A Classification of $\mathrm{SL}(n)$ invariant Valuations

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Abstract

A classification of upper semicontinuous and $\mathrm{SL}(n)$ invariant valuations on the space of n-dimensional convex bodies is established. As a consequence, complete characterizations of centro-affine and L_p affine surface areas are obtained. The proofs make use of a new $\mathrm{SL}(n)$ shaping process for convex bodies.

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In his 1900 ICM Address, David Hilbert asked in his Third Problem if an elementary definition for volume of polytopes is possible. Max Dehn's solution in 1901 makes critical use of the notion of *valuations*, that is, of functions $\Phi : \mathcal{S} \to \mathbb{R}$ that satisfy the inclusion-exclusion relation

$$\Phi(K) + \Phi(L) = \Phi(K \cup L) + \Phi(K \cap L),$$

whenever $K, L, K \cup L, K \cap L \in \mathcal{S}$, where $\mathcal{S}$ is a collection of sets. Dehn solved Hilbert's Third Problem by constructing a rigid motion invariant valuation which vanishes on lower dimensional sets and is not equal to volume (under any normalization). Since then investigations of valuations have been an active and prominent part of mathematics (see [1] – [8], [16], [19], [20], [24], [25], [31] – [34], [53] for some of the more recent results).

Dehn's work has been strengthened considerably by Sydler and Hadwiger. A systematic study of valuations was initiated by Hadwiger, who was in particular interested in classifying valuations on the set, $\mathcal{K}^n$, of convex bodies (compact convex sets) in $\mathbb{R}^n$. Probably the most famous result on valuations is the Hadwiger characterization theorem.

Theorem 1 ([21])**.** *A functional $\Phi : \mathcal{K}^n \to \mathbb{R}$ is a continuous and rigid motion invariant valuation if and only if there are constants $c_0, c_1, \dots, c_n \in \mathbb{R}$ such that*

$$\Phi(K) = c_0\, V_0(K) + \cdots + c_n\, V_n(K)$$

for every $K \in \mathcal{K}^n$.

Here $V_0(K), \dots, V_n(K)$ are the intrinsic volumes of $K \in \mathcal{K}^n$. In particular, $V_0(K)$ is the Euler characteristic (that is, $V_0(K) = 1$ for $K \neq \emptyset$ and $V_0(\emptyset) = 0$), $2\, V_{n-1}(K)$ is the surface area and $V_n(K)$ the volume of K. This result was the starting point for many investigations dealing with characterizations and precise descriptions of classes of valuations having interesting invariance properties (see [21], [26], [47], [48] for more information).

Prior to Hadwiger, Blaschke proved that every continuous, translation and $\mathrm{SL}(n)$ invariant valuation on $\mathcal{K}^3$ is a linear combination of volume and the Euler characteristic. This also follows immediately from Hadwiger's characterization theorem. However, if continuity is weakened to upper semicontinuity, there are more examples and the authors obtained the following result.

Theorem 2 ([29], [35])**.** *A functional $\Phi : \mathcal{K}^n \to \mathbb{R}$ is an upper semicontinuous, translation and $\mathrm{SL}(n)$ invariant valuation if and only if there are constants $c_0, c_1 \in \mathbb{R}$ and $c_2 \geq 0$ such that*

$$\Phi(K) = c_0 \, V_0(K) + c_1 \, V_n(K) + c_2 \, \Omega(K)$$

for every $K \in \mathcal{K}^n$.

The 'new' valuation $\Omega(K)$ in this characterization theorem is the *affine surface area* of a convex body K in $\mathbb{R}^n$. It is defined by

$$\Omega(K) = \int_{\partial K} \kappa(K, x)^{\frac{1}{n+1}} \, dx,$$

where $\kappa(K, x)$ is the generalized Gaussian curvature of ∂K at x. For smooth convex surfaces, this definition is classical. It is also classical that Ω is *equi-affine invariant* for smooth surfaces, that is, Ω is both translation invariant and $\mathrm{SL}(n)$ invariant. The extension of the definition of affine surface area to general convex bodies was obtained much more recently in a series of papers [27], [37], [54]. There it is also proved that Ω is equi-affine invariant on $\mathcal{K}^n$. The long conjectured upper semicontinuity of affine surface area (for smooth surfaces as well as for general convex surfaces) was proved by Lutwak [37] in 1991 and was important in the solution of the affine Plateau problem by Trudinger and Wang [61] in 2005.

The notion of affine surface area is fundamental in affine differential geometry and important results on affine surface area were obtained in recent years (see, for example, [9], [10], [56] – [62]). In addition, since many basic problems in discrete and stochastic geometry are equi-affine invariant, affine surface area has found numerous applications in these fields (see, for example, [11], [12], [18], [50]).

Theorem 2 shows that within the theory of valuations, Ω is the natural notion of surface area for the equi-affine group. This raises the question to obtain the natural notion of surface area for affine groups without assuming translation invariance. In view of Theorem 2, the question therefore is:

> *Is it possible to classify all $\mathrm{SL}(n)$ or $\mathrm{GL}(n)$ invariant valuations on $\mathcal{K}_0^n$?*

Here $\mathcal{K}_0^n$ denotes the space of convex bodies that contain the origin in their interiors.

A complete answer for the centro-affine group $\mathrm{GL}(n)$ is contained in the following theorem.

Theorem 3. *A functional $\Phi : \mathcal{K}_0^n \to \mathbb{R}$ is an upper semicontinuous and* $\mathrm{GL}(n)$ *invariant valuation if and only if there are constants $c_0 \in \mathbb{R}$ and $c_1 \geq 0$ such that*

$$\Phi(K) = c_0\, V_0(K) + c_1\, \Omega_c(K)$$

for every $K \in \mathcal{K}_0^n$.

This theorem establishes a characterization of the *centro-affine surface area* $\Omega_c(K)$. For a convex body $K \in \mathcal{K}_0^n$, the centro-affine surface area is defined by

$$\Omega_c(K) = \int_{\partial K} \kappa_0(K,x)^{\frac{1}{2}}\, d\mu_K(x).$$

Here $d\mu_K(x) = x \cdot u(K,x)\, dx$ is the cone measure on ∂K, $x \cdot u$ denotes the standard inner product of $x, u \in \mathbb{R}^n$, $u(K,x)$ is the exterior normal unit vector to K at $x \in \partial K$, and

$$\kappa_0(K,x) = \frac{\kappa(K,x)}{(x \cdot u(K,x))^{n+1}}.$$

For smooth surfaces, centro-affine surface area is classical and Titeica showed in 1908 the $\mathrm{SL}(n)$ invariance of κ_0. For general convex bodies, Lutwak [39] proved that Ω_c is well defined, $\mathrm{GL}(n)$ invariant and upper semicontinuous. A simple consequence of Theorem 3 is the classical result that polar convex bodies have the same centro-affine surface area (see Section 6).

The classification of $\mathrm{SL}(n)$ invariant valuations leads to a much richer class of valuations. First we state the following complete classification of homogeneous and $\mathrm{SL}(n)$ invariant valuations. Here a functional Φ is called *homogeneous of degree q*, $q \in \mathbb{R}$, if $\Phi(t\,K) = t^q\, \Phi(K)$ for every $t > 0$, $K \in \mathcal{K}_0^n$. Let $K^* = \{x \in \mathbb{R}^n : x \cdot y \leq 1 \text{ for } y \in K\}$ denote the polar body of $K \in \mathcal{K}_0^n$.

Theorem 4. *A functional $\Phi : \mathcal{K}_0^n \to \mathbb{R}$ is an upper semicontinuous and* $\mathrm{SL}(n)$ *invariant valuation that is homogeneous of degree q if and only if there are constants $c_0 \in \mathbb{R}$ and $c_1 \geq 0$ such that*

$$\Phi(K) = \begin{cases} c_0\, V_0(K) + c_1\, \Omega_n(K) & \text{for } q = 0 \\ c_1\, \Omega_p(K) & \text{for } -n < q < n \text{ and } q \neq 0 \\ c_0\, V_n(K) & \text{for } q = n \\ c_0\, V_n(K^*) & \text{for } q = -n \\ 0 & \text{for } q < -n \text{ or } q > n \end{cases}$$

for every $K \in \mathcal{K}_0^n$ where $p = n(n-q)/(n+q)$.

The 'new' valuation $\Omega_p(K)$ in this characterization theorem is the L_p *affine surface area* of a convex body $K \in \mathcal{K}_0^n$. For $p > 1$, L_p affine surface area was defined by Lutwak [39] as the notion corresponding to affine surface area in the L_p Brunn Minkowski theory (see [13], [14], [38] – [46], [57], [58] for contributions to the L_p Brunn Minkowski theory). Lutwak [39] proved that Ω_p is $\mathrm{SL}(n)$ invariant, homogeneous of degree $q = n\,(n-p)/(n+p)$ and upper semicontinuous on $\mathcal{K}_0^n$. Hug [22] defined L_p affine surface areas for every $p > 0$ and obtained the following representation for $K \in \mathcal{K}_0^n$:

$$\Omega_p(K) = \int_{\partial K} \kappa_0(K,x)^{\frac{p}{n+p}}\, d\mu_K(x).$$

Note that $\Omega = \Omega_1$ and $\Omega_c = \Omega_n$, that is, affine and centro-affine surface areas are just special L_p affine surface areas. Geometric interpretations of L_p affine surface areas are obtained in [17], [49], [55], [63], and an application of L_p affine surface areas to partial differential equations is given in [40]. A simple consequence of Theorem 4 is Hug's result [23] that $\Omega_p(K^*) = \Omega_{n^2/p}(K)$ for every $K \in \mathcal{K}_0^n$, $p > 0$ (see Section 6).

In the background of these results is a rather general theorem. We give a complete classification of $\mathrm{SL}(n)$ invariant valuations on $\mathcal{K}_0^n$ which vanish on polytopes. Combined with [32], where a classification of all Borel measurable, homogeneous and $\mathrm{SL}(n)$ invariant valuations on polytopes is obtained, this result implies Theorems 3 and 4. Let $\mathcal{P}_0^n$ denote the set of convex polytopes that contain the origin in their interiors.

Theorem 5. *A functional $\Phi : \mathcal{K}_0^n \to \mathbb{R}$ is an upper semicontinuous and $\mathrm{SL}(n)$ invariant valuation that vanishes on $\mathcal{P}_0^n$ if and only if there is a concave function $\phi : [0,\infty) \to [0,\infty)$ with $\lim_{t\to 0} \phi(t) = \lim_{t\to\infty} \phi(t)/t = 0$ such that*

$$\Phi(K) = \int_{\partial K} \phi(\kappa_0(K,x))\, d\mu_K(x) \tag{1}$$

for every $K \in \mathcal{K}_0^n$.

This theorem shows that each of these 'L_ϕ affine surface areas' is a natural choice for an $\mathrm{SL}(n)$ invariant surface area on $\mathcal{K}_0^n$.

Since in this characterization theorem no translation invariance is assumed and since on the space $\mathcal{K}_0^n$ it is not possible to use many of the standard techniques involving dissections, the proof of Theorem 5 is based on several new constructions. In particular, to allow certain dissections we extend Φ to a valuation on a larger class of sets (see Proposition 10 and Proposition 14). Critical in the proof of Theorem 5 is also a new $\mathrm{SL}(n)$ shaping process for convex bodies (see Section 2). In certain respects, this $\mathrm{SL}(n)$ shaping process behaves similar to the Minkowski addition of a line segment, $K \mapsto K + I$ (which is critical in results for translation invariant valuations). The $\mathrm{SL}(n)$ shaping process is used in two important steps of the proof. First in Proposition 14 to extend valuations defined on $\mathcal{K}_0^n$. In the second application in Lemma 24, the $\mathrm{SL}(n)$ shaping process is used repeatedly in a precisely defined way to obtain in the limit a process that behaves similar to the Minkowski addition of a small ball. This allows us to generalize the classification result first obtained for ε-smooth convex bodies (that is, convex bodies that are Minkowski sums of suitable convex bodies and balls of radius $\varepsilon > 0$) to convex bodies without additional smoothness assumptions.

In the next section, some preliminaries are given. In Section 2, the new $SL(n)$ shaping process is described in detail. In Section 3, we derive the extension result mentioned above. The proof of Theorem 5 is given in Section 4. In Section 5, we derive Theorems 3 and 4 from Theorem 5. In Section 6, two corollaries of Theorems 3 and 4 are derived. In the last section, we show that Theorem 2 can easily be obtained from Theorem 5.

1 Notation and Preliminaries

A general reference on the geometry of convex bodies is the book by Schneider [52]. We work in n-dimensional Euclidean space, $\mathbb{R}^n$, and write $x = (x_1, x_2, \ldots, x_n)$ for vectors $x \in \mathbb{R}^n$. The standard basis in $\mathbb{R}^n$ will be denoted by $e_1, e_2, \ldots, e_n$. Let $x \cdot y$ denote the usual scalar product $x_1y_1 + x_2y_2 + \cdots + x_ny_n$ of two vectors $x, y \in \mathbb{R}^n$, and define the norm $|x| = \sqrt{x \cdot x}$. The unit sphere $\{x \in \mathbb{R}^n : |x| = 1\}$ is denoted by S^{n-1} and the unit ball $\{x \in \mathbb{R}^n : |x| \leq 1\}$ by B^n. For the n-dimensional volume of B^n, we write v_n. For a ball with center $a \in \mathbb{R}^n$ and radius $r > 0$, we write $B^n(a, r)$. We call any dilated and rotated copy of $\{(x_1, \ldots, x_n) \in \mathbb{R}^n : (x_1, \ldots, x_{n-1}) \in B^{n-1}\}$ an *unbounded circular cylinder*. Given $A \subset \mathbb{R}^n$, let $\operatorname{cone}(A)$ denote the smallest closed convex cone with apex at the origin containing A. Given $A_1, A_2, \ldots, A_k \subset \mathbb{R}^n$, we write $[A_1, A_2, \ldots, A_k]$ for their convex hull. For a hyperplane H containing the origin, we denote by H^+ and H^- the complementary closed halfspaces bounded by H.

Let $\mathcal{Q}^n$ be the set of convex polyhedral cones with apex at the origin, and let $\mathcal{Q}^n_j$, where $j = 1, \ldots, n$, be the set of cones $Q \in \mathcal{Q}^n$ bounded by at most j hyperplanes containing the origin with linearly independent normal vectors, that is, $Q = H_1^+ \cap \cdots \cap H_i^+$ with $i \leq j$. Let $\mathcal{K}^n_j$, $j = 1, \ldots, n$, be the set of convex bodies K such that either $K \in \mathcal{K}^n_0$ or there exist $K_0 \in \mathcal{K}^n_0$ and a cone $Q \in \mathcal{Q}^n_j$ such that and

$$K = K_0 \cap Q = K_0 \cap H_1^+ \cap \cdots \cap H_i^+, \ i \leq j. \tag{2}$$

Let $\bar{\mathcal{K}}^n_0$ be the set of convex bodies K such that either K is in $\mathcal{K}^n_0$ or there exist a convex body $K_0 \in \mathcal{K}^n_0$ and a polyhedral cone $Q \in \mathcal{Q}^n$ such that $K = K_0 \cap Q$. Note that $\mathcal{Q}^n_1 \subseteq \cdots \subseteq \mathcal{Q}^n_n \subseteq \mathcal{Q}^n$ and $\mathcal{K}^n_0 \subseteq \mathcal{K}^n_1 \subseteq \cdots \subseteq \mathcal{K}^n_n \subseteq \bar{\mathcal{K}}^n_0$.

Let $\mathcal{D}^n_j$, where $j = 0, \ldots, n-1$, be the set of n-dimensional convex bodies $D = [K \cap H, u, v]$ where $K \in \mathcal{K}^n_j$ is defined by (2), H is a hyperplane with $K \cap H^+, K \cap H^- \in \mathcal{K}^n_{j+1}$, and where $u, v \in K \cap H_1 \cap \cdots \cap H_i$, $u \in H^+ \backslash H$, $v \in H^- \backslash H$. Note that $\mathcal{D}^n_j \subseteq \mathcal{K}^n_j$.

For a hyperplane H containing the origin, we denote by $\mathcal{K}^{n-1}_0(H)$ the set of convex bodies in H that contain the origin in their interiors relative to H and by $\mathcal{P}^{n-1}(H)$ the set of convex polytopes in H. On $\mathcal{K}^n$ as well as on these subspaces we always use the topology induced by the Hausdorff distance and denote by $\delta(K, L)$ the Hausdorff distance of $K, L \in \mathcal{K}^n$. For $K \in \mathcal{K}^n$, we denote its dimension by $\dim K$.

For $v \in \mathbb{R}^n$, $v \neq 0$, let $v^\perp$ be the hyperplane containing the origin with normal vector v. For $v \in \mathbb{R}^n$, $v \neq 0$, and $K \in \bar{\mathcal{K}}^n_0$, we denote the supporting halfspace that contains K and has outer normal vector v by $H^+(K,v)$ and the corresponding supporting hyperplane by $H(K,v)$. For $K \in \bar{\mathcal{K}}_0^n$, let $N(K,x)$ be the *normal cone* of K at $x \in \partial K$, that is, the set of all outer normal vectors v such that $x \in H(K,v)$. For $K \in \bar{\mathcal{K}}_0^n$ and $A \subset \mathbb{R}^n$, we set $N(K,A) = \bigcup_{x\in\partial K\cap A} N(K,x)$ and we define the *tangential continuation* of K with respect to A by

$$H^+(K,A) = \bigcap_{u\in N(K,A)} H^+(K,u).$$

For $K_j, K \in \bar{\mathcal{K}}_0^n$ and D a convex cone, we write $K_j \cap D \xrightarrow{t} K \cap D$ as $j \to \infty$ if the convergence of $K_j \cap D \to K \cap D$ is such that $H^+(K_j, D)$ converges to $H^+(K,D)$ in every large ball.

We require the following result which can be proved as the corresponding statement, Theorem 1, in [30].

Proposition 6. *Suppose $\phi : [0,\infty) \to [0,\infty)$ is concave, $\lim_{t\to 0} \phi(t) = \lim_{t\to\infty} \phi(t)/t = 0$, and define the functional Φ for $K \in \mathcal{K}_0^n$ by*

$$\Phi(K) = \int_{\partial K} \phi(\kappa_0(K,x))\, d\mu_K(x).$$

Then $\Phi : \mathcal{K}_0^n \to [0,\infty)$ is an upper semicontinuous valuation on $\mathcal{K}_0^n$ that vanishes on $\mathcal{P}_0^n$.

We also require the following result on packings on S^{n-1}, which follows immediately from the corresponding Euclidean result. We say that the balls $B^n(x_1,r),\dots,B^n(x_k,r)$ define a packing in S^{n-1}, if $x_i \in S^{n-1}$ for $i = 1,\dots,k$, and if the sets $S^{n-1}\cap B^n(x_i,r)$ have pairwise disjoint relative interiors in S^{n-1}. If $m_{S^{n-1}}(r)$ is the maximum number of balls of radius r that define a packing in S^{n-1}, then

$$v_{n-1}\, m_{S^{n-1}}(r)\, r^{n-1} \to \delta_{n-1}\, n\, v_n \tag{3}$$

as $r \to 0$, where $\delta_{n-1} > 0$ is the packing density of balls in $\mathbb{R}^{n-1}$.

2 The SL(n) shaping process

For $u \in S^{n-1}$ and $v \in u^\perp$, let $T = T(u,v)$ be the linear transformation which maps each point x to $x + (x\cdot u)v$ and thus leaves $u^\perp$ unchanged. Such a transformation is called a *transvection*. Note that $T^{-1}(u,v) = T(u,-v)$ and $T(u,v) \in \mathrm{SL}(n)$. For $T = T(u,v)$ and $x \in \mathbb{R}^n$, we have

$$|Tx - x| \le |v|\,|x| \tag{4}$$

and

$$|Tx| \le (1 + |v|)|x|. \tag{5}$$

For a convex body K, we define

$$K_{T(u,v)} = \bigcup_{s\in[0,1]} T(u,sv)K.$$

We call $K \mapsto K_{T(u,v)}$ the $\mathrm{SL}(n)$ *shaping process*. Note that $K_T = K_{T(u,v)}$ is compact but not necessarily convex. In particular, K_T is not convex if $u^\perp$ intersects K in a smooth part of ∂K.

In two cases, which are of special interest in the following, it turns out that K_T is in fact convex. First, let $K \in \mathcal{D}_0^n$ be such that $K = [L, -t\, e_n, t\, e_n]$, $L \in \mathcal{K}_0^{n-1}(e_n^\perp)$, $t > 0$. Observe that K_T is convex for $T = T(e_n, l\, v)$ if $l \geq 0$ is small, for example, if $K_T \subset L + t[-e_n, e_n]$. Hence there is some constant l_0 depending on L, t and v such that K_T is convex for $0 \leq l \leq l_0$. Second, let $K = L \cap Q \in \bar{\mathcal{K}}_0^n$ where $L \in \mathcal{K}_0^n$ and Q is a polyhedral cone. Observe that $K_{T(u,lv)}$ is convex for all $l \in \mathbb{R}$ if $u^\perp \cap K$ is the origin.

Let Φ be an $\mathrm{SL}(n)$ invariant valuation on $\mathcal{D}_0^n$, resp. $\bar{\mathcal{K}}_0^n$. In the following we are interested in the behaviour of $\Phi(K_{T(u,v)})$ as a function of v. Without loss of generality set $u = e_n$, then fix $v \in e_n^\perp$ and $K \in \mathcal{D}_0^n$ or $K \in \bar{\mathcal{K}}_0^n$. Given two transvections $T = T(e_n, lv)$ and $T' = T'(e_n, l'v)$, $l, l' \geq 0$, we have

$$T' K_T = \bigcup_{s \in [l', l+l']} T(e_n, sv) K$$

and

$$K_{T,T'} = (K_T)_{T'} = K_{T(e_n,(l+l')v)}.$$

Thus for $l, l' \geq 0$,

$$K_{T'} \cap T' K_T = T' K, \;\; K_{T'} \cup T' K_T = K_{T,T'}. \tag{6}$$

Since Φ is a $\mathrm{SL}(n)$ invariant valuation, we obtain

$$\Phi\left(K_{T,T'}\right) = \Phi\left(K_T\right) + \Phi\left(K_{T'}\right) - \Phi\left(K\right)$$

as long as l, l' are chosen such that the occurring sets are convex. Setting $f(l) = \Phi(K_{T(e_n,lv)})$, we obtain

$$f(l + l') = f(l) + f(l') - f(0). \tag{7}$$

Hence $f(l) - f(0)$ is a solution of Cauchy's functional equation.

For the two special cases, we obtain the following results.

Lemma 7. *Let $K = L \cap Q \in \bar{\mathcal{K}}_0^n$ with $L \in \mathcal{K}_0^n$, $Q \in \mathcal{Q}^n$, and let $\Phi : \bar{\mathcal{K}}_0^n \to [0, \infty)$ be a $\mathrm{SL}(n)$ invariant valuation. If $T = T(u, v)$ is a transvection and $u^\perp \cap K = \{0\}$, then*

$$\Phi(K_T) \geq \Phi(K)$$

for every $v \in u^\perp$.

Proof. Since Φ and therefore f are non-negative, the solution of (7) is $f(l) = c\, l + \Phi(K)$ with a suitable constant c. Since $f \geq 0$ and l can be arbitrarily large, this implies $c \geq 0$. □

Lemma 8. *Let $K = [L, -t\, e_n, t\, e_n] \in \mathcal{D}_0^n$ with $L \in \mathcal{K}_0^{n-1}(e_n^\perp)$, $t > 0$, and let $\Phi : \mathcal{K}_0^n \to \mathbb{R}$ be an upper semicontinuous and $\mathrm{SL}(n)$ invariant valuation that* *vanishes on $\mathcal{P}_0^n$. If $T = T(e_n, l\,e_1)$ is a transvection and $L \cap e_1^\perp$ is a polytope, then there is a constant $c_0 \in \mathbb{R}$ depending only on L such that*

$$\Phi(K_T) = c_0\, l + \Phi(K)$$

for every $l \geq 0$ such that K_T is convex.

Proof. Since Φ is upper semicontinuous, also f is upper semicontinuous. The convexity of the sets occurring in (6) is ensured if $l + l' \leq l_0$. Hence for given L, the solution to (7) is given by $f(l) = c(t)\, l + \Phi(K)$ for $l \leq l_0$ where $c : (0, \infty) \to \mathbb{R}$. We prove that $c(t)$ is independent of t, that is, $c(t) = c_0$.

First, we calculate $f(2l) = \Phi(K_{T(e_n, 2le_1)})$. Let $T^+ = T(e_n, le_1)$ and $T^- = T(e_n, -le_1)$. Note that $K_{T^+,T^-} = T(e_n, -le_1) K_{T(e_n,2le_1)} = [L, \pm t\, T^+ e_n, \pm t\, T^- e_n]$. For $\eta > 0$, we define

$$\begin{aligned}
M_1 &= \left[L^+, t\, e_n, t\, T^+ e_n, -\eta\, e_1, -\eta\, e_n\right],\\
M_2 &= \left[L^+, -t\, e_n, -t\, T^- e_n, -\eta\, e_1, \eta\, T^- e_n\right],\\
M_3 &= \left[L^-, -t\, e_n, -t\, T^+ e_n, \eta\, e_1, \eta\, e_n\right],\\
M_4 &= \left[L^-, t\, e_n, t\, T^- e_n, \eta\, e_1, -\eta\, T^- e_n\right],
\end{aligned}$$

where $L^+ = L \cap \{x : x \cdot e_1 \geq 0\}$ and $L^- = L \cap \{x : x \cdot e_1 \leq 0\}$. Let $\eta > 0$ be so small that $K_{T^+,T^-} = M_1 \cup M_2 \cup M_3 \cup M_4$ and let $l > 0$ be so small that $\eta\, T^- e_n \in M_1$.

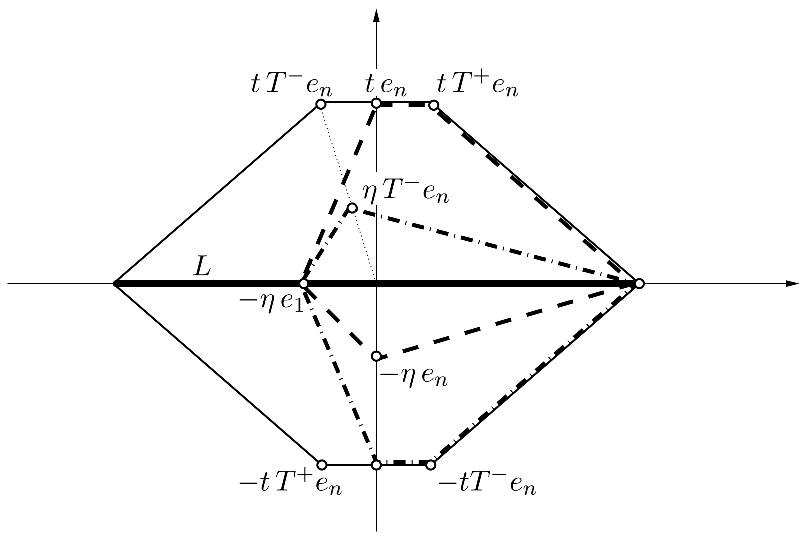

Figure 1: Definition of M_1 (– – –) and M_2 (–·–·–)

Since Φ is a $\mathrm{SL}(n)$ invariant valuation vanishing on polytopes and since $(M_1 \cup M_2) \cap (M_3 \cup M_4) = [L \cap e_1^\perp, \pm\eta e_1, \pm t e_n]$ is a polytope, we have

$$\begin{aligned} f(2\,l) &= \Phi(K_{T^+,T^-}) = \Phi(M_1 \cup M_2) + \Phi(M_3 \cup M_4) \\ &= \sum_{i=1}^4 \Phi(M_i) - \Phi(M_1 \cap M_2) - \Phi(M_3 \cap M_4) \\ &= \sum_{i=1}^4 \Phi(M_i) - \Phi([L^+, -\eta\, e_1, \eta\, T^- e_n, -\eta\, e_n]) - \Phi([L^-, \eta\, e_1, \eta\, e_n, -\eta\, T^- e_n]). \end{aligned}$$

Next, we calculate $f(l) = \Phi(K_{T^+})$. Note that $(M_1 \cup T^+ M_4) \cap (M_3 \cup T^+ M_2) = [L, \pm\eta\, e_n]$. Since Φ is a $\mathrm{SL}(n)$ invariant valuation vanishing on polytopes and since $M_1 \cap T^+ M_4 = [L \cap e_1^\perp, t\, T^+ e_n, \pm\eta\, e_1, t\, e_n, -\eta\, e_n]$ and $M_3 \cap T^+ M_2 = [L \cap e_1^\perp, -t\, T^+ e_n, \pm\eta\, e_1, \eta\, e_n, -t\, e_n]$ are polytopes, we have

$$\begin{aligned} f(l) &= \Phi(K_{T^+}) = \Phi(M_1 \cup T^+ M_4 \cup M_3 \cup T^+ M_2) \\ &= \Phi(M_1 \cup T^+ M_4) + \Phi(M_3 \cup T^+ M_2) - \Phi([L, \pm\eta e_n]) \\ &= \sum_{i=1}^4 \Phi(M_i) - \Phi([L, \pm\eta\, e_n]). \end{aligned}$$

Calculating $f(2\,l) - f(l)$, we obtain

$$c(t)\, l = \Phi([L, \pm\eta\, e_n]) - \Phi([L^+, -\eta\, e_1, \eta\, T^- e_n, -\eta\, e_n]) - \Phi([L^-, \eta\, e_1, \eta\, e_n, -\eta\, T^- e_n]).$$

The right hand side does not depend on t. Therefore $c(t) = c_0$ is a constant. $\square$

The $\mathrm{SL}(n)$ shaping process $K \mapsto K_{T(u,v)}$ behaves in a certain sense similar to the Minkowski addition of a line segment, $K \mapsto K + I$: at each boundary point, the set $K_{T(u,v)}$ is touched from within by an interval parallel to v. In analogy to the fact that Minkowski sums of suitable intervals approximate ellipsoids, we prove in the following that the $\mathrm{SL}(n)$ shaping process with suitable defined transvections can be used to approximate ellipsoids. Let E_λ be the ellipsoid of revolution with semi-major axis of length λ in direction e_n and semi-minor axes of length λ^{-1} in directions $e_1, \ldots, e_{n-1}$. Let $\gamma > 0$.

Lemma 9. *Suppose C is a convex cone such that $e_n \cdot u \geq \gamma$ for any point $u \in C \cap S^{n-1}$. Then there are constants $\alpha(C)$, $\beta(C)$, $\lambda(C)$ such that for $\lambda \geq \lambda(C) > 1$ and k sufficiently large there are transvections $T_{k1} = T(u_{k1}, v_{k1}), \ldots, T_{km_k} = T(u_{km_k}, v_{km_k})$ such that the following holds: $m_k \leq k\, \beta(C)/\lambda^2$, $e_n \cdot u_{kj} \geq \frac{8}{9}\gamma$, $|v_{kj}| \leq \alpha(C)/(k\,\lambda^2)$, and $[0, e_n]_{T_{k1},\ldots,T_{kj}} \subset B^n$ for $j = 1, \ldots, m_k$, and*

$$[0, \lambda\, e_n]_{T_{k1},\ldots,T_{km_k}} \cap D_\lambda \xrightarrow{t} E_\lambda \cap D_\lambda \quad \text{as } k \to \infty$$

where the convex cone D_λ is chosen such that $N(E_\lambda, D_\lambda) = C$.

Proof. We use induction on the dimension and prove the slightly stronger statement that in addition $[0, \lambda\, e_n]_{T_{k1},\ldots,T_{km_k}}$ is the convex hull of the origin and finitely many points on ∂E_λ. We start with the case $n = 2$, where given a vector v, the transvection $T(u, v)$ is uniquely determined by v, if we choose $u = u(v)$ as the unit vector in the orthogonal complement of v that has a smaller angle with e_2.

We introduce the following parametrization: the ellipse $E_\lambda : \lambda^2 x_1^2 + \lambda^{-2} x_2^2 = 1$ is parametrized by $(x_1, x_2) = (-\lambda^{-1}\sin\theta, \lambda\cos\theta)$, $\theta \in (-\pi, \pi]$. For $k = 1, 2, \dots$ and $i \in \{-k+1, \dots, k\}$, set $p_i = (-\lambda^{-1}\sin(\pi i/k), \lambda\cos(\pi i/k))$. Given the convex cone C, let D_λ be such that $N(E_\lambda, D_\lambda) = C$ and let $\widehat{k}$ be the smallest positive integer such that

$$D_\lambda \subset \operatorname{cone}([0, p_{-\widehat{k}}, \dots, p_{\widehat{k}}]).$$

The tangent line to E_λ at a point (x_1, x_2) has normal vector $(\lambda^2 x_1, \lambda^{-2} x_2) = (-\lambda\sin\theta, \lambda^{-1}\cos\theta)$. Since $e_2 \cdot u \geq \gamma$ for any point $u \in C \cap S^{n-1}$, we have

$$\left(\lambda^2 \sin^2\theta + \lambda^{-2}\cos^2\theta\right)^{-\frac{1}{2}} \lambda^{-1}\cos\theta \geq \gamma, \text{ i.e., } \tan^2\theta \leq \frac{1-\gamma^2}{\lambda^4\gamma^2}$$

for all $(x_1, x_2) \in D_\lambda \cap E_\lambda$. This shows that there is a constant $\alpha_1(C)$ such that

$$\frac{\widehat{k}}{k} \leq \frac{\alpha_1(C)}{\lambda^2}. \tag{8}$$

By definition, as $k \to \infty$,

$$[0, p_{-\widehat{k}}, \dots, p_{\widehat{k}}] \cap D_\lambda \xrightarrow{t} E_\lambda \cap D_\lambda. \tag{9}$$

Now there are vectors n_i, w_i, $i = -\widehat{k}+1, \dots, \widehat{k}$, such that the transvection $T(n_i, w_i)$ maps p_{i-1} to p_i. The vector n_i is the unit normal vector to $p_i - p_{i-1}$, and

$$w_i = \frac{p_i - p_{i-1}}{p_{i-1} \cdot n_i} = \frac{p_i - p_{i-1}}{d_i}$$

where $d_i > 0$ is the distance of the line containing p_{i-1}, p_i to the origin. We get

$$[0, \lambda\, e_2]_{T(n_1,w_1),\dots,T(n_{\widehat{k}},w_{\widehat{k}}),T(n_0,-w_0),\dots,T(n_{-(\widehat{k}-1)},-w_{-\widehat{k}+1})} = [0, p_{-\widehat{k}}, \dots, p_{\widehat{k}}] \tag{10}$$

if the left hand side is a convex set. The convexity is guaranteed if the distance of $[p_0, \dots, p_{\widehat{k}}]$ to the lines containing the origin and parallel to $p_i - p_{i-1}$ is positive for $i = 0, \dots, -\widehat{k}+1$. This is the case if the outer unit normal vector $n^E_{-\widehat{k}}$ of E_λ at $p_{-\widehat{k}}$ satisfies $n^E_{-\widehat{k}} \cdot p_{\widehat{k}} > 0$. By

$$n^E_{-\widehat{k}} = \left(\lambda^2 \sin^2\tfrac{\pi\widehat{k}}{k} + \lambda^{-2}\cos^2\tfrac{\pi\widehat{k}}{k}\right)^{-\frac{1}{2}} \begin{pmatrix} \lambda\sin\frac{\pi\widehat{k}}{k} \\ \lambda^{-1}\cos\frac{\pi\widehat{k}}{k} \end{pmatrix} \tag{11}$$

and by (8) we obtain that $n^E_{-\widehat{k}} \cdot p_{\widehat{k}} > 0$ for $\lambda \geq \lambda(C)$ if $\lambda(C)$ is chosen suitably large.

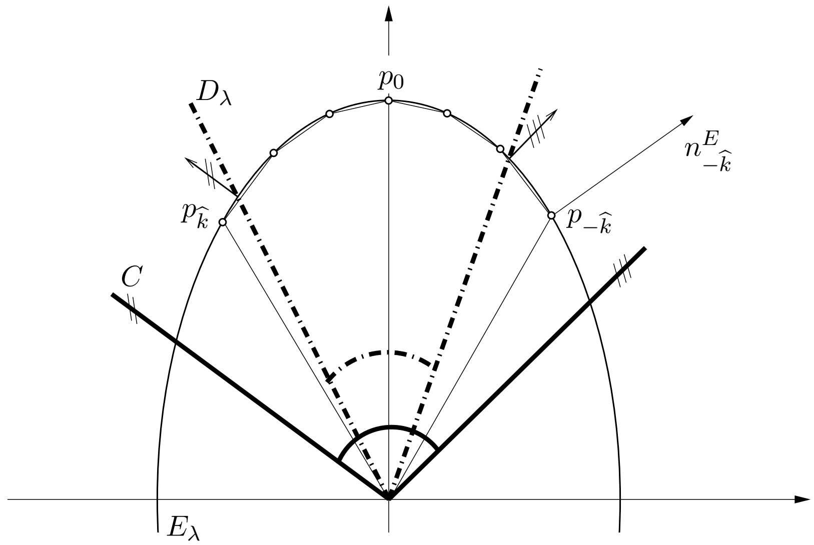

Figure 2: The convexity of (10)

Using (11) and (8), we obtain for $-\widehat{k}+1 \leq i \leq \widehat{k}$,

$$d_i \geq n^E_{-\widehat{k}} \cdot p_{-\widehat{k}} = \left(\lambda^2 \sin^2 \tfrac{\pi \widehat{k}}{k} + \lambda^{-2} \cos^2 \tfrac{\pi \widehat{k}}{k}\right)^{-\frac{1}{2}} \geq \lambda\, \alpha_2(C)$$

with a suitable constant $\alpha_2(C)$, and

$$|p_i - p_{i-1}| = 2 \sin \frac{\pi}{2k} \left| \left(\lambda^{-1} \cos \frac{\pi(2i-1)}{2k}, -\lambda \sin \frac{\pi(2i-1)}{2k} \right) \right| \leq \frac{\alpha_3(C)}{k\,\lambda}$$

for k sufficiently large with a suitable constant $\alpha_3(C)$. Hence there is a constant $\alpha_4(C)$ such that

$$|w_i| \leq \frac{\alpha_4(C)}{k\lambda^2}$$

for k sufficiently large. We set $m_k = 2\widehat{k}$ and define $T_{k1} = T(u_{k1}, v_{k1}) = T(n_1, w_1)$, ..., $T_{km_k} = T(u_{km_k}, v_{km_k}) = T(n_{-\widehat{k}+1}, -w_{-\widehat{k}+1})$ according to (10). Since $\lim_{k\to\infty} p_{\widehat{k}} \in D_\lambda$, for k sufficiently large $e_n \cdot u_{kj} \geq \frac{8}{9}\gamma$. This completes the proof of the lemma in the case $n = 2$.

Assume that the lemma has been proved in dimension $(n-1)$. Given the convex cone C, there are hyperplanes H_1, H_2 supporting C and containing the basis vectors $e_2, \ldots, e_{n-1}$ such that $C \subset H_1^+ \cap H_2^-$. Denote by U_φ the rotation about the axis $H_1 \cap H_2$ with angle φ, and set

$$\tilde{C} = \bigcup_{\varphi \in [-\pi/2, \pi/2]} U_\varphi C \cap e_1^\perp$$

which is the 'spherical projection' of C to $e_1^\perp$.

Since for any point $u \in \tilde{C} \cap S^{n-1}$ we have $e_n \cdot u \geq \gamma$, by induction the following holds for λ and k sufficiently large: there are points $p_{k0}, \ldots, p_{k\tilde{m}_k}$ and transvections $T_{k1} = T(u_{k1}, v_{k1}), \ldots, T_{k\tilde{m}_k} = T(u_{k\tilde{m}_k}, v_{k\tilde{m}_k})$ with $|v_{kj}| \leq \tilde{\alpha}_1(C)/(k\,\lambda^2)$, $\tilde{m}_k \leq k\,\tilde{\alpha}_4(C)/\lambda^2$, $e_n \cdot u_{kj} \geq \frac{8}{9}\gamma$ such that

$$[0, \lambda\, e_n]_{T_{k1},\ldots,T_{k\tilde{m}_k}} = [0, p_{k0}, \ldots, p_{k\tilde{m}_k}] \tag{12}$$

and as $k \to \infty$

$$[0, \lambda\, e_n]_{T_{k1},\dots,T_{k\tilde{m}_k}} \cap \tilde{D}_\lambda \xrightarrow{t} E_\lambda \cap \tilde{D}_\lambda \tag{13}$$

where $N(E_\lambda, \tilde{D}_\lambda) = \tilde{C}$.

We use the paramatrization $(x_1, x_n) = (-\lambda^{-1}\sin\theta, \lambda\cos\theta)$, $\theta \in (-\pi, \pi]$ now in the linear hull of e_1, e_n. As in the case $n = 2$, for λ sufficiently large there are points p_i and vectors n_i, w_i, $i = -\widehat{k}+1, \dots \widehat{k}$, in the linear hull of e_1, e_n such that the transvection $T(n_i, w_i)$ maps p_{i-1} to p_i and leaves the hyperplane spanned by $w_i, e_2, \dots, e_{n-1}$ invariant. Note that as in the planar case there are constants $\alpha_1(C)$ and $\alpha_4(C)$ such that

$$\frac{\widehat{k}}{k} \le \frac{\alpha_1(C)}{\lambda^2} \quad \text{and} \quad |w_i| \le \frac{\alpha_4(C)}{k\lambda^2},$$

and that $e_n \cdot n_i \ge \frac{8}{9}\gamma$ for k sufficiently large. Define the hyperplane H_θ as the linear hull of $e_2, \dots, e_{n-1}$ and $(-\lambda^{-1}\sin\theta, 0, \dots, 0, \lambda\cos\theta)$. Observe that $T(n_i, w_i)$ maps $H_{\pi(i-1)/k} \cap E_\lambda$ onto $H_{\pi i/k} \cap E_\lambda$ and thus the points $p_{k0}, \dots, p_{k\tilde{m}_k}$ successively to points on $H_{\pi i/k} \cap E_\lambda$, $i = -\widehat{k}, \dots, \widehat{k}$. (This is easy to see if E_λ is a ball, applying an affinity shows this for arbitrary λ.) Thus it follows from (9), (10), (12), and (13) that

$$[0, \lambda\, e_n]_{T_{k1},\dots,T_{k\tilde{m}_k}, T(n_1,w_1),\dots,T(n_{\widehat{k}},w_{\widehat{k}}), T(n_0,-w_0),\dots,T(n_{-\widehat{k}+1},-w_{-\widehat{k}+1})} \cap D_\lambda \xrightarrow{t} E_\lambda \cap D_\lambda \tag{14}$$

as $k \to \infty$. We set $m_k = \tilde{m}_k + 2\widehat{k}$ and define $T_{\tilde{m}_k+1} = T(u_1, w_1)$, $\dots$, $T_{m_k} = T(n_{-\widehat{k}+1}, -w_{-\widehat{k}+1})$ according to (14). This completes the proof of the lemma. $\square$

3 An Extension Result

We say that a functional $\Phi : \mathcal{K}_0^n \to [0, \infty)$ is *absolutely continuous on some subset* $\mathcal{L}^n \subset \mathcal{K}_0^n$ if there exists a constant c such that

$$\Phi(L) \le c\, V(L) \quad \text{for every } L \in \mathcal{L}^n.$$

A valuation Φ is called *simple* if $\Phi(K) = 0$ for every K of dimension less than n. We say that $Q_1, \dots, Q_k \in \mathcal{Q}^n$ dissect $Q \in \mathcal{Q}^n$ if $Q = Q_1 \cup \dots \cup Q_k$ and the cones $Q_1, \dots, Q_k$ have pairwise disjoint interiors. We call a simple valuation $\Phi : \bar{\mathcal{K}}_0^n \to [0, \infty)$ *finitely additive* if for every $K \in \bar{\mathcal{K}}_0^n$ we have

$$\Phi(K \cap Q) = \Phi(K \cap Q_1) + \dots + \Phi(K \cap Q_k)$$

when $Q \in \mathcal{Q}^n$ is dissected into $Q_1, \dots, Q_k \in \mathcal{Q}^n$.

The main result of this section is the following proposition.

Proposition 10. *Every valuation $\Phi : \mathcal{K}_0^n \to [0, \infty)$ that is absolutely continuous on $\mathcal{D}_0^n$ can be extended to a simple, finitely additive valuation $\Phi : \bar{\mathcal{K}}_0^n \to [0, \infty)$.*

The proof is contained in the following two lemmas. Recall that $\mathcal{Q}_j^n$ is the set of cones $Q \in \mathcal{Q}^n$ bounded by at most j hyperplanes containing the origin with linearly independent normal vectors, and that $\mathcal{K}_j^n$ is the set of convex bodies K such that either $K \in \mathcal{K}_0^n$ or there exist $K_0 \in \mathcal{K}_0^n$ and a cone $Q \in \mathcal{Q}_j^n$ such that $K = K_0 \cap Q$.

The following lemma is a refinement of results in [32] and [34].

Lemma 11. *Every valuation $\Phi : \mathcal{K}_0^n \to [0, \infty)$ that is absolutely continuous on $\mathcal{D}_0^n$ can be extended to a simple valuation $\Phi : \mathcal{K}_n^n \to [0, \infty)$.*

Proof. Since Φ is absolutely continuous on $\mathcal{D}_0^n$, there is a constant c such that

$$\Phi(D) \le c\, V(D) \quad \text{for every } D \in \mathcal{D}_0^n. \tag{15}$$

Recall that $\mathcal{D}_j^n$ is the set of n-dimensional convex bodies $D = [K \cap H, u, v]$ where $K \in \mathcal{K}_j^n$, H is a hyperplane with $K \cap H^+, K \cap H^- \in \mathcal{K}_{j+1}^n$, and where $u, v \in K \cap H_1 \cap \dots \cap H_i$, $u \in H^+ \backslash H$, $v \in H^- \backslash H$, see (2).

On $\mathcal{K}_j^n$, $j = 1, \dots, n$, we define Φ inductively, starting with $j = 1$, in the following way. The functional Φ is already defined for $K \in \mathcal{K}_{j-1}^n$. Set

$$\Phi(K) = 0 \text{ for } K \in \mathcal{K}_j^n \text{ with } \dim K < n. \tag{16}$$

For $K \in \mathcal{K}_j^n \backslash \mathcal{K}_{j-1}^n$ (defined by (2) with $i = j$) with $\dim K = n$, set

$$\Phi(K) = \lim_{u \to 0} \Phi([K, u]) \tag{17}$$

where $u \in H_j^- \backslash H_j$, $u \in H_1 \cap \dots \cap H_{j-1}$ (and note that $[K, u] \in \mathcal{K}_{j-1}^n$). This implies that Φ is non-negative. We show that Φ is well defined (that is, the limit in (17) exists and does not depend on the choice of H_j), that for $j = 1, \dots, n-1$,

$$\Phi(D) \le c\, V(D) \quad \text{for every } D \in \mathcal{D}_j^n \tag{18}$$

and that Φ has the following additivity properties for $j = 1, \dots, n$. If $K \in \mathcal{K}_{j-1}^n$ and H is a hyperplane such that $K \cap H^+, K \cap H^- \in \mathcal{K}_j^n \backslash \mathcal{K}_{j-1}^n$, then

$$\Phi(K) = \Phi(K \cap H^+) + \Phi(K \cap H^-). \tag{19}$$

And if $K, M, K \cap M, K \cup M \in \mathcal{K}_j^n \backslash \mathcal{K}_{j-1}^n$ are defined by (2) with the same cone Q, then

$$\Phi(K) + \Phi(M) = \Phi(K \cup M) + \Phi(K \cap M). \tag{20}$$

The functional Φ is well defined and a valuation on $\mathcal{K}_0^n$ and (15) holds. Suppose that Φ is well defined by (17) on $\mathcal{K}_{k-1}^n$ and that for $j = k-1$ (18), (19) (if $k > 1$), and (20) hold.

First, we show that the limit in (17) exists for $j = k$. Let $K \in \mathcal{K}_k^n \setminus \mathcal{K}_{k-1}^n$, $K = K_0 \cap H_1^+ \cap \dots \cap H_k^+$, $K_0 \in \mathcal{K}_0^n$, and let $u' \in H_1 \cap \dots \cap H_{k-1}$, $u' \in H_k^- \backslash H_k$ be chosen such that $[K, u] \subseteq [K, u']$ and $-u' \in K$. Applying (20) with $j = k-1$ gives

$$\Phi([K, u]) + \Phi([K \cap H_k, u', -u']) = \Phi([K, u']) + \Phi([K \cap H_k, u, -u']).$$

Since $[K \cap H_k, u', -u'], [K \cap H_k, u, -u'] \in \mathcal{D}_{k-1}^n$ and since (18) holds for $j = k-1$, this implies that

$$|\Phi([K, u]) - \Phi([K, u'])| \le c\, (V([K \cap H_k, u', -u']) + V([K \cap H_k, u, -u'])).$$

Consequently, the limit in (17) exists. If $k = 1$, this shows that Φ is well defined on $\mathcal{K}^n_k$. For $k > 1$ we show that $\Phi(K)$ as defined by (17) does not depend on the choice of the hyperplane H_k. Let $K \in \mathcal{K}^n_k \setminus \mathcal{K}^n_{k-1}$, $K = K_0 \cap H_1^+ \cap \cdots \cap H_k^+$, $K_0 \in \mathcal{K}^n_0$, and let $u \in H_1 \cap \cdots \cap H_{k-1}$, $u \in H_k^- \backslash H_k$. Choose u' in $H_2 \cap \cdots \cap H_k$, $u' \in H_1^- \backslash H_1$. Applying (19) for $j = k-1$ gives

$$\Phi([K, u, u']) = \Phi([K, u, u'] \cap H_k^+) + \Phi([K, u, u'] \cap H_k^-).$$

We have $[K, u, u'] \cap H_k^- = [K \cap H_k, u, u']$ and $u' \in H_k$. Since $[K \cap H_k, u, u'] \in \mathcal{D}^{k-1}_0$, (18) implies that $\lim_{u,u' \to 0} \Phi([K \cap H_k, u, u']) = 0$. Combined with $[K, u, u'] \cap H_k^+ = [K, u']$, we get $\lim_{u,u' \to 0} \Phi([K, u, u']) = \lim_{u' \to 0} \Phi([K, u'])$. Similarly, we get $\lim_{u,u' \to 0} \Phi([K, u, u']) = \lim_{u \to 0} \Phi([K, u])$. Thus Φ is well defined on $\mathcal{K}^n_k$.

Note that since $\Phi(D) \le c\, V(D)$ for $D \in \mathcal{D}^n_{k-1}$, definition (17) implies that for $k < n$

$$\Phi(D) \le c\, V(D) \quad \text{for every } D \in \mathcal{D}^n_k. \tag{21}$$

Next, we show that (19) holds for $j = k \le n$. By the induction hypothesis, (19) holds for $K \in \mathcal{K}^n_{k-2}$. Let $K \in \mathcal{K}^n_{k-1} \setminus \mathcal{K}^n_{k-2}$, that is, there exist $K_0 \in \mathcal{K}^n_0$ and hyperplanes $H_1, \dots, H_{k-1}$ such that $K = K_0 \cap H_1^+ \cap \cdots \cap H_{k-1}^+$. If we choose $u \in H_1 \cap \cdots \cap H_{k-1}$ such that $u \in K \cap H^+ \backslash H$ and $-u \in K \cap H^-$, then K, $[K \cap H, u, -u]$, $[K \cap H^+, -u]$, $[K \cap H^-, u]$ are in $\mathcal{K}^n_{k-1}$ and have the hyperplanes $H_1, \dots, H_{k-1}$ in common. Applying (20) for $j = k-1$ gives

$$\Phi(K) + \Phi([K \cap H, u, -u]) = \Phi([K \cap H^+, -u]) + \Phi([K \cap H^-, u]).$$

By (21) and definition (17), this implies that (19) holds for $j = k$.

Finally, we show that (20) holds for $j = k$. Choose $u \in H_1 \cap \cdots \cap H_{k-1}$, $u \notin H_k$ such that $-u \in K \cap M$. Applying (20) for $j = k-1$ shows that

$$\Phi([K, u]) + \Phi([M, u]) = \Phi([K \cup M, u]) + \Phi([K \cap M, u]).$$

Because of definition (17) this implies that (20) holds for $j = k$. The induction is now completed and Φ is defined on $\mathcal{K}^n_n$.

As last step of the proof, we show that Φ is a simple valuation on $\mathcal{K}^n_n$. That Φ is simple follows from (16). To show that Φ is a valuation, first note that for $Q \in \mathcal{Q}^n_n$ fixed, $\Phi(\,\cdot \cap Q)$ is a valuation on $\mathcal{K}^n_0$ by (20). Next, we show that for $K_0 \in \mathcal{K}^n_0$ fixed, $\Phi(K_0 \cap \cdot\,)$ is a valuation on $\mathcal{Q}^n_n$, that is,

$$\Phi(K_0 \cap Q) + \Phi(K_0 \cap Q') = \Phi(K_0 \cap (Q \cup Q')) + \Phi(K_0 \cap (Q \cap Q')) \tag{22}$$

for $Q, Q', Q \cup Q' \in \mathcal{Q}^n_n$.

If one of the cones Q, Q' is contained in the other, then (22) clearly holds. So suppose that $Q \not\subset Q'$ and $Q' \not\subset Q$. Since $Q \cup Q'$ is convex and the polyhedral cones are bounded by at most n hyperplanes containing the origin, there is a hyperplane H such that $Q' \subset H^-$ and $Q \backslash Q' \subset H^+$. Hence $Q \cap Q' = Q \cap H^-$ and $Q \backslash Q' = Q \cap H^+$, and (22) will follow from

$$\Phi(K_0 \cap Q) = \Phi(K_0 \cap (Q \cap H^+)) + \Phi(K_0 \cap (Q \cap H^-))$$

and

$$\Phi(K_0 \cap (Q \cup Q')) = \Phi(K_0 \cap Q') + \Phi(K_0 \cap (Q \cap H^+)).$$

Thus the following additivity property is sufficient to prove (22). If a cone $Q \in \mathcal{Q}_n^n$ is dissected by a hyperplane H into two cones $Q \cap H^+, Q \cap H^- \in \mathcal{Q}_n^n$, and thus the convex set $K = K_0 \cap Q \in \mathcal{K}_n^n$ is dissected into $K \cap H^+, K \cap H^- \in \mathcal{K}_n^n$, then

$$\Phi(K) = \Phi(K \cap H^+) + \Phi(K \cap H^-). \tag{23}$$

Since the case $K \in \mathcal{K}_{j-1}^n$ and $K \cap H^+, K \cap H^- \in \mathcal{K}_j^n \backslash \mathcal{K}_{j-1}^n$ is already proved by (19), it remains to prove (23) for $K, K \cap H^+, K \cap H^- \in \mathcal{K}_j^n \backslash \mathcal{K}_{j-1}^n$, $j \leq n$. This can be seen in the following way.

Let $K = K_0 \cap Q = K_0 \cap H_1^+ \cap \cdots \cap H_j^+$. Since $Q \cap H^+, Q \cap H^- \in \mathcal{Q}_j^n$, the hyperplane H contains the intersection of two boundary hyperplanes of Q, that is, without loss of generality $H_1 \cap H_2 \subset H$. So $K \cap H^+$ and $K \cap H^-$ are bounded by $H_1, H, H_3, \ldots, H_j$ and $H, H_2, H_3, \ldots, H_j$, respectively. Let $u \in H \cap H_3 \cap \cdots \cap H_j$, $u \in H_1^- \cap H_2^-$. By (19), we have

$$\Phi([K, u]) = \Phi([K \cap H^+, u]) + \Phi([K \cap H^-, u])$$

and because of (17)

$$\Phi(K \cap H^+) = \lim_{u \to 0} \Phi([K \cap H^+, u]), \ \ \Phi(K \cap H^-) = \lim_{u \to 0} \Phi([K \cap H^-, u]).$$

Further, by (19)

$$\begin{aligned} \Phi([K, u]) &= \Phi([K, u] \cap H_1^+) + \Phi([K, u] \cap H_1^-) \\ &= \Phi(K) + \Phi([K, u] \cap H_1^+ \cap H_2^-) + \Phi([K, u] \cap H_1^-). \end{aligned}$$

Note that (18) and (19) imply that $\Phi([L \cap H, u]) \leq c\, V([L \cap H, u])$ for each pyramid $[L \cap H, u]$, $L \in \mathcal{K}_n^n$. In particular, we have

$$\lim_{u \to 0} \Phi([K, u] \cap H_1^-) = \lim_{u \to 0} \Phi([K, u] \cap H_2^-) = 0$$

and by (19)

$$0 \leq \lim_{u \to 0} \Phi([K, u] \cap H_1^+ \cap H_2^-) \leq \lim_{u \to 0} \Phi([K, u] \cap H_2^-) = 0.$$

Combined these equations imply (23).

Next, we derive the following auxiliary result, where we write $Q_L = \mathrm{cone}(L)$ for $L \in \mathcal{K}_n^n$. If $L, L' \in \mathcal{K}_n^n$ differ only within a cone $Q \in \mathcal{Q}_n^n$ and $Q \subset Q_L = Q_{L'}$, then

$$\Phi(L) - \Phi(L') = \Phi(L \cap Q) - \Phi(L' \cap Q). \tag{24}$$

To prove this, note that there are hyperplanes such that

$$Q_L = H_1^+ \cap \cdots \cap H_i^+ \cap \cdots \cap H_j^+, \ Q = H_1^+ \cap \cdots \cap H_i^+ \cap \tilde{H}_{i+1}^+ \cap \cdots \cap \tilde{H}_k^+.$$

This implies that there are $L_i, L_i' \in \mathcal{K}_i^n$ which are bounded by $H_1, \ldots, H_i$ and differ only within Q such that

$$L = L_i \cap H_{i+1}^+ \cap \cdots \cap H_j^+, \ L' = L_i' \cap H_{i+1}^+ \cap \cdots \cap H_j^+.$$

It follows from (19) that

$$\Phi(L_i) - \Phi(L_i') = \Phi(L_i \cap H_{i+1}^+) - \Phi(L_i' \cap H_{i+1}^+) = \cdots = \Phi(L) - \Phi(L')$$

and

$$\Phi(L_i) - \Phi(L_i') = \Phi(L_i \cap \tilde{H}_{i+1}^+) - \Phi(L_i' \cap \tilde{H}_{i+1}^+) = \cdots = \Phi(L \cap Q) - \Phi(L' \cap Q).$$

Combined these equations prove (24).

Finally, let $M, K, M \cup K \in \mathcal{K}_n^n$, and set $Q_M = \text{cone}(M)$, $Q_K = \text{cone}(K)$, and $Q = Q_M \cap Q_K$. By (24) (with $L = M$, $L' = (M \cup K) \cap Q_M$) and by (20) we have

$$\begin{aligned} \Phi(M) - \Phi((K \cup M) \cap Q_M) &= \Phi(M \cap Q) - \Phi((K \cup M) \cap Q) \\ &= -\Phi(K \cap Q) + \Phi(K \cap M) \end{aligned}$$

and the same holds if the roles of M and K are interchanged. Combining this with (20) and (22) we obtain

$$\begin{aligned} \Phi(M) + \Phi(K) - \Phi(K \cap M) &= \Phi((K \cup M) \cap Q_M) + \Phi((K \cup M) \cap Q_K) \\ &\quad -\Phi(K \cap Q) - \Phi(M \cap Q) + \Phi(K \cap M) \\ &= \Phi((K \cup M) \cap Q_M) + \Phi((K \cup M) \cap Q_K) \\ &\quad -\Phi((K \cup M) \cap Q) \\ &= \Phi(K \cup M) \end{aligned}$$

which shows that Φ is a valuation on $\mathcal{K}_n^n$. This completes the proof of the lemma. □

Lemma 12. *Every simple valuation $\Phi : \mathcal{K}_n^n \to [0, \infty)$ can be extended to a simple and finitely additive valuation $\Phi : \bar{\mathcal{K}}_0^n \to [0, \infty)$.*

Proof. Let $K = K_0 \cap Q \in \bar{\mathcal{K}}_0^n$ with $K_0 \in \mathcal{K}_0^n$ and $Q \in \mathcal{Q}^n$. Let Q be full dimensional and let Q be dissected into full dimensional simplicial cones $Q_1, \ldots, Q_k \in \mathcal{Q}^n$ and set $K_i = K \cap Q_i \in \mathcal{K}_n^n$. If $K \notin \mathcal{K}_n^n$, we define

$$\Phi(K) = \Phi(K_1) + \cdots + \Phi(K_k) \tag{25}$$

and show that this definition does not depend on the particular dissection of Q. If $K \in \mathcal{K}_n^n$, then both sides of (25) are defined and we show that (25) holds.

First, suppose that Q does not contain any linear subspace. Then there exists a suitable affine hyperplane H not containing the origin such that $R = K \cap H = Q \cap H$ is an $(n-1)$-dimensional polytope and each $S_i = Q_i \cap H$, $i = 1, \ldots, k$, is an $(n-1)$-dimensional simplex. We need the following notions (see [36]). A finite set of $(n-1)$-dimensional simplices αR is called a *triangulation* of an $(n-1)$-dimensional polytope R if the simplices have pairwise disjoint interiors (relative to H) and their union equals R. An *elementary move* applied to αR is one of the two following operations: a simplex $S \in \alpha R$ is dissected into two $(n-1)$-dimensional simplices S_1, S_2 by an $(n-2)$-dimensional plane containing an $(n-3)$-dimensional face of S; or the reverse, that is, two simplices

$S_1, S_2 \in \alpha R$ are replaced by $S = S_1 \cup S_2$ if S is again a simplex. It is shown in [36] that for every two triangulations αR and $\alpha' R$, there are finitely many elementary moves that transform αR into $\alpha' R$.

Let $\{Q_1, \dots, Q_k\}$ and $\{Q'_1, \dots, Q'_{k'}\}$ be two dissections of Q into simplicial cones and let αR and $\alpha' R$ be the corresponding triangulations. Hence $\alpha R = \{S_1, \dots, S_k\}$, where for the $(n-1)$-dimensional simplices $S_i \in \alpha R$, we have $S_i = K_i \cap H$. If S_i is dissected by an $(n-2)$-dimensional plane $E \subset H$ corresponding to an elementary move into S_i^1, S_i^2, then K_i is dissected by the cone generated by E into $K_i^1, K_i^2 \in \mathcal{K}_n^n$. Since Φ is a simple valuation on $\mathcal{K}_n^n$, we obtain $\Phi(K_i) = \Phi(K_i^1) + \Phi(K_i^2)$. The same argument applies for the reverse move. Since the triangulation $\alpha' R$ can be transformed into the triangulation αR by finitely many elementary moves, using this argument repeatedly shows that (25) does not depend on the choice of the dissection of Q for $K \notin \mathcal{K}_n^n$ and that (25) holds for $K \in \mathcal{K}_n^n$.

Second, suppose that Q contains a linear subspace and let $K \in \mathcal{K}_n^n$. If the subspace is one-dimensional, we choose a suitable hyperplane H containing the origin such that $Q \cap H^+$ and $Q \cap H^-$ do not contain any linear subspace and $K \cap H^+, K \cap H^- \in \mathcal{K}_n^n$. Since Φ is a valuation on $\mathcal{K}_n^n$ and (25) holds for $K \cap H^+$ and $K \cap H^-$, we obtain

$$\Phi(K) = \Phi(K \cap H^+) + \Phi(K \cap H^-) = \sum_{i=1}^{k} (\Phi(K_i \cap H^+) + \Phi(K_i \cap H^-)) = \sum_{i=1}^{k} \Phi(K_i).$$

Using this argument repeatedly, proves that (25) holds for all $K \in \mathcal{K}_n^n$.

Third, suppose that Q contains a linear subspace and let $K \notin \mathcal{K}_n^n$. Given two dissections of Q into simplicial cones, there is always a common refinement of these two dissections. We have already shown that (25) holds in each simplicial cone. Thus the right hand side of (25) does again not depend on the particular dissection of Q and can be taken as a definition of Φ for $K \notin \mathcal{K}_n^n$.

Finally, using a standard dissection argument (as in [36]) it is easy to see that Φ is a finitely additive valuation on $\bar{\mathcal{K}}_0^n$. □

For the extension of Φ that was constructed in the previous proposition, the following two remarks hold true. If Φ is $\mathrm{SL}(n)$ invariant on $\mathcal{K}_0^n$, then the extended valuation is $\mathrm{SL}(n)$ invariant on $\bar{\mathcal{K}}_0^n$. If Φ is upper semicontinuous on $\mathcal{K}_0^n$ and $K_j \to K$ as $j \to \infty$ for $K, K_j \in \mathcal{K}_0^n$, then

$$\Phi(K \cap Q) \geq \limsup_{j \to \infty} \Phi(K_j \cap Q) \tag{26}$$

for every $Q \in \mathcal{Q}^n$.

4 Proof of Theorem 5

Let $\Phi : \mathcal{K}_0^n \to \mathbb{R}$ be an upper semicontinuous and $\mathrm{SL}(n)$ invariant valuation that vanishes on $\mathcal{P}_0^n$. Since every $K \in \mathcal{K}_0^n$ can be approximated by polytopes we have

$$\Phi(K) \geq 0$$

for all $K \in \mathcal{K}_0^n$. We define the function $\phi : [0,\infty) \to [0,\infty)$ by

$$\phi(t) = \frac{1}{n\, v_n}\, \Phi(t^{-\frac{1}{2n}}\, B^n)\, t^{\frac{1}{2}}, \tag{27}$$

that is, ϕ is determined by Φ in such a way that (1) holds for centered balls. The main part of the proof consists of showing that given any function $\phi : [0,\infty) \to [0,\infty)$, there it is at most one upper semicontinuous and $\mathrm{SL}(n)$ invariant valuation $\Phi : \mathcal{K}_0^n \to [0,\infty)$ that vanishes on $\mathcal{P}_0^n$ such that (27) holds.

The proof is by induction in the dimension n. The case $n = 1$ is trivial since $\mathcal{K}_0^1 = \mathcal{P}_0^1$. So, let $n \geq 2$ and assume that Theorem 5 holds in dimension $(n-1)$:

Assumption. *If $\Psi : \mathcal{K}_0^{n-1} \to [0,\infty)$ is an upper semicontinuous and $\mathrm{SL}(n-1)$ invariant valuation that vanishes on $\mathcal{P}_0^{n-1}$, then there is a concave function $\psi : [0,\infty) \to [0,\infty)$ with $\lim_{t\to 0} \psi(t) = \lim_{t\to\infty} \psi(t)/t = 0$ such that*

$$\Psi(K) = \int_{\partial K} \psi(\kappa_0(K,x))\, d\mu_K(x) \tag{28}$$

for every $K \in \mathcal{K}_0^{n-1}$.

We proceed as follows. Let $\Phi : \mathcal{K}_0^n \to [0,\infty)$ be an upper semicontinuous and $\mathrm{SL}(n)$ invariant valuation that vanishes on $\mathcal{P}_0^n$. In Section 4.1, we show that Φ is absolute continuous on $\mathcal{D}_0^n$. Thus Proposition 10 implies that Φ can be extended to $\bar{\mathcal{K}}_0^n$. So, let $\Phi : \bar{\mathcal{K}}_0^n \to [0,\infty)$ be the extended valuation.

Let $\mathcal{E}_0^n \subset \mathcal{K}_0^n$ be the family of convex bodies E which can be represented as

$$E = E_1 \cup \cdots \cup E_m,$$

where the E_i's have pairwise disjoint interiors and every E_i is the intersection of a convex polyhedral cone with either a polytope $P \in \mathcal{P}_0^n$ or a linear image of centered ball or a linear image of an unbounded circular cylinder. In Section 4.2, we show that if ϕ is given, then Φ is uniquely determined on $\mathcal{E}_0^n$. Since Φ is upper semicontinuous on $\mathcal{K}_0^n$, it follows that for every $K \in \mathcal{K}_0^n$,

$$\Phi(K) \geq \sup\{\limsup_{j\to\infty} \Phi(E_j) : E_j \in \mathcal{E}_0^n, E_j \to K\}.$$

In Section 4.3, we show that for every $K \in \mathcal{K}_0^n$ that is ε-smooth, $\varepsilon > 0$,

$$\Phi(K) = \sup\{\limsup_{j\to\infty} \Phi(E_j) : E_j \in \mathcal{E}_0^n, E_j \to K\}. \tag{29}$$

Thus, given ϕ, Φ is uniquely determined for ε-smooth convex bodies. In Section 4.4, we use this result to prove that (29) holds for every $K \in \mathcal{K}_0^n$. Thus, given ϕ, Φ is uniquely determined on $\mathcal{K}_0^n$.

Finally, we show in Section 4.5 that if $\Phi : \mathcal{K}_0^n \to [0,\infty)$ is an upper semicontinuous and $\mathrm{SL}(n)$ invariant valuation that vanishes on $\mathcal{P}_0^n$, then the function ϕ defined in (27) is concave and $\lim_{t\to 0} \phi(t) = \lim_{t\to\infty} \phi(t)/t = 0$. This completes the proof of the theorem.

4.1 Absolute continuity on $\mathcal{D}_0^n$

We identify the hyperplane spanned by $e_1, \dots, e_{n-1}$ with $\mathbb{R}^{n-1}$ and set $I = [-e_n, e_n]$. Note that for $K \in \mathcal{K}_0^{n-1}$, $[K, I] \in \mathcal{D}_0^n$. Define $\Psi : \mathcal{K}_0^{n-1} \to [0, \infty)$ by $\Psi(K) = \Phi([K, I])$. Observe that Ψ is an upper semicontinuous and $\mathrm{SL}(n-1)$ invariant valuation on $\mathcal{K}_0^{n-1}$ that vanishes on $\mathcal{P}_0^{n-1}$. Thus by the induction assumption, (28), there is a concave function $\psi : [0, \infty) \to [0, \infty)$ with $\lim_{t \to 0} \psi(t) = \lim_{t \to \infty} \psi(t)/t = 0$ such that

$$\Psi(K) = \int_{\partial K} \psi(\kappa_0(K, x))\, d\mu_K(x) \tag{30}$$

for every $K \in \mathcal{K}_0^{n-1}$. Since Φ is $\mathrm{SL}(n)$ invariant, $\Phi([K, t\, I]) = \Phi([t^{\frac{1}{n-1}} K, I])$ and it follows from (30) that

$$\Phi([K, t\, I]) = \Psi(t^{\frac{1}{n-1}} K) = t \int_{\partial K} \psi\big(t^{-2} \kappa_0(K, x)\big)\, d\mu_K(x). \tag{31}$$

The main result of this section is the absolute continuity of Φ on $\mathcal{D}_0^n$ stated in Proposition 14, which requires the following lemma.

Lemma 13. *Suppose that $\Phi : \mathcal{K}_0^n \to [0, \infty)$ is an upper semicontinuous and $\mathrm{SL}(n)$ invariant valuation that vanishes on $\mathcal{P}_0^n$, and define $\psi : [0, \infty) \to [0, \infty)$ by (30). Then there exist convex bodies $L_r \in \mathcal{K}_0^n$ and constants $r_0, c_0 > 0$ such that*

$$(1 - r^2)\, B^n \subset L_r \subset B^n \tag{32}$$

and

$$\Phi(L_r) \ge c_0\, \psi\Big(\frac{1}{r^2}\Big) \tag{33}$$

for every $r \in (0, r_0)$.

Proof. For $P \in \mathcal{P}_0^{n-1}$, set $K = P \cap B^{n-1}$. Further assume that $K \cap e_1^\perp$ is a polytope and $\pm e_1 \in K$. Recall from Section 2, that for transvections $T^+ = T(e_n, l e_1)$, $T^- = T(e_n, -l e_1)$, the set $[K, t\, I]_{T^+, T^-}$ is convex if it is contained in $K + t\, I$ which is the case for $0 \le l \le \frac{1}{t}$. Since $T^+ [K, t\, I]_{T^+, T^-} = [K, t\, I]_{T(e_n, 2 l e_1)}$ and $K \cap e_1^\perp$ is a polytope, Lemma 8 shows

$$\Phi([K, t\, I]_{T^+, T^-}) = 2 c_0 l + \Phi([K, t\, I]) \tag{34}$$

with c_0 depending only on K. We set $l = \frac{1}{t}$, use that Φ is non-negative, and obtain by (31) that

$$2 c_0 \ge -t\, \Phi([K, t\, I]) = -t^2 \psi(t^{-2})\, \mu_K(\{x \in \partial K : \kappa_0(K, x) = 1\})$$

for all $t > 0$. Since $\lim_{s \to \infty} \psi(s)/s = 0$, this implies that $c_0 \ge 0$. Combined with (34) this proves

$$\Phi([K, t\, I]_{T^+, T^-}) \ge \Phi([K, t\, I]). \tag{35}$$

To construct L_r, we choose $K_r = P \cap B^{n-1}$ with $P \in \mathcal{P}_0^{n-1}$ and $-e_1 \in K_r$ in the following way: the boundary of K_r contains the cap $S^{n-2} \cap B^{n-1}(e_1, r/2)$ and coincides with the boundary of P outside the cap $S^{n-2} \cap B^{n-1}(e_1, r)$

$$\begin{aligned} &\partial K_r \cap B^{n-1}(e_1, \tfrac{r}{2}) = S^{n-2} \cap B^{n-1}(e_1, \tfrac{r}{2}), \\ &\partial K_r \backslash B^{n-1}(e_1, r) = \partial K_r \cap \{x : x_1 \leq 1 - \tfrac{r^2}{2}\} = \partial P \cap \{x : x_1 \leq 1 - \tfrac{r^2}{2}\}. \end{aligned} \tag{36}$$

Applying the shaping process to $[K_r, t\, I]$ gives a set $[K_r, t\, I]_{T^+, T^-}$ which is convex for $0 \leq l \leq \frac{1}{t}$. We choose t and l such that the point $t\, T^+ e_n = t\, T(e_n, l e_1)\, e_n = t\, e_n + t\, l\, e_1$ lies on $S^{n-1} \cap \partial B^n(e_1, r)$. This is obtained by setting $t^2 = r^2(1 - r^2/4)$ and $l = (1 - r^2/2)/t$. By (31) and (36), we have

$$\Phi([K_r, t\, I]) \geq t\, \psi(t^{-2})\, \mu_{K_r}(P \cap S^{n-2}) \geq 2^{-n+2} v_{n-2}\, r^{n-2} t\, \psi(t^{-2}).$$

Since ψ is monotone increasing and $r/2 \leq t \leq r$, combined with (35) this implies

$$\Phi([K_r, t\, I]_{T^+, T^-}) \geq 2^{-n+1} v_{n-2}\, r^{n-1} \psi(r^{-2}) \tag{37}$$

for all K_r for which (36) holds.

To specify the polytope P, in addition to (36) we require that all supporting hyperplanes of the set $[K_r, \pm t(e_n + l\, e_1), \pm t(e_n - l\, e_1)]$ at boundary points x with $x_1 > 1 - r^2/2$ have distance at least $(1 - r^2)$ from the origin. That this is possible can be seen from the following elementary calculations: if x is on $S^{n-2} \cap \partial B^n(e_1, r)$, then the hyperplane H supporting S^{n-2} at x and containing the point $t(e_n + l\, e_1)$ is given by the equation $(x + t\, e_n) \cdot y = 1$, $y \in \mathbb{R}^n$, and thus has distance $(1 + r^2 - r^4/4)^{-1/2} > 1 - r^2/2$ from the origin. Hence we can choose a polytope P with $P \cap B^n(e_1, r)$ sufficiently close to $S^{n-2} \cap B^n(e_1, r)$ with the proposed property. For abbreviation, set

$$M_r = [K_r, t\, I]_{T^+, T^-} \cap \{x : x_1 \geq -\frac{1}{2}\}.$$

Observe that $\Phi(M_r) = \Phi([K_r, t\, I]_{T^+, T^-})$.

Let Q_r denote the $(n-1)$-dimensional polytope that is the intersection of $[K_r, t\, I]_{[-l, l]}$ and the hyperplane $x_1 = 1 - r^2$. By construction,

$$[M_r, (1 - r^2)\, B^n] = M_r \cup [Q_r, (1 - r^2)\, B^n].$$

Further it is easy to see that

$$[Q_r, (1 - r^2)\, B^n] \backslash (1 - r^2)\, B^n \subset B^n((1 - r^2) e_1, 3r).$$

In the last step, we take a dense packing of balls of radius $3r$ on $(1 - r^2) S^{n-1}$. Denote by x_i, $i = 1, \ldots, m_r$, the midpoints of the balls of this packing and by U_i the rotations such that $U_i(1 - r^2) e_1 = x_i$, $i = 1, \ldots, m_r$. Here $m_r = m_{S^{n-1}}(3r/(1 - r^2))$ and by (3), there is a constant $c_2 > 0$ such that

$$m_r \geq c_2\, r^{-(n-1)}. \tag{38}$$

Let

$$L_r = [\bigcup_{i=1}^{m_r} U_i M_r].$$

Note that the construction implies that (32) holds. That also (33) holds can be seen in the following way. For $i = 1, \dots, m_r - 1$, we have

$$U_i M_r \cap [\bigcup_{j=i+1}^{m_r} U_j M_r] \in \mathcal{P}_0^n.$$

Since Φ is a valuation that vanishes on $\mathcal{P}_0^n$, this implies

$$\Phi(L_r) = \Phi(U_1 M_r) + \Phi[\bigcup_{i=2}^{m_r} U_i M_r] = \cdots = \sum_{i=1}^{m_r} \Phi(U_i M_r).$$

Since Φ is rotation invariant, we obtain $\Phi(L_r) = m_r\, \Phi(M_r)$. Combined with (37) and (38), this proves (33). $\square$

Proposition 14. *Suppose that $\Phi : \mathcal{K}_0^n \to [0, \infty)$ is an upper semicontinuous and $\mathrm{SL}(n)$ invariant valuation that vanishes on $\mathcal{P}_0^n$. Then there is a constant $c = c(\Phi)$ such that*

$$\Phi(D) \le c\, V_n(D)$$

for every $D \in \mathcal{D}_0^n$.

Proof. First, we show that it suffices to prove that there is a constant $c = c(\Phi)$ such that

$$\Phi([K, u, -u]) \le c\, V_n([K, u, -u]) \tag{39}$$

for every $K \in \mathcal{K}_0^{n-1}$ and every $u \in \mathbb{R}^n \backslash \mathbb{R}^{n-1}$. This can be seen in the following way. Let $D = [K, u, v]$, $K \in \mathcal{K}_0^{n-1}$. For $0 < s, t < 1$ sufficiently small, we have $-t\, u \in [K, v]$ and $-s\, v \in [K, u]$. Since Φ is a non-negative valuation, it follows that

$$\begin{aligned} \Phi([K, u, v]) &= \Phi([K, u, -t\, u]) + \Phi([K, -s\, v, v]) - \Phi([K, -s\, v, -t\, u]) \\ &\le \Phi([K, u, -t\, u]) + \Phi([K, -s\, v, v]). \end{aligned} \tag{40}$$

For given t and u, the functional $K \mapsto \Phi([K, u, -t\, u])$ is an upper semicontinuous and $\mathrm{SL}(n-1)$ invariant valuation on $\mathcal{K}_0^{n-1}$ that vanishes on $\mathcal{P}_0^{n-1}$. Thus by the induction assumption, (28), there is a concave function $\psi_{u,t} : [0, \infty) \to [0, \infty)$ such that

$$\Phi([K, u, -t\, u]) = \int_{\partial K} \psi_{u,t}(\kappa_0(K, x))\, d\mu_K(x)$$

for every $K \in \mathcal{K}_0^{n-1}$. The right hand side does not change when K is replaced by $-K$. Therefore $\Phi([K, u, -t\, u]) = \Phi([-K, u, -t\, u])$. Since Φ is $\mathrm{SL}(n)$ invariant, this implies that $\Phi([K,u,-t\,u]) = \Phi([K,t\,u,-u])$. ¿From this and (39), it follows that

$$\begin{aligned}
\Phi([K,u,-t\,u]) &= \frac{1}{2}(\Phi([K,u,-t\,u]) + \Phi([K,t\,u,-u])) \\
&= \frac{1}{2}(\Phi([K,u,-u]) + \Phi([K,t\,u,-t\,u])) \\
&\le \frac{c}{2}(V_n([K,u,-u]) + V_n([K,t\,u,-t\,u])) = c\,V_n([K,u,-t\,u]).
\end{aligned}$$

Thus (39) and (40) imply

$$\Phi([K,u,v]) \le c(V_n([K,u,-t\,u]) + V_n([K,-s\,v,v]). \tag{41}$$

Since $t,s>0$ can be chosen arbitrarily small, it follows that $\Phi([K,u,v]) \le c\,V_n([K,u,v])$.

Next, we prove (39). Since Φ is $\mathrm{SL}(n)$ invariant, it suffices to show that there is a constant $c = c(\Phi)$ such that

$$\Phi([K,-t\,e_n,t\,e_n]) \le c\,V_n([K,-t\,e_n,t\,e_n])$$

for every $K \in \mathcal{K}_0^{n-1}$ and $t>0$.

We use the family L_r, $r>0$, of convex bodies constructed in Lemma 13. Note that by (32), $L_r \to B^n$ as $r \to 0$. Since Φ is upper semicontinuous, this implies that $\limsup_{r\to 0}\Phi(L_r) \le \Phi(B^n)$. Thus it follows from (33) and the concavity of ψ that there is a constant c_1 such that

$\psi(t) \le c_1$ for all $t>0$. Combined with (31) this implies that

$$\Phi([K,-t\,e_n,t\,e_n]) \le t\int_{\partial K} c_1\,d\mu_K(x) = (n-1)\,c_1\,t\,V_{n-1}(K).$$

This concludes the proof of the proposition. □

By Propositions 10 and 14, Φ can be extended to $\bar{\mathcal{K}}_0^n$. We also denote the extended valuation by Φ. The following lemma is used in Section 4.4.

Lemma 15. *Let $\Phi : \mathcal{K}_0^n \to [0,\infty)$ be an upper semicontinuous and $\mathrm{SL}(n)$ invariant valuation that vanishes on $\mathcal{P}_0^n$, and which is extended to $\bar{\mathcal{K}}_0^n$. For $\lambda>0$, let $K_\lambda, K \in \mathcal{K}_0^n$, $Q_\lambda, Q \in \mathcal{Q}_n$ and $Q \subset Q_\lambda$. If $K_\lambda \to K \in \mathcal{K}_0^n$ and $Q_\lambda \cap B^n \to Q \cap B^n$ as $\lambda \to \infty$, then given $\eta>0$ there exists a constant λ_0 such that*

$$\Phi(K_\lambda \cap Q_\lambda) \le \Phi(K_\lambda \cap Q) + \eta$$

for every $\lambda \ge \lambda_0$.

Proof. Let $Q = \bigcap_{j=1}^k H_j^+$. Set $C_{\lambda,j} = K_\lambda \cap Q_\lambda \cap H_j^-$. Since Φ is non-negative it suffices to prove $\Phi(C_{\lambda,j}) \le \frac{1}{k}\eta$ for λ sufficiently large and $j = 1,\dots,k$. Let $C_j = \lim_{\lambda\to\infty} C_{\lambda,j} = K \cap Q \cap H_j$. We choose $u \in H_j$ and $v \notin H_j$ such that $[C_j,u,-v,v] \in \mathcal{K}_0^n$. Proposition 14 implies that there is a constant c such that for all $j = 1,\dots k$

$$\Phi([C_j,u,-v,v]) \le c\,V_n([C_j,u,-v,v]) \le \frac{\eta}{2k}$$

for $u \in H$ and $v \notin H$ suitably small. Since Φ is a non-negative simple valuation and Q_λ are polyhedral cones, we have

$$\Phi(C_{\lambda,j}) \leq \Phi([C_{\lambda,j}, u, -v, v]).$$

Since Φ is upper semicontinuous on $\mathcal{K}_0^n$, for $\eta > 0$ there is a $\lambda_0 > 0$ such that

$$\Phi([C_{\lambda,j}, u, -v, v]) \leq \Phi([C_j, u, -v, v]) + \frac{\eta}{2k}$$

for all $j = 1, \dots k$ and for $\lambda \geq \lambda_0$. This completes the proof of the lemma. $\square$

4.2 Uniqueness on $\mathcal{E}_0^n$

We show that given ϕ, Φ is uniquely determined on $\mathcal{E}_0^n$. We need the following result on valuations on the class $\mathcal{P}(S^{n-1})$ of spherical polytopes. Here a set $P \subset S^{n-1}$ is called a *spherical polytope*, if there is a polyhedral cone $Q \in \mathcal{Q}^n$ such that $P = S^{n-1} \cap Q$. Let σ be the $(n-1)$-dimensional Hausdorff measure. Schneider [51] proved that if $\nu : \mathcal{P}(S^{n-1}) \to \mathbb{R}$ is a rotation invariant, non-negative, and simple valuation, then there is a constant $c \geq 0$ such that $\nu(P) = c\,\sigma(P)$ for every $P \in \mathcal{P}(S^{n-1})$. A simple consequence of Schneider's characterization theorem is the following result, which shows that given ϕ, Φ is determined on intersections of centered balls with convex polyhedral cones.

Lemma 16. *For $t > 0$,*

$$\Phi(t\,B^n \cap Q) = \frac{\Phi(t\,B^n)}{V_n(t\,B^n)}\, V_n(t\,B^n \cap Q)$$

for every $Q \in \mathcal{Q}^n$.

Next, we consider intersections of right circular cylinders with convex polyhedral cones.

Lemma 17. *If Z is an unbounded cylinder, then*

$$\Phi(Z \cap Q) = 0$$

for every $Q \in \mathcal{Q}^n$ such that $Z \cap Q \in \bar{\mathcal{K}}_0^n$.

Proof. Define $\Phi^* : \mathcal{K}_0^n \to \mathbb{R}$ by

$$\Phi^*(K) = \Phi(K^*). \tag{42}$$

Note that

$$(K \cup L)^* = K^* \cap L^*, \quad (K \cap L)^* = K^* \cup L^*$$

for $K, L \in \mathcal{K}_0^n$ having convex union and that $(AK)^* = A^{-t}K^*$ holds for every $K \in \mathcal{K}_0^n$ and every $A \in \mathrm{SL}(n)$, where A^{-t} is the transpose of the inverse of A. Thus Φ^* is an upper semicontinuous and $\mathrm{SL}(n)$ invariant valuation on $\mathcal{K}_0^n$ that vanishes on $\mathcal{P}_0^n$. Applying Proposition 14 for Φ^* shows that there is a constant c^* such that

$$\Phi(B^{n-1} + t\,[-e_n, e_n]) \leq c^*\,\frac{1}{t} \tag{43}$$

for $t > 0$. Since Φ is finitely additive and non-negative and since $Z \cap Q \subset B^{n-1} + t\,[-e_n, e_n]$ for every t sufficiently large, the statement is an immediate consequence of (43). $\square$

4.3 Uniqueness for ε-smooth convex bodies

We prove the following result. Let $\varepsilon > 0$.

Proposition 18. *For every ε-smooth $K \in \mathcal{K}_0^n$,*

$$\Phi(K) = \sup\{\limsup_{j\to\infty} \Phi(E_j) : E_j \in \mathcal{E}_0^n, E_j \to K\}.$$

Without further mentioning we assume in this section that K is ε-smooth for some $\varepsilon > 0$. In the next three subsections, we derive lemmas that are used in the proof of Proposition 18. First, we consider boundary points of K where the generalized Gaussian curvature exists. Here we distinguish between points with positive curvature and points with vanishing curvature. Then we derive a result for general boundary points. The proof of Proposition 18 is contained in Subsection 4.3.4.

4.3.1 Boundary points with positive curvature

We call a family of convex polyhedral cones Q_t, $t > 0$, a *Vitali covering* for $x \in \partial K$ if $x \in \operatorname{int} Q_t$ for $t > 0$, if $Q_{t'} \subset Q_t$ for $0 < t' < t$, if $\operatorname{diam}(\partial K \cap Q_t) \to 0$ as $t \to 0$ and if there exists a constant $q = q(x) > 0$ such that

$$\frac{\sigma(\partial K \cap Q_t)}{\operatorname{diam}(\partial K \cap Q_t)^{n-1}} \geq q$$

for every $t > 0$. Here diam stands for diameter and int for interior.

Lemma 19. *Let $\eta > 0$ be given. For every $x \in \partial K$ with $\kappa(K, x) > 0$, there exist a centered ellipsoid $E = E(x)$, a constant $t(x) > 0$, and a Vitali covering of convex polyhedral cones Q_t for x such that*

$$x \in E \cap \operatorname{int} Q_{t'} \subset K \cap \operatorname{int} Q_{t'}, \tag{44}$$

$$\partial Q_t \cap K \subset H^+(E, Q_{t'}), \tag{45}$$

and

$$\Phi(K \cap Q_t) \leq \Phi(E \cap Q_{t'}) + \eta\, V_n(K \cap Q_t) \tag{46}$$

for every t, $0 < t < t(x)$, where t' is chosen suitably in $(t/3, t)$.

Proof. Since $\kappa(K, x) > 0$, there is a centered ellipsoid $E_0(x)$ which osculates K at x. For given $s > 0$, we choose a centered ellipsoid $E_s^i(x)$ that touches K at x from within in the following way. Let $A_s \in \mathrm{SL}(n)$ be the map which transforms $E_s^i(x)$ into the ball $r_s B^n$ and maps x to $r_s e_1$. Let F_s be the centered ellipsoid with semi-axes $r_s, (1+s)\, r_s, \dots, (1+s)\, r_s$. Note that $r_s B^n$ touches $A_s K$ at $r_s e_1$ from within. Now choose $E_s^i(x)$ sufficiently close to $E_0(x)$ such that $A_s K$ touches the ellipsoid F_s at $r_s e_1$ from within.

We now choose $0 < s < 1/4$ so small that

$$2((1 + 4\sqrt{s})^{2\,n} - 1) \leq \eta. \tag{47}$$

For this fixed s, we set $A = A_s$ and $r = r_s$.

Let R be an $(n-1)$-dimensional polytope chosen such that

$$R \subset B^{n-1} \subset (1+\sqrt{s})\, R. \tag{48}$$

Let R_t be the cone with base $t\,R$ in the support hyperplane to $r\,B^n$ at $r\,e_1$.

We need the following simple estimate.

Claim 19.1. *For $t>0$ sufficiently small, $[F_s \cap R_t, r\,B^n]\backslash r\,B^n \subset R_{(1+4\sqrt{s})\,t}$.*

Proof. Let C_t be the cone with base $t\,B^{n-1}$ in the support hyperplane to $r\,B^n$ at $r\,e_1$. By (48), we have

$$R_t \subset C_t \subset R_{(1+\sqrt{s})t}. \tag{49}$$

Next, we show that

$$[F_s \cap C_t, r\,B^n]\backslash r\,B^n \subset C_{(1+2\sqrt{s})t}. \tag{50}$$

Because of the rotational symmetry, we only have to consider the two-dimensional case. We choose the parameterizations $x(\alpha) = (r\cos\alpha, r\sin\alpha)$ for the circle and $x_s(\alpha) = (r\cos\alpha, r(1+s)\sin\alpha)$ for the ellipse. For $t = r\tan\alpha$, let $\hat{t} = r\tan\hat{\alpha}$ be the smallest number such that

$$[F_s \cap C_t, r\,B^n]\backslash r\,B^n \subset C_{\hat{t}}.$$

The points 0, $x(\hat{\alpha})$ and $x_s(\alpha)$ are the vertices of a triangle with a right angle and $\tan(\hat{\alpha}-\alpha) = \sqrt{s}\sqrt{2+s}\,\sin\alpha$. Using an addition theorem for the tangent, we obtain

$$\hat{t} = r\tan(\alpha + (\hat{\alpha}-\alpha)) \le (1+2\sqrt{s})\,t$$

for $t>0$ and thus $\alpha>0$ sufficiently small. This proves (50). The statement of the claim now follows from (49) and $0 < s \le 1/4$. //

Define $M_t \in \mathcal{K}_0^n$ in the following way. Let m_t be the maximum number such that there are rotations U_i, $i=1,\dots,m_t$, and the sets

$$U_i(r\,B^n \cap R_{(1+4\sqrt{s})t})$$

are pairwise disjoint for $i=1,\dots,m_t$. Since $m_t \ge m_{S^{n-1}}(3t/r)$, we obtain that by (3), there is a constant c (depending on r) such that

$$m_t\, V_n(r\,B^n \cap R_{(1+4\sqrt{s})t}) \ge \frac{1}{c} > 0. \tag{51}$$

We define

$$M_t = [r\,B^n, U_1(A\,K \cap R_t), \dots, U_{m_t}(A\,K \cap R_t)].$$

This construction implies that

$$M_t \to r\,B^n \quad \text{as} \quad t \to 0. \tag{52}$$

Claim 19.1 implies that

$$U_i(\partial(A\,K) \cap R_t) \subset \partial M_t$$

holds for $i=1,\dots,m_t$. We dissect

$$M_t \setminus \bigcup_{i=1}^{m_t} U_i(A\,K \cap R_{(1+4\sqrt{s})t})$$

using convex polyhedral cones $P_1,\dots,P_{k_t}$ whose interiors are disjoint from $U_i(A\,K \cap R_{(1+4\sqrt{s})t})$ for $i=1,\dots,m_t$. Since Φ is finitely additive, simple and non-negative, we obtain

$$\begin{aligned}\Phi(M_t) &= \sum_{i=1}^{m_t} \Phi(U_i(A\,K \cap R_{(1+4\sqrt{s})t})) + \sum_{j=1}^{k_t} \Phi(M_t \cap P_j)\\ &\geq m_t\,\Phi(A\,K \cap R_t) + \sum_{j=1}^{k_t} \Phi(r\,B^n \cap P_j).\end{aligned} \tag{53}$$

On the other hand,

$$\begin{aligned}\Phi(r\,B^n) &= \sum_{i=1}^{m_t} \Phi(U_i(r\,B^n \cap R_{(1+4\sqrt{s})t})) + \sum_{j=1}^{k_t} \Phi(r\,B^n \cap P_j)\\ &= m_t\,\Phi(r\,B^n \cap R_{(1+4\sqrt{s})t}) + \sum_{j=1}^{k_t} \Phi(r\,B^n \cap P_j).\end{aligned} \tag{54}$$

Let $\eta_1 > 0$ be given. Since Φ is upper semicontinuous and by (52), there is a constant $t_1 > 0$ such that

$$\Phi(M_t) \leq \Phi(r\,B^n) + \eta_1$$

for $0 < t < t_1$. Combined with (53), (54) and (51), this implies that

$$\begin{aligned}\Phi(A\,K \cap R_t) &\leq \Phi(r\,B^n \cap R_{(1+4\sqrt{s})t}) + \frac{\eta_1}{m_t}\\ &\leq \Phi(r\,B^n \cap R_{(1+4\sqrt{s})t}) + c\,\eta_1\,V_n(r\,B^n \cap R_{(1+4\sqrt{s})t}).\end{aligned} \tag{55}$$

We need the following simple estimates.

Claim 19.2. *For $t > 0$ sufficiently small, $\partial R_t \cap F_s \subset H^+(r\,B^n, R_{\frac{t}{1+4\sqrt{s}}})$.*

Proof. As before, let C_t the cone with base $t\,B^{n-1}$ in the support hyperplane to $r\,B^n$. We show that

$$H^+(r\,B^n, C_{\frac{t}{1+2\sqrt{s}}}) \cap \partial C_t \cap F_s = \emptyset. \tag{56}$$

Because of the rotational symmetry, we only have to consider the two-dimensional case. We choose the parameterizations $x(\alpha) = (r\cos\alpha, r\sin\alpha)$ for the circle and $x_s(\alpha) = (r\cos\alpha, r(1+s)\sin\alpha)$ for the ellipse. For $t = r\tan\alpha$, let $t' = r\tan\alpha'$ be the biggest number such that

$$H^+(r\,B^n, C_{t'}) \cap \partial C_t \cap F_s = \emptyset.$$

The points 0, $x(\alpha')$ and $x_s(\alpha)$ are the vertices of a triangle with a right angle and $r\tan(\alpha-\alpha')=\sqrt{s}\sqrt{2+s}\,\sin\alpha$. Therefore the estimate

$$t=r\tan(\alpha+(\alpha-\alpha'))\le(1+2\sqrt{s})\,t'$$

holds for $t>0$ sufficiently small. This proves (56). Because of (48) we have

$$R_{\frac{t}{1+4\sqrt{s}}}\subset R_{\frac{t}{(1+2\sqrt{s})(1+\sqrt{s})}}\subset C_{\frac{t}{(1+2\sqrt{s})(1+\sqrt{s})}}.$$

Hence (56) implies the statement of the claim. //

Claim 19.3. *For $t>0$ sufficiently small and $a>1$,*

$$V_n(r\,B^n\cap R_{a\,t})\le a^n V_n(r\,B^n\cap R_t)\le a^{2n}V_n(r\,B^n\cap R_{\frac{t}{a}}).$$

Proof. Let $V_{n-1}(R)$ denote the $(n-1)$-dimensional volume of R. We have

$$\frac{V_n(r\,B^n\cap R_t)}{t^{n-1}}\to\frac{r}{n}\,V_{n-1}(R)$$

as $t\to 0$. Therefore

$$\frac{1}{a}\,\frac{r}{n}\,V_{n-1}(R)\,t^{n-1}\le V_n(r\,B^n\cap R_t)\le a\,\frac{r}{n}\,V_{n-1}(R)\,t^{n-1}$$

for $t>0$ sufficiently small. This implies the statement of the claim. //

By Lemma 16 and Claim 19.3, we obtain from (55) and (47) with a suitable η_1 that for $t>0$ sufficiently small

$$\begin{aligned}\Phi(AK\cap R_t)\le\ &(1+4\sqrt{s})^{2n}\Phi(rB^n\cap R_{\frac{t}{1+4\sqrt{s}}})+(1+4\sqrt{s})^{2n}c\,\eta_1\,V_n(rB^n\cap R_{\frac{t}{1+4\sqrt{s}}})\\ \le\ &\Phi(rB^n\cap R_{t'})+((1+4\sqrt{s})^{2n}-1+(1+4\sqrt{s})^{2n}c\,\eta_1)V_n(rB^n\cap R_{t'})\\ \le\ &\Phi(rB^n\cap R_{t'})+\eta\,V_n(A\,K\cap R_t)\end{aligned}$$

where $t'=t/(1+4\sqrt{s})$. Transforming back shows that (46) holds for $Q_t=A^{-1}(R_t)$. Claim 19.2 implies that (45) holds. The family Q_t is a Vitali covering, since A only depends on x. This completes the proof of the lemma. $\square$

4.3.2 Boundary points with curvature zero

First, we prove the following lemma.

Lemma 20.

$$\lim_{t\to\infty}\frac{\Phi(t\,B^n)}{V_n(t\,B^n)}=0.$$

Proof. Let I_n be the cube with vertices at $(\pm 1, \dots, \pm 1)$. Since I_n is a polytope, $\Phi(I_n) = 0$. We construct L_t in the following way. Let Q be a polyhedral cone with apex at the origin generated by one of the facets F of I_n. Let E_t be ellipsoids of volume $V_n(t\,B^n)$ such that the vertices of F lie on E_t and $E_t \cap Q \to I_n \cap Q$ as $t \to \infty$. Let L_t be obtained by taking $2\,n$ suitably rotated copies of $E_t \cap Q$ such that $L_t \to I_n$ as $t \to \infty$. We have

$$\Phi(L_t) = 2\,n\,\Phi(E_t \cap Q)$$

and by Lemma 16

$$\Phi(E_t \cap Q) = \frac{V_n(E_t \cap Q)}{V_n(E_t)} \Phi(E_t) = V_n(E_t \cap Q) \frac{\Phi(t\,B^n)}{V_n(t\,B^n)}.$$

Since Φ is upper semicontinuous and since $L_t \to I_n$ as $t \to \infty$,

$$0 = \Phi(I_n) \geq \limsup_{t \to \infty} \Phi(L_t).$$

Combined with $V_n(E_t \cap Q) \to V_n(I_n \cap Q) = 2^{n-1}/n$ as $t \to \infty$, this completes the proof of the lemma. $\square$

Lemma 21. *Let $\eta > 0$ be given. For every $x \in \partial K$ with $\kappa(K, x) = 0$, there exist a $t(x) > 0$ and a Vitali covering of convex polyhedral cones Q_t for x such that*

$$\Phi(K \cap Q_t) \leq \eta\, V_n(K \cap Q_t) \tag{57}$$

for every t, $0 < t < t(x)$.

Proof. Let $\eta_1 > 0$ be given. By Lemma 20, it is possible to choose $r > 0$ so large that

$$\Phi(r\,B^n) \leq \eta_1 V_n(r\,B^n). \tag{58}$$

Since $\kappa(K, x) = 0$, there is a centered ellipsoid E which touches K from within at x such that $V_n(E) = V_n(rB^n)$. Let $A \in \mathrm{SL}(n)$ map this ellipsoid to $r\,B^n$ and the point x to e_1.

Let R be an $(n-1)$-dimensional polytope chosen such that

$$R \subset B^{n-1} \subset 2\,R \tag{59}$$

and let R_t be the cone with base $t\,R$ in the support hyperplane to $r\,B^n$ at $r\,e_1$. By H_r we denote the support hyperplane to $r\,B^n$ at $r\,e_1$.

We need the following simple estimate.

Claim 21.1. *For $t > 0$ sufficiently small, $[H_r \cap R_t, r\,B^n] \backslash r\,B^n \subset R_{6t}$.*

Proof. Let C_t the cone with base $t\,B^{n-1}$ in the support hyperplane to $r\,B^n$ at $r\,e_1$. By (59), we have

$$R_t \subset C_t \subset R_{2t}. \tag{60}$$

It is easy to see that

$$[H_r \cap C_t, r\,B^n] \backslash r\,B^n \subset C_{3t}.$$

The statement of the claim now follows from (60). //

Define $M_t \in \mathcal{K}_0^n$ in the following way. Let m_t be the maximum number such that there are rotations U_i, $i = 1, \dots, m_t$, and the sets

$$U_i(r\, B^n \cap R_{6t})$$

are pairwise disjoint for $i = 1, \dots, m_t$. By (3), there is a constant c such that

$$m_t\, V_n(r\, B^n \cap R_{6t}) \geq \frac{1}{c} > 0. \tag{61}$$

We define

$$M_t = [r\, B^n, U_1(A\, K \cap R_t), \dots, U_{m_t}(A\, K \cap R_t)].$$

This construction implies that

$$M_t \to r\, B^n \quad \text{as} \quad t \to 0. \tag{62}$$

Claim 21.1 implies that

$$U_i(\partial(A\, K) \cap R_t) \subset \partial M_t$$

holds for $i = 1, \dots, m_t$. We dissect

$$M_t \backslash \bigcup_{i=1}^{m_t} U_i(A\, K \cap R_{6t})$$

using convex polyhedral cones $P_1, \dots, P_{k_t}$ whose interiors are disjoint from $U_i(A\, K \cap R_{6t})$ for $i = 1, \dots, m_t$. Since Φ is finitely additive, simple and non-negative, we obtain

$$\begin{aligned} \Phi(M_t) &= \sum_{i=1}^{m_t} \Phi(U_i(A\, K \cap R_{6t})) + \sum_{j=1}^{k_t} \Phi(M_t \cap P_j) \\ &\geq m_t\, \Phi(A\, K \cap R_t) + \sum_{j=1}^{k_t} \Phi(r\, B^n \cap P_j). \end{aligned} \tag{63}$$

On the other hand,

$$\begin{aligned} \Phi(r\, B^n) &= \sum_{i=1}^{m_t} \Phi(U_i(r\, B^n \cap R_{6t})) + \sum_{j=1}^{k_t} \Phi(r\, B^n \cap P_j) \\ &= m_t\, \Phi(r\, B^n \cap R_{6t}) + \sum_{j=1}^{k_t} \Phi(r\, B^n \cap P_j). \end{aligned} \tag{64}$$

Let $\eta_2 > 0$ be given. Since Φ is upper semicontinuous and by (62), there is a constant $t(x) > 0$ such that

$$\Phi(M_t) \leq \Phi(r\, B^n) + \eta_2$$

for $0 < t < t(x)$. Combined with (63), (64), (61), (58), Lemma 16 and Claim 19.3 this implies that

$$\begin{aligned} \Phi(A\,K \cap R_t) &\leq \Phi(r\,B^n \cap R_{6t}) + \frac{\eta_2}{m_t} \\ &\leq \Phi(r\,B^n \cap R_{6t}) + c\,\eta_2\,V_n(r\,B^n \cap R_{6t}) \\ &\leq (\eta_1 + c\,\eta_2)V_n(r\,B^n \cap R_{6t}) \\ &\leq 6^{2n}(\eta_1 + c\,\eta_2)V_n(r\,B^n \cap R_{t/6}) \\ &\leq \eta\,V_n(A\,K \cap R_t). \end{aligned}$$

Transforming back shows that (57) holds for $Q_t = A^{-1}(R_t)$. The family Q_t is a Vitali covering, since A only depends on x. This completes the proof of the lemma. □

4.3.3 An absolute continuity property

The main result of this section is Proposition 23, which requires the following lemma.

Lemma 22. *For every ε-smooth convex body $K \in \mathcal{K}_0^n$, there are constants c'_K, d_K, and $r_K > 0$ such that for $0 < r < r_K$*

$$\Phi(K \cap \operatorname{cone}(I(x,r))) \leq c'_K\,r^{n-1} \tag{65}$$

and

$$r^{n-1} \leq d_K\,V_n(K \cap \operatorname{cone}(I(x,r))) \tag{66}$$

for every $x \in \partial K$ and every $(n-1)$-dimensional closed cube $I(x,r)$ of side length $2r$ centered at x lying in the support hyperplane to K at x.

Proof. Observe that there are positive numbers $\alpha = \alpha(K)$ and $\beta = \beta(K) \leq \varepsilon$ with the following properties: First, each boundary point x of K is touched from within by a centered ellipsoid E_x of volume $V_n(\beta B^n)$. Second, each linear map which maps the ellipsoid E_x to βB^n, maps each $(n-1)$-dimensional cube I of sidelength 2 and centered at x lying in the support hyperplane to K at x into a cube which is of sidelength at most α. Third, $\beta B^n \subset K$. These properties follow since K is ε-smooth and from the fact that there are two real positive numbers bounding the distance between the origin and all points on ∂K from below and above.

For $y \in \beta S^{n-1}$, let $A_y \in \mathrm{SL}(n)$ be such a linear map which maps the ellipsoid of volume $V_n(\beta B^n)$ to $\beta\,B^n$ and which maps x to y. We show that the convex hull of $A_y(K \cap \operatorname{cone}(I(x,r)))$ and $\beta\,B^n$ differs only in a small neighbourhood around y from $\beta\,B^n$:

$$[\beta\,B^n, A_y(K \cap \operatorname{cone}(I(x,r)))] \setminus \beta\,B^n \subset B^n(y, 2\,\alpha\,\sqrt{n}\,r) \tag{67}$$

for $r > 0$ sufficiently small. Let $z \in A_y(\partial K \cap \operatorname{cone}(I(x,r)))$. Let $\bar{z} \in \beta\,S^{n-1}$ be a point such that the line through z and $\bar{z}$ is tangent to $\beta\,B^n$. Since $z \notin \beta\,B^n$, we have $|z - \bar{z}|^2 \leq 4(|z| - \beta)\beta$ for $r > 0$ sufficiently small. Thus

$$[\beta\,B^n, z] \setminus \beta\,B^n \;\subset\; B^n(z, 2\sqrt{|z| - \beta}\sqrt{\beta}).$$

Since z lies between the tangent hyperplane to $\beta\, B^n$ at y and $\beta\, B^n$, we have $|z| - \beta \le \frac{1}{\beta}\, |z-y|^2$. Combined with $A_y(I(x,r)) \subset B^n(y, \alpha\sqrt{n}\, r)$, this implies that (67) holds.

Define $M_r \in \mathcal{K}_0^n$ in the following way. Let m_r be the maximum number of points $y_1, \dots, y_{m_r} \in \beta\, S^{n-1}$ such that the sets

$$\beta\, S^{n-1} \cap B^n(y_i, 2\alpha\, \sqrt{n}\, r)$$

are pairwise disjoint for $i = 1, \dots, m_r$. Since $m_r = m_{S^{n-1}}(2\,\alpha\, \sqrt{n}\, r/\beta)$, it follows from (3) that there is an $r_0(\beta) > 0$ and a constant c such that

$$m_r\, \big(\frac{\alpha\, r}{\beta}\big)^{n-1} \ge \frac{1}{c} > 0 \tag{68}$$

for $r \le r_0$. We define

$$M_r = [\beta\, B^n, A_{y_1}(K \cap \operatorname{cone}(I(x,r))), \dots, A_{y_{m_r}}(K \cap \operatorname{cone}(I(x,r)))]$$

and obtain $M_r \to \beta\, B^n$ as $r \to 0$. Since Φ is upper semicontinuous, this implies that

$$\Phi(\beta\, B^n) \ge \frac{1}{2}\, \Phi(M_r) \tag{69}$$

for $0 < r \le r_1$ with a suitable $r_1 > 0$.

By (67) and our construction of M_r, the sets $A_{y_i}(K \cap \operatorname{cone}(I(x,r)))$ are contained in the boundary of M_r and are pairwise disjoint for $i = 1, \dots, m_r$. Since Φ is non-negative and $\mathrm{SL}(n)$ invariant, this and the definition of M_r imply that

$$\Phi(M_r) \ge \sum_{i=1}^{m_r} \Phi(A_{y_i}(K \cap \operatorname{cone}(I(x,r)))) = m_r\, \Phi(K \cap \operatorname{cone}(I(x,r))).$$

¿From this combined with (69) and (68) it follows that

$$\Phi(K \cap \operatorname{cone}(I(x,r))) \le 2\, \Phi(\beta\, B^n)\, m_r^{-1} \le 2\, c\, \Phi(\beta\, B^n)\, \big(\frac{\alpha\, r}{\beta}\big)^{n-1}$$

for $0 < r \le r_K = \min\{r_0, r_1\}$. This proves (65).

The inequality (66) follows because K is ε-smooth with $\varepsilon \ge \beta$ and hence

$$V_n(K \cap \operatorname{cone}(I(x,r))) \ge V_n(\beta B^n \cap \operatorname{cone}(I(x,r))) \ge \lambda(\beta) r^{n-1}$$

for $r \le r_2(\beta)$ with some $\lambda(\beta) > 0$. □

In the proof of the following result, we use an argument well known from Vitali's lemma.

Proposition 23. *For every ε-smooth convex body K, there is a constant c_K such that*

$$\Phi(K \cap Q) \le c_K\, V_n(K \cap Q)$$

for every $Q \in \mathcal{Q}^n$.

Proof. We need the following estimates. Since $K \in \mathcal{K}_0^n$ is fixed and every cube $I(x,r)$ is contained in a support hyperplane of K at x, there are constants $\rho_c > \rho_i$ such that for every $x \in \partial K$ and $r > 0$,

$$\text{cone}(B(\frac{x}{|x|}, \rho_i\, r))) \subset \text{cone}(I(x,r)) \subset \text{cone}(B(\frac{x}{|x|}, \rho_c\, r)).$$

Let $\text{cone}(I(x,r))$ intersect $\text{cone}(I(y,s))$ and let $s \geq r$. We have $B(\frac{y}{|y|}, \rho_c\, s) \cap B(\frac{x}{|x|}, \rho_c\, r) \neq \emptyset$ and consequently

$$B(\frac{x}{|x|}, \rho_c\, r) \subset B(\frac{y}{|y|}, 3\,\rho_c\, s).$$

Thus

$$\text{cone}(I(x,r)) \subset \text{cone}(I(y, 3\,\frac{\rho_c}{\rho_i}\, s)) \tag{70}$$

whenever $\text{cone}(I(x,r)) \cap \text{cone}(I(y,s)) \neq \emptyset$ and $s \geq r$.

Now, let $Q \in \mathcal{Q}^n$. We choose an open set U such that

$$Q \subset U \quad \text{and} \quad V_n(U) \leq 2\, V(K \cap Q). \tag{71}$$

Let $\mathcal{J}$ be the family of all closed $(n-1)$-dimensional cubes $I = I(x,r)$ contained in the support hyperplane to K at x with center $x \in \partial K \cap Q$, side length $2r$, $0 < r \leq r_K$, and $\partial K \cap \text{cone}(I) \subset \partial K \cap U$. The relative interiors of $\partial K \cap \text{cone}(I)$ for $I \in \mathcal{J}$ form an open covering of $\partial K \cap Q$. Since $\partial K \cap Q$ is compact, we can choose a finite subcovering and denote by $\mathcal{I} \subset \mathcal{J}$ the set of closed cubes corresponding to this subcovering. We choose a suitable subset of $\mathcal{I}$ in the following way. Let I_1 be the cube with largest sidelength in $\mathcal{I}$. In the jth step, we choose the cube with largest sidelength in $\mathcal{I} \backslash \{I_1, \ldots, I_{j-1}\}$ that is disjoint from $\bigcup_{i=1}^{j-1} \text{cone}(I_i)$. Let $I_1, \ldots, I_k$ be the cubes obtained in this way. The corresponding cones are pairwise disjoint and we obtain by (71)

$$\sum_{i=1}^{k} V_n(K \cap \text{cone}(I_i)) \leq V_n(U) \leq 2\, V_n(K \cap Q). \tag{72}$$

By (70)

$$\partial K \cap Q \subset \bigcup_{i=1}^{k} \text{cone}(I(x_i, 3\,\frac{\rho_c}{\rho_i}\, r_i)).$$

Since Φ is non-negative, this implies that

$$\Phi(K \cap Q) \leq \sum_{i=1}^{k} \Phi(K \cap \text{cone}(I(x_i, 3\,\frac{\rho_c}{\rho_i}\, r_i))),$$

and applying (65) and (66) now shows that

$$\begin{aligned} \Phi(K \cap Q) \quad &\leq \quad c_K' \left(3\,\frac{\rho_c}{\rho_i}\right)^{n-1} \sum_{i=1}^{k} r_i^{n-1} \\ &\leq \quad c_K'\, d_K \left(3\,\frac{\rho_c}{\rho_i}\right)^{n-1} \sum_{i=1}^{k} V_n(K \cap \text{cone}(I(x_i, r_i))). \end{aligned}$$

Combined with (72), this completes the proof of the proposition. $\square$

4.3.4 Proof of Proposition 18

Let $P^i, P^c \in \mathcal{P}_0^n$ be such that $P^i \subset \operatorname{int} K \subset P^c$. For every choice of such P^i and P^c and every $\alpha > 0$, we construct a convex body $E \in \mathcal{E}_0^n$ such that $P^i \subset E \subset P^c$ and such that

$$\Phi(K) \leq \Phi(E) + \alpha\, V_n(K) \tag{73}$$

holds. This shows that there is always a convex body $E \in \mathcal{E}_0^n$ arbitrarily close to K such that $\Phi(E)$ is almost as large as $\Phi(K)$. Since Φ is upper semicontinuous, this proves Proposition 18.

We use Alexandrov's theorem (see, for example, [52]): Suppose $N \subset \partial K$ is the set of normal points of ∂K, that is, the set of points where the generalized Gaussian curvature exists. Then

$$\sigma(N) = \sigma(\partial K). \tag{74}$$

We also use the following result from measure theory (see, for example, [15]). Let $N \subset \mathbb{R}^n$ be a set of finite $(n-1)$-dimensional Hausdorff measure $\sigma(N)$ and denote the diameter of set V by $\operatorname{diam} V$. We call a collection $\mathcal{V}$ of sets a *Vitali class* for N, if for every $x \in N$ and every $\delta > 0$, there is a $V \in \mathcal{V}$ such that $x \in V$, $0 < \operatorname{diam} V \leq \delta$, and

$$\frac{\sigma(V)}{(\operatorname{diam} V)^{n-1}} \geq q(x) > 0,$$

where $q(x)$ depends only on x. A version of Vitali's covering theorem (c.f. [15]) states the following: If $\mathcal{V}$ is a Vitali class of closed sets for N, then, for every $\eta > 0$, there are pairwise disjoint $V_1, \dots, V_k \in \mathcal{V}$ such that $\sigma(N) \leq \sum_{i=1}^k \sigma(V_i) + \eta$.

Let $N \subset \partial K$ be the set of normal points of ∂K and let $\mathcal{V}$ be the collection of sets

$$V_t(x) = \partial K \cap Q_t(x)$$

for $x \in N$ and $0 < t \leq t(x)$, where $Q_t = Q_t(x)$ are the polyhedral cones from Lemma 19 and Lemma 21. If $\kappa(K, x) > 0$, let

$$E_t(x) = H^+(E(x), Q_{t'}(x)),$$

where $E(x)$ and t' are chosen as in Lemma 19. If $\kappa(K, x) = 0$, set $Q_{t'}(x) = Q_t(x)$ and let $E_t(x)$ be the support halfspace of K at x. For given $x \in N$, choose $t_1(x) \leq t(x)$ so small that

$$P^i \subset E_t(x) \tag{75}$$

for $0 < t < t_1(x)$. For $\kappa(K, x) = 0$, it is trivial that this is possible; for $\kappa(K, x) > 0$, this is possible since the ellipsoid $E(x)$ touches K at x from within.

Let c_K be the constant from Proposition 23 and set

$$\eta = \frac{\alpha\, V_n(K)}{4\, c_K}. \tag{76}$$

Let $\mathcal{V}$ be the collection of sets

$$V_t(x) = \partial K \cap Q_t(x)$$

for $x \in N$ and $0 < t \leq t_1(x)$. Since for $x \in N$ the sets $Q_t(x)$ are Vitali coverings for x, $\mathcal{V}$ is a Vitali class for N. For $\eta_1 > 0$, Vitali's covering theorem shows that there are pairwise disjoint $V_{t_1}(x_1),\ldots, V_{t_m}(x_m) \in \mathcal{V}$ such that $\sigma(N) \leq \sum_{i=1}^m \sigma(V_{t_i}(x_i)) + \eta_1$. Since we can choose $\eta_1 > 0$ suitably, this shows that

$$V_n(K) - \sum_{i=1}^m V_n(Q_{t_i}(x_i)) \leq \eta. \tag{77}$$

By (45) we have that for $i \neq j$, $\partial E_{t_i}(x_i)$ does not intersect $\partial E_{t_j}(x_j)$ within K. Without loss of generality, let $P^c \in \mathcal{P}_0^n$ be so close to K that for every i, j, $i \neq j$, $\partial E_{t_i}(x_i)$ does not intersect $\partial E_{t_j}(x_j)$ within P^c. Define

$$E = \bigcap_{i=1}^m E_{t_i}(x_i) \cap P^c. \tag{78}$$

Our construction implies that $E \in \mathcal{E}_0^n$.

Next, dissect $P^c \setminus \bigcup_{i=1}^m Q_{t_i}(x_i)$ with polyhedral cones $Q_1, \ldots, Q_k$. We have

$$\Phi(K) = \sum_{i=1}^m \Phi(K \cap Q_{t_i}(x_i)) + \sum_{j=1}^k \Phi(K \cap Q_j). \tag{79}$$

Our definition of $E_{t_i}(x_i)$ implies that for a normal point x_i with vanishing curvature, $E_{t_i}(x_i)$ is a polytope. For a normal point x_i with positive curvature, $E_{t_i}(x_i)$ consists of a piece of an ellipsoid, which lies in $K \cap Q_{t_i}(x_i)$, and pieces of cylinders and polytopes. Since Φ vanishes on cylinders and polytopes, we have

$$\Phi(E \cap Q_{t_i}(x_i)) = \Phi(E_{t_i}(x_i) \cap Q_{t_i}(x_i)).$$

Combined with the fact that $E \cap Q_j$ does not meet any cone $Q_{t_i}(x_i)$ and thus is a polytope or a piece of a cylinder for $j = 1, \ldots, k$, we therefore have

$$\Phi(E) = \sum_{i=1}^m \Phi(E_{t_i}(x_i) \cap Q_{t_i}(x_i)).$$

Using this and (46) we obtain

$$\begin{aligned} \sum_{i=1}^m \Phi(K \cap Q_{t_i}(x_i)) &\leq \sum_{i=1}^m \Big(\Phi\left(E_{t_i}(x_i) \cap Q_{t_i}(x_i)\right) + \frac{\alpha}{2}\, V_n(K \cap Q_{t_i}(x_i))\Big) \\ &\leq \Phi(E) + \frac{\alpha}{2}\, V_n(K). \end{aligned} \tag{80}$$

Proposition 23 shows that

$$\sum_{j=1}^k \Phi(K \cap Q_j) \leq c_K \sum_{j=1}^k V_n(K \cap Q_j).$$

By (74), our choice of the Q_j's and (77) imply that

$$\sum_{j=1}^k V_n(K \cap Q_j) \leq \eta.$$

Consequently, we have by our definition of η in (76)

$$\sum_{j=1}^{k} \Phi(K \cap Q_j) \leq \frac{\alpha}{2} V_n(K). \tag{81}$$

By (79), (80), and (81) we now obtain

$$\Phi(K) \leq \Phi(E) + \alpha V_n(K).$$

Since (75) and (78) imply that $P^i \subset E \subset P^c$, this implies that (73) holds. Thus Proposition 18 is proved.

4.4 Uniqueness for general convex bodies

The main result of this section is that for every $K \in \mathcal{K}_0^n$,

$$\Phi(K) = \sup\{\limsup_{j\to\infty} \Phi(E_j) : E_j \in \mathcal{E}_0^n, E_j \to K\}. \tag{82}$$

For $K \in \mathcal{K}_0^n$, let $r(K) = \max\{r > 0 : r\, B^n \subset K\}$ be the inradius of K and let $\mathcal{Q}(K)$ be the set of polyhedral cones Q such that there is a point $x \in \partial K$ with $Q \subset \operatorname{cone}(B(x, \frac{1}{5}r(K))$. By (26), Proposition 18 immediately implies (82) if for all cones $Q \in \mathcal{Q}(K)$ there exist ε-smooth convex bodies K_ε, $\varepsilon > 0$, such that $\lim_{\varepsilon\to 0} K_\varepsilon = K$ and

$$\Phi(K \cap Q) \leq \limsup_{\varepsilon\to 0} \Phi(K_\varepsilon \cap Q).$$

Hence (82) is a consequence of the following proposition.

Proposition 24. *If $K \in \mathcal{K}_0^n$ and $Q \in \mathcal{Q}(K)$, then given $\eta, \eta' > 0$ there is an ε-smooth $K_\varepsilon \in \mathcal{K}_0^n$, $\varepsilon > 0$, such that*

$$\Phi(K \cap Q) \leq \Phi(K_\varepsilon \cap Q) + \eta$$

and $\delta(K, K_\varepsilon) < \eta'$.

Proof. For the construction of K_ε, we use the $\mathrm{SL}(n)$ shaping process introduced in Section 2. Let $x_Q \in \partial K$ be a point such that $Q \subset \operatorname{cone}(B(x_Q, \frac{1}{5}r(K))$ and assume without loss of generality that $e_n = x_Q/|x_Q|$. Let $\tau \in [0, \frac{1}{2})$. Define

$$C_\tau = \left\{u \in S^{n-1} :\ x_Q \cdot u \geq (1-\tau) r(K)\right\}.$$

Let L be a convex body such that $L \supset r(K)B^n$ and let $x \in \partial L \cap B(x_Q, \tau r(K))$. (In the following, L is close to K and hence this intersection is not empty.) The support hyperplane of L at x does not intersect $r(K)B^n$. Since

$$C_\tau = \{u \in S^{n-1} :\ H^+(B(x_Q, \tau r(K)), u) \supset r(K)B^n\},$$

this implies

$$N(L, B(x_Q, \tau r(K)) \subset C_\tau. \tag{83}$$

Note that for $0 < \tau < \frac{1}{2}$ it follows from the definition of C_τ that for all $x \in B(x_Q, \tau r(K))$ and $u \in C_\tau \cap S^{n-1}$

$$x \cdot u \geq (1 - 2\tau)\, r(K). \tag{84}$$

We apply Lemma 9 with $C = C_{\frac{1}{4}}$, that is, $\gamma = \frac{3}{4}\frac{r(K)}{|x_Q|}$. Let $\lambda \geq \lambda(C)$ be so large that

$$\left(e^{\frac{\alpha(C)\,\beta(C)}{\lambda^4}} - 1\right) \max_{x \in K \cap Q} |x| \leq \min\left\{\tfrac{1}{2}\eta',\ \tfrac{1}{20} r(K)\right\} \tag{85}$$

where α, β, λ are the constants from Lemma 9. Let $T_{ki} = T(u_{ki}, v_{ki})$ be the transvections from Lemma 9. Observe that Lemma 9 implies that we have $e_n \cdot u_{ki} \geq \frac{8}{9}\frac{3}{4}\frac{r(K)}{|x_Q|}$ and thus $u_{ki} \in C_{\frac{1}{3}}$. It follows from (4), (5) and Lemma 9 that for $x \in K \cap Q$, $k = 1, \ldots,$ and $l = 1, \ldots, m_k$,

$$\begin{aligned}
|T_{kl} \cdots T_{k1}x - x| &\leq \sum_{i=1}^{l} |T_{ki} \cdots T_{k1}x - T_{k,i-1} \cdots T_{k1}x| \\
&\leq \frac{\alpha(C)}{k\,\lambda^2} \sum_{i=1}^{m_k} |T_{k,i-1} \cdots T_{k1}x| \\
&\leq \frac{\alpha(C)}{k\,\lambda^2} \sum_{i=1}^{m_k} \big(1 + \frac{\alpha(C)}{k\,\lambda^2}\big)^{i-1} |x| \\
&\leq \left(\big(1 + \frac{\alpha(C)}{k\,\lambda^2}\big)^{\frac{k\beta(C)}{\lambda^2}} - 1\right)|x| \\
&\leq \left(e^{\frac{\alpha(C)\,\beta(C)}{\lambda^4}} - 1\right)|x|.
\end{aligned} \tag{86}$$

By (85), we obtain that for $x \in \partial K \cap Q$, $k = 1, \ldots,$ and $l = 1, \ldots, m_k$

$$|T_{kl} \cdots T_{k1}x - x_Q| \leq \frac{1}{20} r(K) + \frac{1}{5} r(K) \leq \frac{1}{4} r(K).$$

Hence we have

$$(\partial K \cap Q)_{T_{k1}, \ldots, T_{kl}} \subset B(x_Q, \frac{1}{4} r(K)). \tag{87}$$

Since $u_{ki} \in C_{\frac{1}{3}}$ for $i = 1, \ldots, m_k$, we obtain from (84) that

$$T_{kl} \cdots T_{k1}x \cdot u_{k,l+1} \geq \frac{1}{3} r(K)$$

for all $x \in \partial K \cap Q$, $l = 0, \ldots, m_k - 1$, that is, the distance of $u_{k,l+1}^\perp$ to $(\partial K \cap Q)_{T_{k1}, \ldots, T_{kl}}$ is positive. Thus (as explained in Section 2)

$$M_{\lambda,k} = M_{T_{k1}, \ldots, T_{km_k}}$$

is convex, where we set $M = K \cap Q$. A point x is called an *upper boundary point* of $M_{\lambda,k}$ if $x \in \partial K_{T_{k1}, \ldots, T_{km_k}} \cap \operatorname{int} Q_{T_{k1}, \ldots, T_{km_k}}$.

The following property turns out to be important. For $Q \in \mathcal{Q}^n$, $L \subset \mathbb{R}^n$ and $Z \in \bar{\mathcal{K}}_0^n$, we say that a translated copy of Z touches $x \in \partial L \cap Q$ from within if there is a vector z such that $x \in z + Z \subset L$. Let $T = T_{k,l+1}$.

Claim 24.1. *If $Z \subset \frac{1}{3}\, r(K)B^n$ and each $x \in \partial K_{T_{k1},\dots,T_{kl}} \cap Q_{T_{k1},\dots,T_{kl}}$ is touched by a translated copy of Z from within, then each $y \in (\partial K_{T_{k1},\dots,T_{kl}} \cap Q_{T_{k1},\dots,T_{kl}})_T$ is touched by a translated copy of Z_T from within.*

Proof. Each x is mapped by T to $x + (x \cdot u)v$. Hence, since $u \in C_{\frac{1}{3}}$, (84) and (87) imply that the distance of Tx and x is at least $\frac{1}{3}r(K)\,|v|$. For $z \in Z$, the distance of z to $Tz = z + (z \cdot u)v$ is at most $\frac{1}{3}r(K)\,|v|$, since $Z \subset \frac{1}{3}r(K)B^n$. This implies the statement of the claim. //

Note that a translated copy of $[0, \frac{1}{3}r(K)e_n]$ touches every $x \in \partial K \cap Q$ from within. Let

$$Z_{\lambda,k} = [0, \lambda\, e_n]_{T_{k1},\dots,T_{km_k}}.$$

By Lemma 9, $[0, e_n]_{T_{k1},\dots,T_{kl}} \subset B^n$ for $l = 1, \dots, m_k$. Therefore Claim 24.1 implies that each upper boundary point of $M_{\lambda,k}$ is touched by a translated copy of the polytope $(r(K)/(3\lambda))Z_{\lambda,k}$ from within. Let $k \to \infty$. Since by (85) and (86) the sets $M_{\lambda,k}$ are contained in $M + \frac{1}{2}\eta' B^n$, there is a convergent subsequence, also denoted by $M_{\lambda,k}$, such that $M_{\lambda,k} \to M_\lambda$. Note that Lemma 9 implies that $Z_{\lambda,k} \cap D_\lambda \xrightarrow{t} E_\lambda \cap D_\lambda$ as $k \to \infty$, where $N(E_\lambda, D_\lambda) = C_{\frac{1}{4}}$. Hence at each upper boundary point x of M_λ a translated copy of $(r(K)/(3\lambda))E_\lambda \cap D_\lambda$ touches M_λ from within. To prove the smoothness of M_λ we show that such x is always touched by points of the ellipsoid $(r(K)/(3\lambda))E_\lambda$ contained in the interior of D_λ.

We denote by $N(M_{\lambda,k}) = N(K_{T_{k1},\cdots,T_{km_k}}, Q_{T_{k1},\cdots,T_{km_k}})$ the set of outer normal vectors to $M_{\lambda,k}$, and define the normal cone $N(M_\lambda)$ in the same way. Note that $r(K_T) \geq r(K)$ for all transvections T. ¿From (83) and (87) it follows that the normal cones $N(M_{\lambda,k})$ and $N(M_\lambda)$ are contained in $C_{\frac{1}{4}}$. Since $N(E_\lambda, D_\lambda) = C_{\frac{1}{4}}$, we obtain that all points on the upper boundary of M_λ are touched by a translated copy of $(r(K)/(3\lambda))E_\lambda$ from within and thus are ε-smooth with $\varepsilon = r(K)/(3\lambda^4)$.

Because of (86) it is possible to choose polyhedral cones Q_λ such that $M_{\lambda,k} \subset Q_\lambda$ and $Q_\lambda \cap B^n \to Q \cap B^n$ as $\lambda \to \infty$. We define

$$\bar{M}_{\lambda,k} = \bigcap_{u \in N(M_{\lambda,k})} H^+(M_{\lambda,k}, u),$$

that is, $\bar{M}_{\lambda,k}$ is the tangential continuation of $M_{\lambda,k}$. Note that $\bar{M}_{\lambda,k} \cap Q_\lambda \to \bar{M}_\lambda \cap Q_\lambda$ as $k \to \infty$, where $\bar{M}_\lambda$ is the tangential continuation of M_λ.

Since $\bar{M}_\lambda$ is ε-smooth, we can choose an ε-smooth $K_\varepsilon \in \mathcal{K}_0^n$ such that $K_\varepsilon \cap Q_\lambda = \bar{M}_\lambda \cap Q_\lambda$. By (85) and (86) we see that $\delta(M, M_{\lambda,k}) < \frac{1}{2}\eta'$ for all k. Thus K_ε can be chosen such that $\delta(K, K_\varepsilon) < \eta'$. Since Φ is non-negative, we obtain by Lemma 7

$$\Phi(M) \leq \Phi(M_{\lambda,k}) \leq \Phi(\bar{M}_{\lambda,k} \cap Q_\lambda). \tag{88}$$

Since Φ is upper semicontinuous, (26) and (88) imply that

$$\Phi(M) \leq \lim_{k \to \infty} \Phi(\bar{M}_{\lambda,k} \cap Q_\lambda) \leq \Phi(\bar{M}_\lambda \cap Q_\lambda). \tag{89}$$

We apply Lemma 15 and see that for λ sufficiently large

$$\Phi(\bar{M}_\lambda \cap Q_\lambda) \leq \Phi(K_\varepsilon \cap Q) + \eta.$$

Combined with (89), this completes the proof of the proposition. $\square$

4.5 Properties of ϕ

As last step in the proof of Theorem 5, we need the following result.

Proposition 25. *Let $\Phi : \mathcal{K}_0^n \to \mathbb{R}$ be an upper semicontinuous and* $\mathrm{SL}(n)$ *invariant valuation that vanishes on $\mathcal{P}_0^n$ and define $\phi : [0,\infty) \to [0,\infty)$ by*

$$\phi(t) = \frac{1}{n\, v_n}\, \Phi(t^{-\frac{1}{2n}}\, B^n)\, t^{\frac{1}{2}}.$$

Then $\phi : [0,\infty) \to [0,\infty)$ is concave and $\lim_{t\to 0} \phi(t) = \lim_{t\to\infty} \phi(t)/t = 0$.

Proof. First, note that Lemma 20 implies

$$\lim_{t\to 0} \phi(t) = \lim_{t\to 0} \frac{\Phi(t^{-\frac{1}{2n}} B^n)\, t^{\frac{1}{2}}}{n\, v_n} = \frac{1}{n} \lim_{s\to\infty} \frac{\Phi(s\, B^n)}{V_n(s\, B^n)} = 0.$$

Note that Lemma 20 also holds for Φ^*, where as in (42) $\Phi^*(K) = \Phi(K^*)$. This implies that

$$\lim_{t\to\infty} \frac{\phi(t)}{t} = \lim_{t\to\infty} \frac{\Phi(t^{-\frac{1}{2n}} B^n)}{n\, v_n\, t^{\frac{1}{2}}} = \frac{1}{n} \lim_{s\to\infty} \frac{\Phi^*(s\, B^n)}{V_n(s\, B^n)} = 0.$$

It remains to show that ϕ is concave. Note that by Lemma 16 and (27), for a centered ball B and $Q \in \mathcal{Q}^n$,

$$\Phi(B \cap Q) = n\, V_n(B \cap Q)\, \phi(v_n^2 V_n(B)^{-2}).$$

Thus if $L = \bigcup L_j \in \mathcal{E}_0^n$, where $L_j = E_j \cap Q_j$, $Q_j \in \mathcal{Q}^n$ have pairwise disjoint interiors, and E_j are ellipsoids, then

$$\Phi(L) = \sum \Phi(L_j) = \sum n\, V_n(L_j)\, \phi(v_n^2 V_n(E_j)^{-2}). \tag{90}$$

We start by proving the case $n = 2$. Let $s > 0$ and let L_j be a sector of the circle $s\, B^2$. By (90), we have

$$\Phi(L_j) = 2V_2(L_j)\phi(\pi^2 V_2(s\, B^2)^{-2}) = 2V_2(L_j)\phi(s^{-4}). \tag{91}$$

We approximate $s\, B^2$ by pieces of suitable ellipses in the following way. At the points $(s\cos((2k+1)\pi/(2m)), s\sin((2k+1)\pi/(2m))$, $k = 0,\ldots,m-1$, on the boundary of $s\, B^2$ rotated copies E_k^t, $k = 0,\ldots,m-1$, of a centered ellipse of area $\pi\, t^2 > \pi\, s^2$ which contains $s\, B^2$ touch the boundary of $s\, B^2$ from the exterior. Thus the angle between the semi-minor axis of this ellipse and the x_1-axis is $(2k+1)\pi/(2m)$. The intersection $\bigcap_{k=0,\ldots,m-1} E_k^t$ contains $s\, B^2$ and the boundary of this intersection is smooth except for $2\, m$ points where the ellipse

∂E_k^t intersects the next rotated copy ∂E_{k+1}^t. Then we choose rotated copies E_k^r of a centered ellipse of area $\pi\, r^2 < \pi\, s^2$ which are contained in $\bigcap_{k=0,\ldots,m-1} E_k^t$ and which touch the intersection $E_{k-1}^t \cap E_k^t$ from the interior close to the two points $\partial E_{k-1}^t \cap \partial E_k^t$. The angle between the semi-major axis of this ellipse and the x_1-axis is $k\pi/m$. The smaller ellipse E_k^r touches the ellipse E_{k-1}^t in two points denoted by $\pm p_{k,k-1}$ and the ellipse E_k^t in $\pm p_{k,k}$.

Define L_m to be the convex hull of the boundary of E_k^t between the points $p_{k,k}, p_{k+1,k}$ and between $-p_{k,k}, -p_{k+1,k}$ and the boundary of E_k^r between the points $p_{k,k-1}, p_{k,k}$ and between $-p_{k,k-1}, -p_{k,k}$ for $k = 0, \ldots, m-1$. Clearly as $m \to \infty$ the convex sets L_m converge to $s\, B^2$. Denote by L_m^r the convex hull of the origin and the sector of ∂E_0^r between the x_1-axis and the point $p_{0,0}$, and by L_m^t the convex hull of the origin and the sector of ∂E_0^t between the point $p_{0,0}$ and the point $(s\cos(\pi/(2m)), s\sin(\pi/(2m)))$, the endpoint of the semi-minor axis of E_0^t.

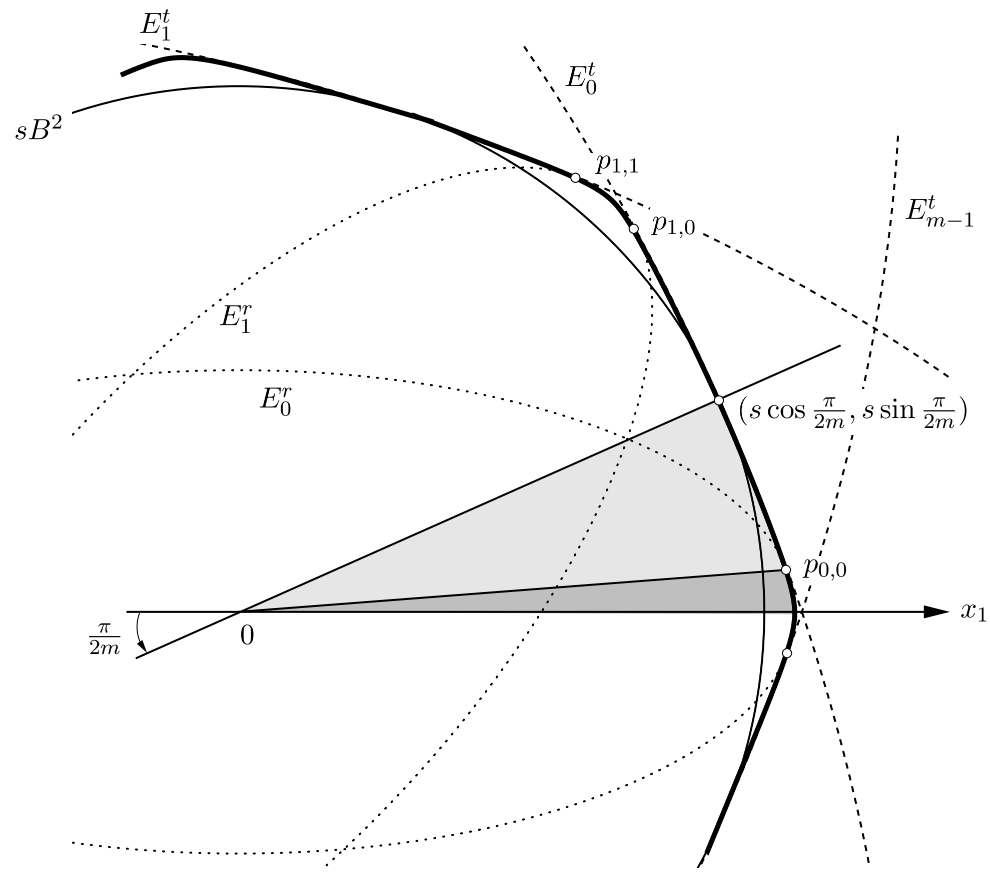

Figure 3: L_m^t, L_m^r

Since Φ is a simple valuation, we obtain

$$\Phi(L_m) = 4\, m\, (\Phi(L_m^r) + \Phi(L_m^t)). \tag{92}$$

Thus to compute $\Phi(L_m)$ it suffices by (91) to compute $V_2(L_m^t)$ and $V_2(L_m^r)$. First we compute the coordinates of the point $p_{0,0} = (p_1, p_2)$, where the ellipse

$$E_0^r: \ \frac{x_1^2}{r^2 l^2} + \frac{l^2 x_2^2}{r^2} = 1$$

with a suitable parameter l touches the ellipse

$$E_0^t: \ \frac{1}{s^2}\big(x_1 \cos\frac{\pi}{2m} + x_2 \sin\frac{\pi}{2m}\big)^2 + \frac{s^2}{t^4}\big(-x_1 \sin\frac{\pi}{2m} + x_2 \cos\frac{\pi}{2m}\big)^2 = 1.$$

It is easy to see that $s/r < l < s/(r\cos(\pi/(2m)))$ since the semi-major axis of E_0^r is larger than s and smaller than the intersection of E_0^t with the x_1-axis. Thus $l = s/r + O(m^{-2})$ as $m \to \infty$. Further p_1 is between $s\cos(\pi/(2m))$ and $s/\cos(\pi/(2m))$ which shows $p_1 = s + O(m^{-2})$ as $m \to \infty$.

The value for p_2 is computed using that the normal vector

$$n_r = 2\left(-\frac{1}{s} + O(\frac{1}{m^2}),\ p_2\,\frac{s^2}{r^4} + O(\frac{1}{m^2})\right)$$

to E_r^0 at $p_{0,0}$ must coincide with the normal vector

$$n_r = 2\left(-\frac{1}{s} + O(\frac{1}{m^2}),\ s^3\left(\frac{1}{s^4} - \frac{1}{t^4}\right)\frac{\pi}{2m} + p_2\,\frac{s^2}{t^4} + O(\frac{1}{m^2})\right)$$

to E_t^0 at $p_{0,0}$. This implies

$$p_{0,0} = (p_1, p_2) = \left(s + O(\frac{1}{m^2}),\ s\,\frac{r^4}{s^4}\,\frac{t^4 - s^4}{t^4 - r^4}\,\frac{\pi}{2m} + O(\frac{1}{m^2})\right)$$

as $m \to \infty$.

Hence

$$V_2(L_m^r) = \frac{1}{2}\,p_1 p_2 + O(\frac{1}{m^2}) = \frac{1}{2}\,\frac{r^4(t^4 - s^4)}{s^2(t^4 - r^4)}\,\frac{\pi}{2m} + O(\frac{1}{m^2}) \tag{93}$$

and

$$V_2(L_m^t) = \frac{1}{2}\,s^2\,\frac{\pi}{2m} - V_2(L_m^r) + O(\frac{1}{m^2}) = \frac{1}{2}\,\frac{t^4(s^4 - r^4)}{s^2(t^4 - r^4)}\,\frac{\pi}{2m} + O(\frac{1}{m^2}). \tag{94}$$

By (91) and (92) we thus obtain

$$\Phi(L_m) = 4m\left(\frac{r^2(t^4 - s^4)}{s^2(t^4 - r^4)}\,\frac{\pi}{2m}\,r^2\phi(r^{-4}) + \frac{t^2(s^4 - r^4)}{s^2(t^4 - r^4)}\,\frac{\pi}{2m}\,t^2\phi(t^{-4})\right) + O(\frac{1}{m})$$

as $m \to \infty$. The upper semicontinuity implies that $\Phi(s\,B^2) \geq \lim_{m\to\infty}\Phi(L_m)$ and hence

$$2\pi s^2\phi(s^{-4}) \geq 2\pi\frac{r^4(t^4 - s^4)}{s^2(t^4 - r^4)}\phi(r^{-4}) + 2\pi\frac{t^4(s^4 - r^4)}{s^2(t^4 - r^4)}\phi(t^{-4}).$$

Put $r' = r^{-4}$, $s' = s^{-4}$, and $t' = t^{-4}$. We obtain for $t' < s' < r'$

$$\phi(s') \geq \frac{s'-t'}{r'-t'}\,\phi(r') + \left(1 - \frac{s'-t'}{r'-t'}\right)\phi(t')$$

which shows that ϕ is a concave function.

To prove that ϕ is concave for $n \geq 3$, we supplement the circle $s\,B^2$ and the planar ellipses E_k^r and E_k^t in such a way that we obtain an n-dimensional ball and n-dimensional ellipsoids: the planar figures are left unchanged and in directions of the x_i-axes, $i = 3, \ldots, n$, we add semi-axes of length s. Thus the ellipsoid E_k^t is a rotated copy of the ellipsoid

$$\frac{x_1^2}{s^2} + \frac{s^2 x_2^2}{t^4} + \frac{x_3^2}{s^2} + \cdots + \frac{x_n^2}{s^2} = 1 \tag{95}$$

where we rotate this ellipsoid in the x_1-x_2-plane by an angle $(2k+1)\,\pi/(2m)$. Analogously the ellipsoid E_k^r is a rotated copy of the ellipsoid

$$\frac{x_1^2}{r^2 l^2} + \frac{l^2 x_2^2}{r^2} + \frac{x_3^2}{s^2} + \cdots + \frac{x_n^2}{s^2} = 1 \tag{96}$$

where l was computed above and we rotate this ellipsoid in the x_1-x_2-plane by an angle $k\pi/m$.

The ellipsoids E_k^t contain the ball $s\,B^n$ and touch it from the exterior along the intersection of the ball with the hyperplanes $x_2 = 0$ rotated in the x_1-x_2-plane by an angle $(2k+1)\pi/(2m)$. The ellipsoids E_k^r are contained in $E_{k-1}^t \cap E_k^t$. They touch E_{k-1}^t from the interior along the intersection of E_k^r with the hyperplanes containing the origin, $\pm p_{k,k-1}$, and the x_i-axes, $i = 3, \ldots, n$. Analogously they touch E_k^t from the interior along the intersection of E_k^r with the hyperplanes containing the origin, $\pm p_{k,k}$, and the x_i-axes, $i = 3, \ldots, n$.

As before we define L_m to be the convex hull of the boundary of E_k^t between the hyperplanes containing the points $\pm p_{k,k}, \pm p_{k+1,k}$ and the boundary of E_k^r between the hyperplanes containing the points $\pm p_{k,k-1}, \pm p_{k,k}$. As $m \to \infty$, L_m converge to $s\,B^n$. Define L_m^r and L_m^t in the same way as before. Since Φ is a simple valuation we obtain

$$\Phi(L_m) = 4\,m\,(\Phi(L_m^r) + \Phi(L_m^t)).$$

Since the ratio $V_n(L_m^r) : V_n(E^r)$ coincides with the ratio $V_2(L_m^r) : V_2(E^r)$ in the planar case, (93) implies

$$\Phi(L_m^r) \quad = \quad n v_n \frac{r^2(t^4-s^4)}{s^2(t^4-r^4)} \frac{1}{4m}\, s^{n-2} r^2 \phi(s^{-2(n-2)} r^{-4}) + O(\frac{1}{m^2})$$

and analogously by (94)

$$\Phi(L_m^t) = n v_n \frac{t^2(s^4-r^4)}{s^2(t^4-r^4)} \frac{1}{4m}\, s^{n-2} t^2 \phi(s^{-2(n-2)} r^{-4}) + O(\frac{1}{m^2}).$$

This implies for $m \to \infty$

$$s^n \phi(s^{-2n}) \geq \frac{r^4(t^4-s^4)}{s^4(t^4-r^4)}\, s^n \phi(s^{-2(n-2)} r^{-4}) + \frac{t^4(s^4-r^4)}{s^4(t^4-r^4)}\, s^n \phi(s^{-2(n-2)} t^{-4}).$$

Put $r' = s^{-2(n-2)}r^{-4}$, $s' = s^{-2n}$, and $t' = s^{-2(n-2)}t^{-4}$. We obtain for $t' < s' < r'$

$$\phi(s') \geq \frac{s'-t'}{r'-t'}\,\phi(r') + \left(1 - \frac{s'-t'}{r'-t'}\right)\phi(t')$$

which shows that ϕ is a concave function for all $n \geq 2$. $\square$

5 Proof of Theorems 3 and 4

Theorem 3 is a special case of Theorem 4. To prove Theorem 4, we use the following result.

Theorem 26 ([32]). *A functional $\Psi : \mathcal{P}_0^n \to \mathbb{R}$ is a Borel measurable,* $\mathrm{SL}(n)$ *invariant valuation that is homogeneous of degree q if and only if there is a constant $c_0 \in \mathbb{R}$ such that*

$$\Psi(P) = \begin{cases} c_0\, V_0(P) & \text{for } q = 0 \\ c_0\, V_n(P) & \text{for } q = n \\ c_0\, V_n(P^*) & \text{for } q = -n \\ 0 & \text{else} \end{cases}$$

for every $P \in \mathcal{P}_0^n$.

Suppose $\Psi : \mathcal{K}_0^n \to \mathbb{R}$ is an upper semicontinuous, $\mathrm{SL}(n)$ invariant valuation that is homogeneous of degree q. Then Ψ restricted to $\mathcal{P}_0^n$ is Borel measurable and we apply Theorem 26. If $q = 0, -n, n$, we set $\Phi(K) = \Psi(K) - c_0$, $\Phi(K) = \Psi(K) - c_0\, V_n(K^*)$ and $\Phi(K) = \Psi(K) - c_0\, V_n(K)$, respectively, and in all other cases, we set $\Phi = \Psi$. Hence $\Phi : \mathcal{K}_0^n \to \mathbb{R}$ is an upper semicontinuous, $\mathrm{SL}(n)$ invariant valuation that vanishes on $\mathcal{P}_0^n$ and it is homogeneous of degree q. We apply Theorem 5 and obtain that there is a concave function $\phi : [0, \infty) \to [0, \infty)$ with $\lim_{t\to 0} \phi(t) = \lim_{t\to\infty} \phi(t)/t = 0$ such that

$$\Phi(K) = \int_{\partial K} \phi(\kappa_0(K, x))\, d\mu_K(x) \tag{97}$$

for every $K \in \mathcal{K}_0^n$. Since Φ is homogeneous of degree q, (27) implies that

$$\phi(t) = \frac{\Phi(B^n)}{n\, v_n}\, t^{\frac{-q+n}{2n}}. \tag{98}$$

If $q \leq -n$ or $q \geq n$, $\lim_{t\to 0} \phi(t) = \lim_{t\to\infty} \phi(t)/t = 0$ and (98) imply that $\Phi(B^n) = 0$. Thus, in both cases, we obtain from (97) and (98) that $\Phi(K) = 0$ for every $K \in \mathcal{K}_0^n$. For $-n < q < n$, we obtain from (97) and (98) that there is a constant $c_1 \geq 0$ such that $\Phi(K) = c_1\, \Omega_p(K)$ for every $K \in \mathcal{K}_0^n$ where $p = n(n-q)/(n+q)$. This concludes the proof of Theorem 4.

6 Corollaries

Theorem 3 allows to obtain a simple proof of the following classical result.

Corollary 27. *For $K \in \mathcal{K}_0^n$, $\Omega_c(K^*) = \Omega_c(K)$.*

Proof. Set $\Psi(K) = \Omega_c(K^*)$. As explained in the proof of Lemma 17, Ψ is an upper semicontinuous and $\mathrm{GL}(n)$ invariant valuation that vanishes on polytopes. Therefore by Theorem 3, there is a constant $c \ge 0$ such that $\Psi(K) = c\,\Omega_c(K)$. Since for the unit ball $B^n = (B^n)^*$, $c = 1$. □

Theorem 4 allows to obtain a simple proof of the following result of Hug [23].

Corollary 28. *For $K \in \mathcal{K}_0^n$ and $p > 0$, $\Omega_p(K^*) = \Omega_{n^2/p}(K)$.*

Proof. Set $\Psi(K) = \Omega_p(K^*)$. As explained in the proof of Lemma 17, Ψ is an upper semicontinuous and $\mathrm{SL}(n)$ invariant valuation that is homogeneous of degree $-n\,(n-p)/(n+p)$ and vanishes on polytopes. Therefore by Theorem 4, there is a constant $c \ge 0$ such that $\Psi(K) = c\,\Omega_r(K)$ where $r = n^2/p$. Since for the unit ball, B^n, all L_p-affine surface areas coincide, we have $\Omega_p(B^n) = \Omega_p((B^n)^*) = c\,\Omega_r(B^n) = c\,\Omega_p(B^n)$. Thus $c = 1$.

□

7 A new proof of Theorem 2

Since Φ is translation invariant, there is a constant c_0 such that $\Phi(K) = c_0$ for every singleton $K = \{x\}$. Then $\Phi_0 = \Phi - c_0\,V_0$ is an upper semicontinuous and equi-affine invariant valuation and it vanishes on singletons.

Since Φ_0 is equi-affine invariant, for every i-dimensional simplex S of i-dimensional volume x, $\Phi_0(S)$ depends only on i and x, that is, $\Phi_0(S) = f_i(x)$. For $i \le n-1$, two simplices of the same dimension are always (in $\mathbb{R}^n$) affine images of each other and thus $f_i(x) = a_i$ with some constant a_i, $i = 1, \dots, n-1$. Dissecting S into simplices S_1 and S_2 gives

$$\Phi_0(S) + \Phi_0(S_1 \cap S_2) = \Phi_0(S_1) + \Phi_0(S_2).$$

If S is one-dimensional, then $f_1(x_1 + x_2) = f_1(x_1) + f_1(x_2)$. Thus $f_1 = a_1 = 0$. By induction on the dimension of S, we obtain that $a_i = 0$ for $i \le n-1$. Thus Φ vanishes on simplices of dimension less than n. For $i = n$, we have $f_n(x_1 + x_2) = f_n(x_1) + f_n(x_2)$. Thus f_n is a solution of Cauchy's functional equation and there is a constant c_1 such that $\Phi_0(S) = c_1\,V_n(S)$ for all simplices S.

Set $\Psi = \Phi_0 - c_1\,V_n$. Then Ψ is an upper semicontinuous and $\mathrm{SL}(n)$ invariant valuation on $\mathcal{K}^n$ that vanishes on simplices. Since each polytope can be dissected into simplices, Ψ vanishes on $\mathcal{P}^n$ and is non-negative on $\mathcal{K}^n$. For every hyperplane H through the origin, Ψ is $\mathrm{GL}(n-1)$ invariant on $\mathcal{K}^{n-1}(H)$. Thus $0 \le \Psi(K) = \lim_{s\to 0} \Psi(sK) \le \Psi(0) = 0$ for $K \in \mathcal{K}^{n-1}(H)$. Since Ψ is translation invariant, this implies that Ψ is simple on $\mathcal{K}^n$.

Theorem 5 implies that there is a concave function $\psi : [0,\infty) \to [0,\infty)$ with $\lim_{t\to 0} \psi(t) = \lim_{t\to\infty} \psi(t)/t = 0$ such that

$$\Psi(K) = \int_{\partial K} \psi(\kappa_0(K,x))\, d\mu_K(x)$$

for every $K \in \mathcal{K}_0^n$.

For $0 < r < 1$, let $H_r^- = \{x \in \mathbb{R}^n : x \cdot e_1 \geq r\}$ and for $s > 0$, let $K = [B^n \cap H_r^-, -s\, e_1]$. We have $\Psi(K) = \psi(1)\, \sigma(B^n \cap H_r^-)$. For $t \in (-r, s)$, $K + t\, e_1 \in \mathcal{K}_0^n$, and

$$\Psi(K + t\, e_1) = \int_{S^{n-1} \cap H_r^-} \psi\Big(\frac{1}{(1 + t\, e_1 \cdot v)^{n+1}}\Big)\, (1 + t\, e_1 \cdot v)\, dv.$$

Since Φ is translation invariant, as $r \to 1$ we obtain

$$\psi\Big(\frac{1}{(1+t)^{n+1}}\Big)\, (1+t) = \psi(1).$$

Since $s > 0$ is arbitrary, this implies $\psi(t) = \psi(1)\, t^{\frac{1}{n+1}}$ for $t \geq 0$. This completes the proof of the Theorem.

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