on $\mathbb{R}^n$ that is concentrated on $\partial K$. By (8), we obtain
\[
C(K, \partial K) = n |K^*|.
\]
It follows from (9) that for every closed set $\beta \subset \mathbb{R}^n$,
\[
\limsup_{j \to \infty} C(K_j, \beta) \leq C(K, \beta).
\]
Let $C_0(K, \cdot) : B(\mathbb{R}^n) \to [0, \infty)$ be the 0-th curvature measure of the convex body $K$ (see [41], Section 4.2). For $\beta \in B(\mathbb{R}^n)$, we have
\[
C_0(K, \beta) = \mathcal{H}(\sigma(K, \beta)).
\]
We decompose the measure $C_0(K, \cdot)$ into measures absolutely continuous and singular with respect to $\mathcal{H}$, say, $C_0(K, \cdot) = C_0^a(K, \cdot) + C_0^s(K, \cdot)$. Note that
\[
\frac{dC_0^a(K, \cdot)}{d\mathcal{H}} = \kappa(K, \cdot).
\]
Let \( \text{reg } K \) denote the set of regular boundary points of \( K \), that is, boundary points with a unique exterior unit normal vector. From (12), we obtain for \( \omega \subset \text{reg } K \) and \( \omega \in \mathcal{B}(\mathbb{R}^n) \),

\[
C(K, \omega) = \int_{\partial(K, \omega)} \frac{d\mathcal{H}(u)}{h(K, u)^n} = \int_{\omega} \frac{dC_0(K, x)}{(x \cdot u(K, x))^n}.
\] (14)

We decompose the measure \( C(K, \cdot) \) into measures absolutely continuous and singular with respect to the measure \( \mu_K \), say, \( C(K, \cdot) = C^a(K, \cdot) + C^s(K, \cdot) \). The singular part is concentrated on a \( \mu_K \) null set \( \omega_0 \subset \partial K \), that is, for \( \beta \in \mathcal{B}(\mathbb{R}^n) \)

\[
C^s(K, \beta \setminus \omega_0) = 0.
\] (15)

Since \( C^a(K, \cdot) \) is concentrated on \( \text{reg } K \), (13) and (14) imply for \( \omega \subset \partial K \) and \( \omega \in \mathcal{B}(\mathbb{R}^n) \),

\[
C^a(K, \omega) = \int_{\omega} \frac{\kappa(K, x)}{(x \cdot u(K, x))^n} d\mathcal{H}(x) = \int_{\omega} \kappa_0(K, x) d\mu_K(x).
\] (16)

Combined with (10), this implies

\[
\int_{\partial K} \kappa_0(K, x) d\mu_K(x) \leq n |K^*|.
\] (17)

Hug [20] proved that for almost all \( x \in \partial K \),

\[
\kappa(K, x) = \left( \frac{x}{|x|} \cdot u_K(x) \right)^{n+1} f(K^*, \frac{x}{|x|}).
\]

Hence we have for almost all \( y \in \partial K^* \),

\[
\kappa_0(K^*, y) = a_0(K, \frac{y}{|y|}).
\] (18)

Here \(|x|\) denotes the length of \( x \).
2 Proof of Theorems 3 and 8

Let \( \phi \in \text{Conc}(0, \infty) \) and \( K \in \mathcal{K}_c^n \). By definition (3), Jensen’s inequality, (17), and the monotonicity of \( \phi \), we obtain

\[
\Omega_\phi(K) = \int_{\partial K} \phi(\kappa_0(K, x)) \, d\mu_K(x)
\leq n |K| \phi\left(\frac{1}{n |K|} \int_{\partial K} \kappa_0(K, x) \, d\mu_K(x)\right)
\leq n |K| \phi\left(\frac{|K^*|}{|K|}\right).
\]

For origin-centered ellipsoids, \( \kappa_0(K, \cdot) \) is constant and there is equality in the above inequalities. Now we use the Blaschke-Santaló inequality: for \( K \in \mathcal{K}_c^n \)
\[
|K| |K^*| \leq |B^n|^2
\]
with equality precisely for origin-centered ellipsoids (see, for example, [28]). Here \( B^n \) is the unit ball in \( \mathbb{R}^n \). We obtain

\[
\Omega_\phi(K) \leq n |K| \phi\left(\frac{|K^*|}{|K|}\right) \leq n |K| \phi\left(\frac{|B^n|^2}{|K|^2}\right) = \Omega_\phi(B_K). \tag{19}
\]

For \( \phi \) strictly increasing, equality in the second inequality of (19) holds if and only if there is equality in the Blaschke-Santaló inequality, that is, precisely for ellipsoids. This completes the proof of Theorem 3 and the proof of Theorem 8 follows along similar lines.

3 Proof of Theorems 4 and 9

Define \( \Omega_\phi^* \) on \( \mathcal{K}_0^n \) by \( \Omega_\phi^*(K) := \Omega_\phi(K^*) \). Since \( \Omega_\phi \) is upper semicontinuous, so is \( \Omega_\phi^* \). For \( K, L, K \cup L \in \mathcal{K}_0^n \), we have

\[
(K \cup L)^* = K^* \cap L^* \quad \text{and} \quad (K \cap L)^* = K^* \cup L^*.
\]

Since \( \Omega_\phi \) is a valuation, this implies that

\[
\Omega_\phi^*(K) + \Omega_\phi^*(L) = \Omega_\phi(K^*) + \Omega_\phi(L^*)
= \Omega_\phi((K \cup L)^*) + \Omega_\phi((K \cap L)^*)
= \Omega_\phi^*(K \cap L) + \Omega_\phi^*(K \cup L).
\]

9
that is, $\Omega^*_\phi$ is a valuation on $K^n_0$. For $A \in SL(n)$ and $K \in K^n_0$, we have $(AK)^* = A^{-t} K^*$, where $A^{-t}$ denotes the inverse of the transpose of $A$. Since $\Omega_\phi$ is SL($n$) invariant, this implies $\Omega^*_\phi(AK) = \Omega^*_\phi(K)$, that is, $\Omega^*_\phi : K^n_0 \to \mathbb{R}$ is SL($n$) invariant. Since $\Omega_\phi$ vanishes on polytopes, so does $\Omega^*_\phi$. Therefore $\Omega^*_\phi$ satisfies the assumptions of Theorem 2. Thus there exists $\alpha \in \text{Conc}(0, \infty)$ such that $\Omega^*_\phi = \Omega^* \alpha$. Let $B^n$ denote the unit ball in $\mathbb{R}^n$. For $r > 0$, we obtain from (3) that

$$\Omega_\alpha(rB^n) = n |B^n| r^n \alpha\left(\frac{1}{r^{2n}}\right)$$

and

$$\Omega^*_\phi(rB^n) = \Omega_\phi\left(\frac{1}{r}B^n\right) = \frac{n |B^n|}{r^n} \phi(\frac{r^{2n}}{r^n}).$$

This shows that $\alpha = \phi^*$ and completes the proof of Theorem 4. The proof of Theorem 9 follows along the lines of the proof that $\Omega^*_\phi$ satisfies the assumptions of Theorem 2.

4 Proof of Theorems 5 and 10

Define $y : S^{n-1} \to \partial K^*$ by $u \mapsto \rho(K^*, u)$ $u$. Note that this is a Lipschitz function. For the Jacobian $Jy$ of $y$, we have a.e. on $S^{n-1}$,

$$Jy(u) = \frac{\rho(K^*, u)^{n-1}}{u \cdot u_{K^*}(\rho(K^*, u) u)}$$

(see, for example, [20]). By the area formula (see, for example, [9]), we have for every a.e. defined function $g : S^{n-1} \to [0, \infty]$,

$$\int_{S^{n-1}} g(u) Jy(u) d\mathcal{H}(u) = \int_{\partial K^*} g\left(\frac{y}{|y|}\right) d\mathcal{H}(y).$$

Setting

$$g(u) = \frac{\tau(a_0(K, u))}{h(K, u)^n Jy(u)}$$

for $\tau : [0, \infty] \to [0, \infty]$, we get by (6), (7), (8), and (18),

$$\int_{S^{n-1}} \tau(a_0(K, u)) d\nu_K(u) = \int_{S^{n-1}} \tau(a_0(K, u)) \frac{d\mathcal{H}(u)}{h(K, u)^n}
= \int_{\partial K^*} \tau(\kappa_0(K^*, y)) \frac{\frac{y}{|y|} \cdot u_{K^*}(y)}{\rho(K^*, \frac{y}{|y|})^{n-1}} \rho(K^*, \frac{y}{|y|})^n d\mathcal{H}(y)
= \int_{\partial K^*} \tau(\kappa_0(K^*, y)) d\mu_{K^*}(y).$$

10
For $\tau \in \text{Conv}(0, \infty)$, this implies Theorem 10. To obtain Theorem 5, we set $\tau = \phi_* \in \text{Conc}(0, \infty)$ and apply Theorem 4.

5 Proof of Theorem 6

Let $\psi \in \text{Conv}(0, \infty)$ and $K \in \mathcal{K}_0^n$. Note that $\psi$ is strictly decreasing and positive. By definition (4), the Jensen inequality, (17), and the monotonicity of $\psi$, we obtain

$$
\Omega_\psi(K) = \int_{\partial K} \psi(\kappa_0(K, x)) \, d\mu_K(x)
\geq n |K| \psi\left(\frac{1}{n |K|} \int_{\partial K} \kappa_0(K, x) \, d\mu_K(x)\right)
\geq n |K| \psi\left(\frac{|K*|}{|K|}\right).
$$

This shows that $\Omega_\psi(K) > 0$. The $\text{SL}(n)$ invariance of $\Omega_\psi$ follows immediately from the definition. So does the fact that $\Omega_\psi(P) = \infty$ for $P \in \mathcal{P}_0^n$.

Next, we show that $\Omega_\psi$ is a valuation on $\mathcal{K}_0^n$, that is, for $K, L \in \mathcal{K}_0^n$ such that $K \cup L \in \mathcal{K}_0^n$,

$$
\Omega_\psi(K \cup L) + \Omega_\psi(K \cap L) = \Omega_\psi(K) + \Omega_\psi(L). \quad (21)
$$

Let $K^c = \{x \in \mathbb{R}^n : x \notin K\}$ and let int $K$ denote the interior of $K$. We follow Schütt [42] (see also [14]) and work with the decompositions

$$
\partial(K \cup L) = (\partial K \cap \partial L) \cup (\partial K \cap L^c) \cup (\partial L \cap K^c),
\partial(K \cap L) = (\partial K \cap \partial L) \cup (\partial K \cap \text{int} \, L) \cup (\partial L \cap \text{int} \, K),
\partial K = (\partial K \cap \partial L) \cup (\partial K \cap L^c) \cup (\partial K \cap \text{int} \, L),
\partial L = (\partial K \cap \partial L) \cup (\partial L \cap K^c) \cup (\partial L \cap \text{int} \, K),
$$

where all unions on the right hand side are disjoint. Note that for $x$ such that the curvatures $\kappa_0(K, x), \kappa_0(L, x), \kappa_0(K \cup L, x), \kappa_0(K \cap L, x)$ exist,

$$
u(K, x) = u(L, x) = u(K \cup L, x) = u(K \cap L, x) \quad (22)
$$

and

$$
\kappa_0(K \cup L, x) = \min\{\kappa_0(K, x), \kappa_0(L, x)\},
\kappa_0(K \cap L, x) = \max\{\kappa_0(K, x), \kappa_0(L, x)\}. \quad (23)
$$

To prove (21), we use (4), split the involved integrals using the above decompositions, and use (22) and (23).
Finally, we show that $\Omega_\psi$ is lower semicontinuous on $K^n_0$. The proof complements the proofs in [22] and [30]. Let $K \in K^n_0$ and $\varepsilon > 0$ be chosen. Since $\kappa_0(K, \cdot)$ is measurable a.e. on $\partial K$ and since the set $\omega_0$, where the singular part of $C(K, \cdot)$ is concentrated, is a $\mu_K$ null set, we can choose by Lusin’s theorem (see, for example, [9]) pairwise disjoint closed sets $\omega_l \subset \partial K$, $l \in \mathbb{N}$, such that $\kappa_0(K, \cdot)$ is continuous as a function restricted to $\omega_l$, such that for every $l \in \mathbb{N}$,

$$\omega_l \cap \omega_0 = \emptyset \quad (24)$$

and such that

$$\mu_K \left( \bigcup_{l=1}^\infty \omega_l \right) = \mu_K(\partial K). \quad (25)$$

For $\omega \subset \mathbb{R}^n$, let $\tilde{\omega}$ be the cone generated by $\omega$, that is, $\tilde{\omega} = \{tx \in \mathbb{R}^n : t \geq 0, x \in \omega \}$. Note that $\tilde{\omega}$ is closed and that $\partial K \cap \tilde{\omega} = \tilde{\omega}$.

Let $K_j$ be a sequence of convex bodies converging to $K$. First, we show that for $l \in \mathbb{N}$,

$$\liminf_{j \to \infty} \int_{\partial K_j \cap \tilde{\omega}_l} \psi(\kappa_0(K_j, x)) \, d\mu_{K_j}(x) \geq \int_{\partial K \cap \tilde{\omega}_l} \psi(\kappa_0(K, x)) \, d\mu_{K}(x). \quad (26)$$

Let $\eta > 0$ be chosen. We choose a monotone sequence $t_i \in (0, \infty)$, $i \in \mathbb{Z}$, $\lim_{i \to -\infty} t_i = 0$, $\lim_{i \to \infty} t_i = \infty$, such that

$$\max_{i \in \mathbb{Z}} |\psi(t_{i+1}) - \psi(t_i)| \leq \eta \quad (27)$$

and such that for $i \in \mathbb{Z}$, $j \geq 0$,

$$\mu_{K_j}(\{x \in \partial K_j : \kappa_0(K_j, x) = t_i\}) = 0, \quad (28)$$

where $K_0 = K$. This is possible, since $\mu_{K_j}(\{x \in K_j : \kappa_0(K_j, x) = t\}) > 0$ holds only for countably many $t$. Set

$$\omega_{li} = \{x \in \omega_l : t_i \leq \kappa_0(K, x) \leq t_{i+1}\}.$$ 

Since $\kappa_0(K, \cdot)$ is continuous on $\omega_l$ and $\omega_l$ is closed, the sets $\tilde{\omega}_{li}$ are closed for $i \in \mathbb{Z}$. This implies by (11) that

$$\limsup_{j \to \infty} C(K_j, \tilde{\omega}_{li}) \leq C(K, \tilde{\omega}_{li}). \quad (29)$$

By (24), (15), and the definition of $\omega_{li}$,

$$C(K, \tilde{\omega}_{li}) = C^a(K, \tilde{\omega}_{li}) \leq t_{i+1} \mu_K(\partial K \cap \tilde{\omega}_{li}). \quad (30)$$
By (16),
\[
\int_{\partial K_j \cap \omega_l} \kappa_0(K_j, x) \, d\mu_{K_j}(x) \leq C(K_j, \bar{\omega}_l).
\]
(31)

Using the monotonicity of $\psi$, we obtain
\[
\int_{\omega_l} \psi(\kappa_0(K, x)) \, d\mu_K(x) \leq \sum_{i \in \mathbb{Z}} \int_{\omega_i} \psi(\kappa_0(K, x)) \, d\mu_K(x) \leq \sum_{i \in \mathbb{Z}} \psi(t_i) \mu_K(\omega_i).
\]
(32)

Using (28), the Jensen inequality, (31), and the monotonicity of $\psi$, we obtain
\[
\int_{\partial K_j \cap \omega_l} \psi(\kappa_0(K_j, x)) \, d\mu_{K_j}(x) = \sum_{i \in \mathbb{Z}}' \int_{\partial K_j \cap \omega_l} \psi(\kappa_0(K_j, x)) \, d\mu_{K_j}(x)
\geq \sum_{i \in \mathbb{Z}}' \psi \left( \frac{C(K_j, \bar{\omega}_l)}{\mu_{K_j}(\partial K_j \cap \bar{\omega}_l)} \right) \mu_{K_j}(\partial K_j \cap \bar{\omega}_l)
\]
where the $'$ indicates that we sum only over $\bar{\omega}_l$ with $\mu_{K_j}(\partial K_j \cap \bar{\omega}_l) \neq 0$.

Since
\[
\liminf_{j \to \infty} \sum_{i \in \mathbb{Z}}' \psi \left( \frac{C(K_j, \bar{\omega}_l)}{\mu_{K_j}(\partial K_j \cap \bar{\omega}_l)} \right) \mu_{K_j}(\partial K_j \cap \bar{\omega}_l)
\geq \sum_{i \in \mathbb{Z}}' \psi \left( \limsup_{j \to \infty} \left( \frac{C(K_j, \bar{\omega}_l)}{\mu_{K_j}(\partial K_j \cap \bar{\omega}_l)} \right) \right) \liminf_{j \to \infty} \mu_{K_j}(\partial K_j \cap \bar{\omega}_l),
\]
we obtain by (29), (30), (32), (27), and (28) that
\[
\liminf_{j \to \infty} \int_{\partial K_j \cap \omega_l} \psi(\kappa_0(K_j, x)) \, d\mu_{K_j}(x)
\geq \sum_{i \in \mathbb{Z}}' \psi \left( \frac{C(K, \bar{\omega}_l)}{\mu_K(\partial K \cap \bar{\omega}_l)} \right) \mu_K(\partial K \cap \bar{\omega}_l)
\geq \sum_{i \in \mathbb{Z}} \psi(t_{i+1}) \mu_K(\partial K \cap \bar{\omega}_l)
\geq \sum_{i \in \mathbb{Z}} \psi(t_{i+1}) \mu_K(\partial K \cap \bar{\omega}_l) - \sum_{i \in \mathbb{Z}} (\psi(t_i) - \psi(t_{i+1})) \mu_K(\partial K \cap \bar{\omega}_l)
\geq \int_{\partial K \cap \omega_l} \psi(\kappa_0(K, x)) \, d\mu_K(x) - \eta \mu_K(\partial K \cap \bar{\omega}_l).
Since \( \eta > 0 \) is arbitrary, this proves (26).

Finally, (28) and (26) imply

\[
\lim inf_{j \to \infty} \int_{\partial K_j} \psi(\kappa_0(K_j, x)) \, d\mu_{K_j}(x) = \lim inf_{j \to \infty} \sum_{l=1}^{\infty} \int_{\partial K_j \cap \bar{\omega}_l} \psi(\kappa_0(K_j, x)) \, d\mu_{K_j}(x) \\
\geq \sum_{l=1}^{\infty} \lim inf_{j \to \infty} \int_{\partial K_j \cap \bar{\omega}_l} \psi(\kappa_0(K_j, x)) \, d\mu_{K_j}(x) \\
\geq \int_{\partial K} \psi(\kappa_0(K, x)) \, d\mu_K(x).
\]

This completes the proof of the theorem.

6 Open problems

The affine surface areas \( \Omega_\psi \) and \( \Omega^*_{\psi} \) for \( \psi \in \text{Conv}(0, \infty) \) are lower semicontinuous and \( \text{SL}(n) \) invariant valuations. More general examples of such functionals are

\[
\Psi = \Omega_{\psi_1} + \Omega^*_{\psi_2} - \Omega_{\phi}
\]

for \( \psi_1, \psi_2 \in \text{Conv}(0, \infty) \) and \( \phi \in \text{Conc}(0, \infty) \). Additional examples are the following continuous functionals

\[
K \mapsto c_0 + c_1 |K| + c_2 |K^*|
\]

for \( c_0, c_1, c_2 \in \mathbb{R} \). In view of Theorem 2, this gives rise to the following

Conjecture 1. If \( \Psi : \mathcal{K}_0^n \to (-\infty, \infty] \) is a lower semicontinuous and \( \text{SL}(n) \) invariant valuation, then there exist \( \psi_1, \psi_2 \in \text{Conv}(0, \infty), \phi \in \text{Conc}(0, \infty) \), and \( c_0, c_1, c_2 \in \mathbb{R} \) such that

\[
\Psi(K) = c_0 + c_1 |K| + c_2 |K^*| + \Omega_{\psi_1}(K) + \Omega^*_{\psi_2}(K) - \Omega_{\phi}(K)
\]

for every \( K \in \mathcal{K}_0^n \).

The following special case of the above conjecture is of particular interest.

Conjecture 2. If \( \Psi : \mathcal{K}_0^n \to (-\infty, \infty] \) is a lower semicontinuous and \( \text{SL}(n) \) invariant valuation that is homogeneous of degree \( q < -n \) or \( q > n \), then there exists \( c \geq 0 \) such that

\[
\Psi(K) = c \Omega_p(K)
\]

for every \( K \in \mathcal{K}_0^n \), where \( p = n(n - q)/(n + q) \).
References


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