Valuations on Function Spaces

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Dedicated to Peter M. Gruber on the occasion of his 70th birthday

Abstract

A short survey is given on classification results for valuations on function spaces. Real valued, matrix valued and convex body valued valuations on Lebesgue spaces and on Sobolev spaces are considered.

Key words: Valuation, Lebesgue space, Sobolev space, LYZ operator, Fisher information matrix

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A function Z defined on a lattice $(\mathcal{L}, \vee, \wedge)$ and taking values in an Abelian semigroup is called a *valuation* if

$$\mathrm{Z}(f \vee g) + \mathrm{Z}(f \wedge g) = \mathrm{Z}(f) + \mathrm{Z}(g) \tag{1}$$

for all $f, g \in \mathcal{L}$ (see, for example, [7]). A function Z defined on a subset $\mathcal{S}$ of the set $\mathcal{L}$ is called a valuation on $\mathcal{S}$ if (1) holds whenever $f, g, f \vee g, f \wedge g \in \mathcal{S}$.

The classical case are valuations on convex bodies (compact convex sets) in $\mathbb{R}^n$. Here valuations are defined on $\mathcal{K}^n$, the space of convex bodies in $\mathbb{R}^n$, which is equipped with the topology coming from the Hausdorff metric. The operations $\vee$ and $\wedge$ are the usual union and intersection. Results on valuations on convex polytopes start with Dehn's solution of Hilbert's Third Problem in 1901. In the 1950s, a systematic study of valuations was initiated by Hadwiger, who was in particular interested in classifying valuations on $\mathcal{K}^n$. Probably the most famous result on valuations is the Hadwiger characterization theorem.

Theorem 1 (Hadwiger [29]). *A functional* $\mathrm{Z} : \mathcal{K}^n \to \mathbb{R}$ *is a continuous and rigid motion invariant valuation if and only if there are constants* c_0, $c_1, \ldots, c_n \in \mathbb{R}$ *such that*

$$\mathrm{Z}(K) = c_0 \, V_0(K) + \cdots + c_n \, V_n(K)$$

for every $K \in \mathcal{K}^n$.

Here $V_0(K), \ldots, V_n(K)$ are the intrinsic volumes of $K \in \mathcal{K}^n$. In particular, $V_0(K)$ is the Euler characteristic (that is, $V_0(K) = 1$ for $K \neq \varnothing$ and $V_0(\varnothing) = 0$), $2V_{n-1}(K)$ is the surface area and $V_n(K)$ the volume of K. Hadwiger's result was the starting point for many investigations dealing with characterizations and precise descriptions of classes of valuations having interesting invariance properties (see [29, 35, 59, 60] for more information).

We remark that $(\mathcal{K}^n, \cup, \cap)$ is not a lattice but that there is a natural lattice structure on the whole of $\mathcal{K}^n$, when $\wedge$ is defined as intersection and $\vee$ as convex hull of the union. Gruber [19] established a classification of the endomorphisms of this lattice (see also [20]). Let $\mathcal{K}_0^n$ denote the space of convex bodies in $\mathbb{R}^n$ that contain the origin in their interiors. The lattice $(\mathcal{K}_0^n, \vee, \wedge)$ was studied by Böröczky & Schneider [9], where the main focus is again on the classification of endomorphisms and a characterization of polarity is obtained (see also [4, 5]). However, note that these endomorphisms are not valuations on $(\mathcal{K}_0^n, \vee, \wedge)$ in the sense of definition (1).

For a space of real valued functions, we define the operations $\vee$ and $\wedge$ as pointwise maximum and minimum, respectively. When we identify a convex body K in $\mathbb{R}^n$ with its indicator function $\mathbb{1}_K$ or its support function h_K (where $h_K(u) = h(K, u) = \max\{u \cdot x : x \in K\}$ and $u \cdot x$ is the standard inner product of $u, x \in \mathbb{R}^n$), we see that valuations on convex bodies can be considered as valuations on suitable function spaces.

In this short survey, results on valuations on some of the standard function spaces are collected. An emphasis is put on classification theorems of invariant real valued valuations and of matrix valued and convex body valued valuations that are compatible with the action of some transformation group.

1 Valuations on Lebesgue Spaces

For a space X with measure μ, define $L^p(X, \mu)$ as the space of measurable functions $f : X \to \mathbb{R}$ such that

$$\|f\|_p = \left(\int_X |f(x)|^p \, d\mu(x) \right)^{1/p} < \infty.$$

We say that $f_j \to f$ in $L^p(X, \mu)$, if $\|f_j - f\|_p \to 0$. Let $L^p(\mathbb{R}^n)$ denote the L^p space with respect to the Lebesgue measure dx and $L^p(\mathbb{S}^{n-1})$ the L^p space on the unit sphere $\mathbb{S}^{n-1}$ with respect to the spherical Lebesgue measure.

1.1 Real valued valuations

Andy Tsang [69] obtained classification results for valuations on $L^p(X, \mu)$ for a non-atomic measure space X. Here we state some of the consequences of his results. Let $p \geqslant 1$.

Theorem 2 (Tsang [69]). *A functional* $Z : L^p(\mathbb{R}^n) \to \mathbb{R}$ *is a continuous and translation invariant valuation if and only if there exists a continuous function* $\zeta : \mathbb{R} \to \mathbb{R}$ *with the property that* $|\zeta(t)| \leqslant \gamma |t|^p$ *for all* $t \in \mathbb{R}$ *for some* $\gamma \geqslant 0$ *such that*

$$Z(f) = \int_{\mathbb{R}^n} (\zeta \circ f)(x) \, dx$$

for every $f \in L^p(\mathbb{R}^n)$.

Hassane Kone [36] has recently extended the above theorem to Orlicz spaces.

Theorem 3 (Tsang [69]). *A functional* $Z : L^p(\mathbb{S}^{n-1}) \to \mathbb{R}$ *is a continuous and rotation invariant valuation if and only if there exists a continuous function* $\zeta : \mathbb{R} \to \mathbb{R}$ *with the properties that* $\zeta(0) = 0$ *and* $|\zeta(t)| \leqslant \delta + \gamma |t|^p$ *for all* $t \in \mathbb{R}$ *for some* $\delta, \gamma \geqslant 0$ *such that*

$$Z(f) = \int_{\mathbb{S}^{n-1}} (\zeta \circ f)(u) \, du$$

for every $f \in L^p(\mathbb{S}^{n-1})$.

Let $\mathcal{S}^p(\mathbb{R}^n)$ be the space of sets $S \subset \mathbb{R}^n$ which are star shaped with respect to the origin and whose radial function

$$\rho_S(u) = \rho(S, u) = \max\{\lambda \geqslant 0 : \lambda u \in S\}$$

is in $L^p(\mathbb{S}^{n-1})$. These sets are called L^p stars. The space $\mathcal{S}^p(\mathbb{R}^n)$ is equipped with the topology so that $S_j \to S$ if $\|\rho_{S_j} - \rho_S\|_p \to 0$. In [33, 34], Dan Klain obtained classification results for valuations on L^n stars. The operations $\vee$ and $\wedge$ are union and intersection, respectively, and hence correspond to the pointwise maximum and minimum for radial functions. Here we state one of Klain's results that can be obtained as a simple variation of Theorem 3 by restricting Tsang's result to non-negative functions.

Theorem 4 (Klain [34]). *A functional* $Z : \mathcal{S}^n(\mathbb{R}^n) \to \mathbb{R}$ *is a continuous and rotation invariant valuation if and only if there exists a continuous function* $\zeta : [0, \infty) \to \mathbb{R}$ *with the properties that* $\zeta(0) = 0$ *and* $|\zeta(t)| \leqslant \delta + \gamma |t|^n$ *for all* $t \in \mathbb{R}$ *for some* $\delta, \gamma \geqslant 0$ *such that*

$$Z(S) = \int_{\mathbb{S}^{n-1}} (\zeta \circ \rho_S)(u) \, du$$

for every $S \in \mathcal{S}^n(\mathbb{R}^n)$.

If the valuation Z in Theorem 4 is in addition positively homogeneous of degree p (that is, $Z(rS) = r^p Z(S)$ for all $r > 0$ and $S \in \mathcal{S}^n(\mathbb{R}^n)$), then

$\zeta(t) = c \, t^p$ with $c \in \mathbb{R}$ and $0 \leqslant p \leqslant n$ and hence Z is a dual mixed volume (as defined by Lutwak [46]). The theory of dual mixed volumes is also called dual Brunn Minkowski theory and the theory of mixed (and intrinsic) volumes is called Brunn Minkowski theory. Hence the Hadwiger theorem belongs to the Brunn Minkowski theory and Theorem 4 is its analogue within the dual Brunn Minkowski theory.

1.2 Matrix valued valuations

On convex bodies also vector and tensor valued valuations attracted a lot of interest (see, for example, [2, 3, 30, 31, 37, 39]). Here we state a result for matrix valued valuations on the space $L^2(\mathbb{R}^n, |x|^2 dx)$ of measurable functions $f : \mathbb{R}^n \to \mathbb{R}$ with finite second moments, where $|x|$ is the Euclidean norm of $x \in \mathbb{R}^n$. For a measurable function $f : \mathbb{R}^n \to \mathbb{R}$, the *moment matrix*, $\mathrm{K}(f)$, is the $n \times n$ matrix with (not necessarily finite) entries,

$$\mathrm{K}_{ij}(f) = \int_{\mathbb{R}^n} f(x) \, x_i \, x_j \, dx.$$

If f is a probability density with mean zero, then $\mathrm{K}(f)$ is the covariance matrix of f. Let $\mathbb{M}^n$ denote the space of real symmetric $n \times n$ matrices. An operator $\mathrm{Z} : L^2(\mathbb{R}^n, |x|^2 dx) \to \mathbb{M}^n$ is called $\mathrm{SL}(n)$ covariant if

$$\mathrm{Z}(f \circ \phi^{-1}) = \phi \, \mathrm{Z}(f) \, \phi^t$$

for all $f \in L^2(\mathbb{R}^n, |x|^2 dx)$ and $\phi \in \mathrm{SL}(n)$.

Theorem 5 (Ludwig [45]). *An operator* $\mathrm{Z} : L^2(\mathbb{R}^n, |x|^2 dx) \to \mathbb{M}^n$ *is a continuous and* $\mathrm{SL}(n)$ *covariant valuation if and only if there exists a continuous function* $\zeta : \mathbb{R} \to \mathbb{R}$ *with the property that* $|\zeta(t)| \leqslant \gamma \, |t|$ *for all* $t \in \mathbb{R}$ *for some* $\gamma \geqslant 0$ *such that*

$$\mathrm{Z}(f) = \mathrm{K}(\zeta \circ f)$$

for every $f \in L^2(\mathbb{R}^n, |x|^2 dx)$.

The proof makes use of a classification of matrix valued valuations on convex bodies which in turn uses ideas of Haberl [24].

1.3 Convex body valued valuations

Convex body valued and star body valued valuations on $\mathcal{K}^n$ and $\mathcal{K}_0^n$ have become more and more important. See [1, 3, 22–25, 38, 41, 61, 62, 64–66, 73] for some of the recent results.

The most important convex body valued valuations on convex bodies are the projection operator $\Pi : \mathcal{K}^n \to \mathcal{K}^n$ and the moment operator $\mathrm{M} : \mathcal{K}_0^n \to \mathcal{K}^n$. Both are so called *Minkowski valuations*, that is, they are valuations with values in the Abelian semi-group $\langle \mathcal{K}^n, + \rangle$, where + stands

for Minkowski (or vector) addition (that is, $K + L = \{x + y : x \in K, y \in L\}$).
We remark that besides Minkowski addition, also the so called L^p Minkowski
addition and L^p Minkowski valuations have turned out to be very impor-
tant (see, for example, [10,11,27,40,48–50,54,56,57,67,68]). Since there are
no results yet on L^p Minkowski valuations on function spaces, we will not
consider them in this survey.

The *projection body*, ΠK, of a convex body K in $\mathbb{R}^n$ is given by

$$h(\Pi K, u) = V_{n-1}(K|u^\perp) \quad \text{for} \quad u \in \mathbb{S}^{n-1},$$

where V_{n-1} denotes $(n-1)$-dimensional volume and $K|u^\perp$ the image of the
orthogonal projection of K onto the subspace orthogonal to u. Projection
bodies were introduced by Minkowski at the turn of the last century and
have proved to be very useful in many ways and subjects (cf. [17]).

The *moment body*, $\mathrm{M}\,K$, of a convex body $K \in \mathcal{K}_0^n$ is given by

$$h(\mathrm{M}\,K, u) = \int_K |u \cdot x| \, dx = \frac{1}{n+1} \int_{\mathbb{S}^{n-1}} |u \cdot v| \, \rho_K^{n+1}(v) \, dv. \qquad (2)$$

When divided by the volume of K, the moment body of K is called *centroid
body* and is a classical and important notion going back to Dupin (see [17,
47]).

Tsang [70] studied convex body valued valuations on function spaces.
Here the natural space to consider is the space $L^1(\mathbb{R}^n, |x| \, dx)$ of measurable
functions $f : \mathbb{R}^n \to \mathbb{R}$ of finite first moments. We give a special case of
Tsang's classification theorem with modified notation.

Let $\mathcal{S}_c^p(\mathbb{R}^n)$ denote the space of origin-symmetric L^p stars in $\mathbb{R}^n$. For a
function $f \in L^1(\mathbb{R}^n, |x| \, dx)$, define the origin-symmetric and star-shaped set
$\{f\}_{n+1}$ by

$$\rho^{n+1}(\{f\}_{n+1}, u) = \frac{1}{2} \int_{-\infty}^{\infty} |f(ru)| \, d|r|^{n+1}, \qquad (3)$$

where $d|r|^{n+1} = (n+1)|r|^n d|r|$. Note that

$$\frac{1}{n+1} \int_{\mathbb{R}^n} |f(x)| \, |x| \, dx = \frac{1}{2} \int_{\mathbb{S}^{n-1}} \int_{-\infty}^{\infty} |f(ru)| \, |r|^n \, d|r| \, du = \int_{\mathbb{S}^{n-1}} \rho^{n+1}(\{f\}_{n+1}, u) \, du.$$

Thus $\{f\}_{n+1} \in \mathcal{S}_c^{n+1}(\mathbb{R}^n)$. The operator $f \mapsto \{f\}_{n+1}$ and its generalizations
have turned out to be important for many results (see, for example, [6,8,14,
18,32,55]).

Let $\mathcal{K}_c^n$ denote the space of origin-symmetric convex bodies in $\mathbb{R}^n$. An
operator $\mathrm{Z} : L^1(\mathbb{R}^n, |x| \, dx) \to \mathcal{K}_c^n$ is called $\mathrm{GL}(n)$ covariant of weight q if

$$\mathrm{Z}(f \circ \phi^{-1}) = |\det \phi|^q \, \phi \, \mathrm{Z}(f) \qquad (4)$$

for all $f \in L^1(\mathbb{R}^n, |x| \, dx)$. It is called $\mathrm{GL}(n)$ covariant if it is $\mathrm{GL}(n)$ covariant
of some weight $q \in \mathbb{R}$.

Theorem 6 (Tsang [70]). *An operator* $Z : L^1(\mathbb{R}^n, |x|\, dx) \to \mathcal{K}_c^n$ *is a continuous Minkowski valuation which is* $\mathrm{GL}(n)$ *covariant of weight 1 if and only if there exists a continuous function* $\zeta : \mathbb{R} \to \mathbb{R}$ *with the property that* $0 \leqslant \zeta(t) \leqslant \gamma|t|$ *for all* $t \in \mathbb{R}$ *and some* $\gamma \geqslant 0$ *such that*

$$Z(f) = \mathrm{M}\left\{\zeta \circ f\right\}_{n+1}$$

for every $f \in L^1(\mathbb{R}^n, |x|\, dx)$.

Here the operator M is extended from $\mathcal{K}_0^n$ to $\mathcal{S}^{n+1}(\mathbb{R}^n)$ using the right hand side of (2), that is, for $u \in \mathbb{S}^{n-1}$

$$h(\mathrm{M}\left\{\zeta \circ f\right\}_{n+1}, u) = \int_{\mathbb{R}^n} |u \cdot x|\, (\zeta \circ f)(x)\, dx.$$

The proof of Theorem 6 makes use of a classification of Minkowski valuations on the space of convex bodies containing the origin from [40].

2 Valuations on Sobolev Spaces

Let $W^{1,p}(\mathbb{R}^n)$, where $p \geqslant 1$, denote the Sobolev space of real valued functions in $L^p(\mathbb{R}^n)$ whose weak partial derivatives are also in $L^p(\mathbb{R}^n)$. We remark that on Sobolev spaces it is an open problem to establish a classification of all continuous and rigid motion invariant valuations.

2.1 Matrix valued valuations

The following result is a partial analogue of Theorem 5. We say that an operator $Z : W^{1,2}(\mathbb{R}^n) \to \mathbb{M}^n$ is $\mathrm{GL}(n)$ contravariant if for some $q \in \mathbb{R}$

$$Z(f \circ \phi^{-1}) = |\det \phi|^q\, \phi^{-t}\, Z(f)\, \phi^{-1}$$

for all $f \in W^{1,2}(\mathbb{R}^n)$ and $\phi \in \mathrm{GL}(n)$. It is called homogeneous if for some $r \in \mathbb{R}$ we have $Z(tf) = |t|^r\, Z(f)$ for all $t \in \mathbb{R}$ and $f \in W^{1,2}(\mathbb{R}^n)$. It is called affinely contravariant if it is $\mathrm{GL}(n)$ contravariant, translation invariant and homogeneous.

Theorem 7 (Ludwig [43]). *An operator* $Z : W^{1,2}(\mathbb{R}^n) \to \mathbb{M}^n$, *where* $n \geqslant 3$, *is a continuous and affinely contravariant valuation if and only if there is a constant* $c \in \mathbb{R}$ *such that*
$$Z(f) = c\, \mathrm{J}(f^2)$$
for every $f \in W^{1,2}(\mathbb{R}^n)$.

Here $\mathrm{J}(g)$ is the *Fisher information matrix* of a weakly differentiable function $g : \mathbb{R}^n \to [0, \infty)$, that is, the $n \times n$ matrix with entries

$$\mathrm{J}_{ij}(g) = \int_{\mathbb{R}^n} \frac{\partial \log g(x)}{\partial x_i}\, \frac{\partial \log g(x)}{\partial x_j}\, g(x)\, dx. \tag{5}$$

The Fisher information matrix plays an important role in information theory and statistics (see [13, 15]). In general, Fisher information is a measure of the minimum error in the maximum likelihood estimate of a parameter in a distribution. The Fisher information matrix (5) describes such an error for a random vector of density g with respect to a location parameter. The proof of Theorem 7 makes use of the connection of the Fisher information matrix and the so called LYZ ellipsoid established by Lutwak, Yang & Zhang [51,53] and a classification of matrix valued valuations in [39].

2.2 Convex body valued valuations

Define the notion of $\mathrm{GL}(n)$ covariance for operators on the Sobolev space $W^{1,1}(\mathbb{R}^n)$ as for operators on the space $L^1(\mathbb{R}^n, |x| \, dx)$ (cf. (4)). An operator on $W^{1,1}(\mathbb{R}^n)$ is called affinely covariant if it is $\mathrm{GL}(n)$ covariant, translation invariant and homogeneous.

On the space of origin-symmetric convex bodies, there is a second important addition besides Minkowski addition, the so called Blaschke addition. The Blaschke sum of origin-symmetric convex bodies $K, L \subset \mathbb{R}^n$ with non-empty interiors is the origin-symmetric convex body $K \# L$ such that

$$S(K \# L, \cdot) = S(K, \cdot) + S(L, \cdot), \tag{6}$$

where $S(K, \cdot)$ is the surface area measure of K. For a Borel set $\omega \subset \mathbb{S}^{n-1}$ and a convex body K, the surface area measure $S(K, \omega)$ is the $(n-1)$-dimensional Hausdorff measure of the set of all boundary points of K at which there exists a unit normal vector of K belonging to ω. By the solution to the Minkowski problem, the convex body $K \# L$ is well defined by (6) (see [21] or [63]).

Let $\mathcal{K}_e^n$ denote the space of origin-symmetric convex bodies, where we identify convex bodies with the same surface area measure and use the topology induced by weak convergence of surface area measures. Note that when restricted to convex bodies with non-empty interiors, the spaces $\mathcal{K}_c^n$ and $\mathcal{K}_e^n$ coincide. An operator $\mathrm{Z} : W^{1,1}(\mathbb{R}^n) \to \mathcal{K}_e^n$ is called a Blaschke valuation if it is a valuation when the addition on $\mathcal{K}_e^n$ is Blaschke addition.

Theorem 8 (Ludwig [44]). *An operator $\mathrm{Z} : W^{1,1}(\mathbb{R}^n) \to \mathcal{K}_e^n$, where $n \geqslant 3$, is a continuous and affinely covariant Blaschke valuation if and only if there is a constant $c \geqslant 0$ such that*

$$\mathrm{Z}(f) = c \langle f \rangle$$

for every $f \in W^{1,1}(\mathbb{R}^n)$.

The operator $f \mapsto \langle f \rangle$ was introduced by Lutwak, Yang & Zhang [56] and is called the LYZ operator. For a function $f \in W^{1,1}(\mathbb{R}^n)$ and $f \not\equiv 0$, the *LYZ*

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body $\langle f \rangle$ is defined in [56] as the unique origin-symmetric convex body in $\mathbb{R}^n$ such that

$$\int_{\mathbb{S}^{n-1}} g(u)\, dS(\langle f \rangle, u) = \int_{\mathbb{R}^n} g(\nabla f(x))\, dx \qquad (7)$$

for every even continuous function $g : \mathbb{R}^n \to \mathbb{R}$ that is positively homogeneous of degree 1. The LYZ body for $f \equiv 0$ is a convex body with identically vanishing surface area measure and hence identified with $\{0\}$. Equation (7) is a functional version of the classical even Minkowski problem. The proof of Theorem 8 makes use of a classification of convex body valued valuations in [42].

To describe an important application of the LYZ body, we need some more background information. Let $\|\cdot\|$ denote a norm on $\mathbb{R}^n$ that is normalized so that its unit ball has the same volume, v_n, as the n-dimensional Euclidean unit ball. For such a norm, the sharp Gagliardo-Nirenberg-Sobolev inequality states that for every $f \in W^{1,1}(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} \|\nabla f(x)\|_* \, dx \geqslant n\, v_n^{1/n}\, \|f\|_{\frac{n}{n-1}} \qquad (8)$$

where $\|\cdot\|_*$ is the dual norm of $\|\cdot\|$, that is, $\|v\|_* = \sup\{x \cdot v : \|x\| \leqslant 1\}$ for $v \in \mathbb{R}^n$. The sharp Sobolev inequality (8) was established by Federer & Fleming [16] and Maz'ya [58] for Euclidean norms and by Gromov for general norms. Note that the right hand side of (8) does not depend on the norm $\|\cdot\|$. Hence for a given $f \in W^{1,1}(\mathbb{R}^n)$, we may ask for its *optimal Sobolev norm*, that is, for the norm that minimizes the left-hand side of (8) among all norms whose unit balls have volume v_n. This natural and important question was first asked by Lutwak, Yang & Zhang [56], who showed that the optimal Sobolev norm is up to normalization the norm whose unit ball is the LYZ body $\langle f \rangle$.

An operator $Z : W^{1,1}(\mathbb{R}^n) \to \mathcal{K}^n$ is called $\mathrm{GL}(n)$ contravariant if for some $q \in \mathbb{R}$

$$Z(f \circ \phi^{-1}) = |\det \phi|^q \, \phi^{-t}\, Z(f)$$

for all $f \in W^{1,1}(\mathbb{R}^n)$ and $\phi \in \mathrm{GL}(n)$. It is called affinely contravariant if it is $\mathrm{GL}(n)$ contravariant, translation invariant and homogeneous.

Theorem 9 (Ludwig [44])**.** *An operator* $Z : W^{1,1}(\mathbb{R}^n) \to \mathcal{K}_c^n$, *where* $n \geqslant 3$, *is a continuous and affinely contravariant Minkowski valuation if and only if there is a constant* $c \geqslant 0$ *such that*

$$Z(f) = c\, \Pi\langle f \rangle$$

for every $f \in W^{1,1}(\mathbb{R}^n)$.

Note that it follows from the definition of projection bodies and surface area measures that for $f \in W^{1,1}(\mathbb{R}^n)$ and $v \in \mathbb{S}^{n-1}$

$$h(\Pi\langle f \rangle, v) = \frac{1}{2}\int_{\mathbb{R}^n} |v \cdot \nabla f(x)|\, dx.$$

The convex body $\Pi\langle f\rangle$ has proved to be critical for affine analytic inequalities. In particular, the affine Sobolev-Zhang inequality [74] is a volume inequality for the polar body of $\Pi\langle f\rangle$ which strengthens and implies the Euclidean case of the L^1 Sobolev inequality (see also [12, 26, 52, 56]).

Very recently, Tuo Wang [71] has studied the LYZ operator $f \mapsto \langle f\rangle$ on the space, $\mathrm{BV}(\mathbb{R}^n)$, of functions of bounded variation. On $\mathrm{BV}(\mathbb{R}^n)$, the LYZ operator is not a valuation anymore but Wang [72] established a characterization as an affinely covariant Blaschke semi-valuation.

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