Anisotropic Fractional Sobolev Norms

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Abstract

Bourgain, Brezis & Mironescu showed that (with suitable scaling) the fractional Sobolev \( s \)-seminorm of a function \( f \in W^{1,p}(\mathbb{R}^n) \) converges to the Sobolev seminorm of \( f \) as \( s \to 1^- \). The anisotropic \( s \)-seminorms of \( f \) defined by a norm on \( \mathbb{R}^n \) with unit ball \( K \) are shown to converge to the anisotropic Sobolev seminorm of \( f \) defined by the norm with unit ball \( Z_p^* K \), the polar \( L_p \) moment body of \( K \). The limiting behavior for \( s \to 0^+ \) is also determined (extending results by Maz'ya & Shaposhnikova).

For \( p \geq 1 \) and \( 0 < s < 1 \), Gagliardo introduced the fractional Sobolev \( s \)-seminorm of a function \( f \in L^p(\Omega) \) as

\[
\|f\|_{W^{s,p}(\Omega)}^p = \int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x-y|^{n+ps}} \, dx \, dy,
\]  

(1)

where \( |\cdot| \) denotes the Euclidean norm on \( \mathbb{R}^n \) and \( \Omega \subset \mathbb{R}^n \). This seminorm turned out to be critical in the study of traces of Sobolev functions in the Sobolev space \( W^{1,p}(\Omega) \) (cf. [11]). Fractional Sobolev norms have found numerous applications within mathematics and applied mathematics (cf. [3, 7, 27]).

The limiting behavior of fractional Sobolov \( s \)-seminorms as \( s \to 1^- \) and \( s \to 0^+ \) turns out to be very interesting. Bourgain, Brezis & Mironescu [2] showed that

\[
\lim_{s \to 1^-} (1 - s) \|f\|_{W^{s,p}(\Omega)}^p = \alpha_{n,p} \|f\|_{W^{1,p}(\Omega)}^p
\]

(2)

for \( f \in W^{1,p}(\Omega) \) and \( \Omega \subset \mathbb{R}^n \) a smooth and bounded domain, where \( \alpha_{n,p} \) is a constant depending on \( n \) and \( p \), and

\[
\|f\|_{W^{1,p}(\Omega)} = \left( \int_\Omega |\nabla f(x)|^p \, dx \right)^{1/p}
\]

(3)

is the Sobolev seminorm of \( f \).

Maz'ya & Shaposhnikova [28] showed that if \( f \in W^{s,p}(\mathbb{R}^n) \) for all \( s \in (0, 1) \), where \( W^{s,p}(\mathbb{R}^n) \) are the functions in \( L^p(\mathbb{R}^n) \) with finite Gagliardo seminorm (1) with \( \Omega = \mathbb{R}^n \), then

\[
\lim_{s \to 0^+} s \|f\|_{W^{s,p}(\mathbb{R}^n)}^p = \frac{2n}{p} |B| \|f\|_{L_p}^p.
\]

(4)
where $B \subset \mathbb{R}^n$ is $n$-dimensional Euclidean unit ball, $|B|$ its $n$-dimensional volume and $|f|_p$ the $L^p$ norm of $f$ on $\mathbb{R}^n$.

An anisotropic Sobolev seminorm is obtained by replacing the Euclidean norm $|\cdot|$ in (3) by an arbitrary norm $\|\cdot\|_L$ with unit ball $L$. We set

$$\|f\|_{W^{1,p},K} = \left( \int_{\mathbb{R}^n} \|\nabla f(x)\|^p_{K^*} \, dx \right)^{1/p},$$

where $K^* = \{ v \in \mathbb{R}^n : v \cdot x \leq 1 \text{ for all } x \in K \}$ is the polar body of $K$. Anisotropic Sobolev seminorms have attracted increased interest in recent years (cf. [1,5,9,13]).

A natural question is to study the limiting behavior of anisotropic $s$-seminorms as $s \to 1^−$ and $s \to 0^+$. While one might suspect that the limit as $s \to 1^−$ of the anisotropic $s$-seminorms defined using a norm with unit ball $K$ is the Sobolev seminorm with the same unit ball, this turns out not to be true in general.

**Theorem 1.** If $f \in W^{1,p}(\mathbb{R}^n)$ has compact support, then

$$\lim_{s \to 1^-} (1-s) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^p}{\|x - y\|_K^{n+sp}} \, dx \, dy = \int_{\mathbb{R}^n} \|\nabla f(x)\|_{Z^*_p K}^p \, dx$$

where $Z^*_p K$ is the polar $L_p$ moment body of $K$.

For the Euclidean $s$-seminorms and the Euclidean unit ball $B$, the convex body $Z^*_p B$ is just a multiple of $B$. Hence Theorem 1 recovers the result by Bourgain, Brezis & Mironescu (2) including the value of the constant $\alpha_{n,p}$. For a convex body $K \subset \mathbb{R}^n$, the polar $L_p$ moment body is the unit ball of the norm defined by

$$\|v\|_{Z^*_p K}^p = \frac{n + p}{2} \int_{K} |v \cdot x|^p \, dx$$

for $v \in \mathbb{R}^n$.

The polar body of $Z^*_1 K$, the convex body $Z^*_1 K$, is the moment body of $K$. The convex body

$$\frac{2}{(n + 1)|K|} Z_1 K$$

is the centroid body of $K$, a classical concept that goes back at least to Dupin (cf. [12]). If we intersect the origin-symmetric convex body $K$ by halfspaces orthogonal to $u \in S^{n-1}$, then the centroids of these intersections trace out the boundary of twice the centroid body of $K$, which explains the name centroid body. The name moment body comes from the fact that the corresponding moment vectors trace out the boundary (of a constant multiple) of $Z^*_1 K$. Centroid bodies play an important role within the affine geometry of convex bodies (cf. [12,20]) and moment bodies within the theory of valuations on convex bodies (see [14,17,18]).
The polar body of $Z_p^* K$, the convex body $Z_p K$, is the $L_p$ moment body of $K$ and

$$\frac{2}{(n+p)|K|} Z_p K$$

is the $L_p$ centroid body of $K$, a concept introduced by Lutwak & Zhang [26]. $L_p$ centroid bodies and $L_p$ moment bodies have found important applications within convex geometry, probability theory, and the local theory of Banach spaces (cf. [10, 15–17, 21–25, 29–32]).

For $p > 1$, it follows from Bourgain, Brezis & Mironescu [2, Theorem 2] that (5) also holds for $f \in L^p(\Omega)$ in the sense that if $f \notin W^{1,p}(\Omega)$, then both sides of (5) are infinite. For $p = 1$, it follows from [2, Theorem 3'] that a corresponding result holds for $f \notin BV(\mathbb{R}^n)$ (see also Dávila [6]). In [19], the limiting behavior of fractional anisotropic Sobolev seminorms on $BV(\mathbb{R}^n)$ is discussed using fractional anisotropic perimeters. Ponce [33] obtained several extensions of the results in [2], from which Theorem 1 can also be deduced if anisotropic $s$-seminorms are used. The proof given in this paper is independent of Ponce’s results. It makes use of the one-dimensional case of the Bourgain, Brezis & Mironescu Theorem (2) and the Blaschke-Petkanschin Formula from integral geometry.

Corresponding to the result of Maz’ya & Shaposhnikova (4), we obtain the following result.

**Theorem 2.** If $f \in W^{s,p}(\mathbb{R}^n)$ for all $s \in (0, 1)$ and $f$ has compact support, then

$$\lim_{s \to 0^+} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^p}{\|x - y\|^{n+sp}} \, dx \, dy = \frac{2n}{p} |K| |f|_p^p.$$ 

The proof of Theorem 2 is based on the one-dimensional case of (4) and the Blaschke-Petkanschin Formula.

### 1 Preliminaries

We state the Blaschke-Petkanschin Formula (cf. [34, Theorem 7.2.7]) in the case in which it will be used. Let $H^k$ denote the $k$-dimensional Hausdorff measure on $\mathbb{R}^n$ and let $\text{Aff}(n, 1)$ denote the affine Grassmannian of lines in $\mathbb{R}^n$. If $g : \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$ is Lebesgue measurable, then

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} g(x, y) \, dH^n(x) \, dH^n(y) = \int_{\text{Aff}(n, 1)} \int_L \int_L g(x, y) |x-y|^{n-1} \, dH^1(x) \, dH^1(y) \, dL,$$ 

where $dL$ denotes integration with respect to a suitably normalized rigid motion invariant Haar measure on $\text{Aff}(n, 1)$. This measure can be described in the following way.
Any line \( L \in \text{Aff}(n, 1) \) can be parameterized using one of its direction unit vectors \( u \in S^{n-1} \) and its base point \( x \in u^\perp \), where \( u^\perp \) is the hyperplane orthogonal to \( u \), as \( L = \{ x + \lambda u : \lambda \in \mathbb{R} \} \). Hence, for \( h : \text{Aff}(n, 1) \to [0, \infty) \) measurable,

\[
\int_{\text{Aff}(n, 1)} h(L) \, dL = \frac{1}{2} \int_{S^{n-1}} \int_{u^\perp} h(x + L_u) \, dH^{n-1}(x) \, dH^{n-1}(u),
\]

where \( L_u = \{ \lambda u : \lambda \in \mathbb{R} \} \).

For \( f \in W^{1,p}(\mathbb{R}^n) \), we denote by \( \bar{f} \) its precise representative (cf. [8, Section 1.7.1]). We require the following result. For every \( u \in S^{n-1} \), the precise representative \( \bar{f} \) is absolutely continuous on the lines \( L = \{ x + \lambda u : \lambda \in \mathbb{R} \} \) for \( H^{n-1} \)-a.e. \( x \in u^\perp \) and its first-order (classical) partial derivatives belong to \( L^p(\mathbb{R}^n) \) (cf. [8, Section 4.9.2, Theorem 2]). Hence we have for the restriction of \( \bar{f} \) to \( L \),

\[
\bar{f}|_L \in W^{1,p}(L)
\]

for a.e. line \( L \) parallel to \( u \).

We require the following one-dimensional case of (2).

**Proposition 1** ([2]). If \( g \in W^{1,p}(\mathbb{R}) \) has compact support, then

\[
\lim_{s \to 1^-} (1-s) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{|g(x) - g(y)|^p}{|x - y|^{1+ps}} \, dx \, dy = \frac{2}{p} \|g\|^p_{W^{1,p}(\mathbb{R})}.
\]

We require the following one-dimensional case of (4).

**Proposition 2** ([28]). If \( g \in W^{s,p}(\mathbb{R}) \) for all \( s \in (0, 1) \), then

\[
\lim_{s \to 0^+} s \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{|g(x) - g(y)|^p}{|x - y|^{1+ps}} \, dx \, dy = \frac{4}{p} \|g\|^p_{p'}.
\]

We also need the following result. The proof is based on the one-dimensional case of some estimates from [2]. Let \( \text{diam}(C) = \sup\{|x - y| : x \in C, y \in C\} \) denote the diameter of \( C \subset \mathbb{R} \).

**Lemma 1.** If \( g \in W^{1,p}(\mathbb{R}) \) has compact support \( C \), then there exists a constant \( \gamma_p \) depending only on \( p \) such that

\[
(1-s) \int_{-\infty}^{\infty} \frac{|g(x) - g(y)|^p}{|x - y|^{1+ps}} \, dx \, dy \leq \gamma_p \max(1, \text{diam}(C))^p \|g\|^p_{W^{1,p}(\mathbb{R})}
\]

for all \( 1/2 \leq s < 1 \).
Proof. If \( g \in W^{1,p}(\mathbb{R}) \) is smooth, then for \( h \in \mathbb{R} \)

\[
g(x + h) - g(x) = h \int_0^1 g'(x + th) \, dt.
\]

Hence for \( h \in \mathbb{R} \),

\[
\int_{-\infty}^{\infty} |g(x + h) - g(x)|^p \, dx \leq |h|^p \|g\|_{W^{1,p}(\mathbb{R})}^p.
\] (9)

The same estimate is obtained for \( g \in W^{1,p}(\mathbb{R}) \) by approximation (cf. [4, Proposition 9.3]). Let the support of \( g \) be contained in \([-r, r]\), where \( r \geq 1 \). By (9) we get

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{|g(x) - g(y)|^p}{|x - y|^{1+ps}} \, dx \, dy = \int_{-2r}^{-2r} \int_{-\infty}^{\infty} \frac{|g(x + h) - g(x)|^p}{|h|^{1+ps}} \, dx \, dh
\]

\[
\leq \int_{-2r}^{-2r} |h|^{-(1-p(1-s))} \, dh \|g\|_{W^{1,p}(\mathbb{R})}^p
\]

\[
\leq \frac{2}{p(1-s)} \left( \frac{2r}{p(1-s)} \right) \|g\|_{W^{1,p}(\mathbb{R})}^p.
\]

This completes the proof of the lemma. \( \square \)

The following estimate is used in the proof of Theorem 2.

Lemma 2. If \( g \in W^{s,p}(\mathbb{R}) \) for all \( s \in (0, 1) \), then

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{|g(x) - g(y)|^p}{|x - y|^{1+ps}} \, dx \, dy \leq \frac{2^{p+1}}{ps} |g|^p_p + \|g\|_{W^{s',p}(\mathbb{R})}^p
\]

for all \( 0 < s \leq s' < 1 \).

Proof. Note that

\[
\int_{\mathbb{R}} \int_{\{|x-y| \geq 1\}} \frac{|g(x)|^p}{|x - y|^{1+ps}} \, dx \, dy \leq \int_{\{|z| \geq 1\}} \frac{dz}{|z|^{1+ps}} |g|^p_p = \frac{2}{ps} |g|^p_p.
\]

Hence, by Jensen’s inequality,

\[
\int \int_{\{|x-y| \geq 1\}} \frac{|g(x) - g(y)|^p}{|x - y|^{1+ps}} \, dx \, dy \leq 2^{p-1} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{|g(x)|^p + |g(y)|^p}{|x - y|^{1+ps}} \, dx \, dy \leq \frac{2^{p+1}}{ps} |g|^p_p.
\]

On the other hand,

\[
\int \int_{\{|x-y| < 1\}} \frac{|g(x) - g(y)|^p}{|x - y|^{1+sp}} \, dx \, dy \leq \int \int_{\{|x-y| < 1\}} \frac{|g(x) - g(y)|^p}{|x - y|^{1+s'p}} \, dx \, dy
\]

for \( 0 < s < s' \). \( \square \)
2 Proof of Theorem 1

By the Blaschke-Petkantschin Formula (6), we have
\begin{equation}
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^p}{\|x - y\|^{n+ps}_K} \, dx \, dy = \int_{\text{Aff}(n,1)} \|u(L)\|_{K}^{(n+ps)} \int_{L} \int_{L} \frac{|f(x) - f(y)|^p}{\|x - y\|^{1+ps}_1} \, dH^1(x) \, dH^1(y) \, dL.
\end{equation}

By Proposition 1 and (8), we have
\begin{equation}
\lim_{s \to 1^-} (1 - s) \int_{L} \int_{L} \frac{|f(x) - f(y)|^p}{\|x - y\|^{1+ps}_1} \, dx \, dy = \frac{2}{p} \int_{L} |\nabla f(x) \cdot u|^p \, dH^1(x)
\end{equation}
for a.e. line $L$ parallel to $u \in S^{n-1}$.

By Fubini’s Theorem, the definition of the measure on the affine Grassmannian (7) and the polar coordinate formula, we get
\begin{align*}
\frac{2}{p} \int_{\text{Aff}(n,1)} \|u(L)\|_{K}^{(n+ps)} & \int_{L} |\nabla f(x) \cdot u|^p \, dH^1(x) \, dL \\
= \frac{1}{p} \int_{S^{n-1} \setminus u^\perp} \|u\|_{K}^{(n+ps)} \int_{y^+L_u} |\nabla f(x) \cdot u|^p \, dH^1(x) \, dH^{n-1}(y) \, dH^{n-1}(u) \\
= \frac{1}{p} \int_{S^{n-1}} \|u\|_{K}^{(n+ps)} |\nabla f(x) \cdot u|^p \, dH^n(x) \, dH^{n-1}(u) \\
= \frac{n + p}{p} \int_{\mathbb{R}^n} |\nabla f(x) \cdot y|^p \, dH^n(x) \, dH^{n}(y).
\end{align*}

Using Fubini’s Theorem and the definition of the $L_p$ moment body of $K$, we obtain
\begin{equation}
\int_{\text{Aff}(n,1)} \|u(L)\|_{K}^{(n+ps)} \int_{L} |\nabla f(x) \cdot u|^p \, dH^1(x) \, dL = \int_{\mathbb{R}^n} \|\nabla f(x)\|^p_{Z^*_pK} \, dx.
\end{equation}

So, in particular, we have
\begin{equation}
\int_{\text{Aff}(n,1)} \int_{L} |\nabla f(x) \cdot u|^p \, dH^1(x) \, dL = \alpha_{n,p} \int_{\mathbb{R}^n} |\nabla f(x)|^p \, dx < \infty,
\end{equation}
where $\alpha_{n,p}$ is a constant.

Using the Dominated Convergence Theorem combined with Lemma 1 and (13), we obtain from (10), (11) and (12) that
\begin{equation}
\lim_{s \to 1^-} (1 - s) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^p}{\|x - y\|^{n+ps}_K} \, dx \, dy = \int_{\mathbb{R}^n} \|\nabla f(x)\|^p_{Z^*_pK} \, dx.
\end{equation}

This concludes the proof of the theorem.
3 Proof of Theorem 2

By the Blaschke-Petkantschin Formula (6), we obtain
\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f(x) - f(y)|^p \|x - y\|_K^{-(n+ps)} dx \, dy = \int_{\text{Aff}(n,1)} \|u(L)\|_K^{-n} \int_L \frac{|f(x) - f(y)|^p}{|x - y|^s+1} dx \, dy \, dL.
\]

Thus we obtain by the Dominated Convergence Theorem, Lemma 2 and Proposition 2 that
\[
\lim_{s \to 0^+} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f(x) - f(y)|^p \|x - y\|_K^{n+ps} dx \, dy = \frac{4}{p} \int_{\text{Aff}(n,1)} \|u(L)\|_K^{-n} \int_L |f(x)|^p dH^1(x) \, dL.
\]

By Fubini’s Theorem, the definition of the measure on the affine Grassmannian (7) and the polar coordinate formula for volume, we get
\[
\frac{4}{p} \int_{\text{Aff}(n,1)} \|u(L)\|_K^{-n} \int_L |f(x)|^p dH^1(x) \, dL = \frac{2}{p} \int_{S^{n-1}} \int_{u^L} \|u\|_K^{-n} \int_{y+Lu} |f(x)|^p dH^n(x) \, dH^{n-1}(u)
\]
\[
\quad = \frac{2}{p} \int_{S^{n-1}} \int_{\mathbb{R}^n} \|u\|_K^{-n} |f(x)|^p dH^n(x) \, dH^{n-1}(u)
\]
\[
\quad = \frac{2n}{p} |K| |f|^p.
\]

This concludes the proof of the theorem.

References


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