Asymptotic approximation of smooth convex bodies by general polytopes

Monika Ludwig

1 Introduction and statement of results

For the optimal approximation of convex bodies by inscribed or circumscribed polytopes there are precise asymptotic results with respect to different notions of distance. In this paper we want to derive some results on optimal approximation without restricting the polytopes to be inscribed or circumscribed.

Let $\mathcal{P}_n$ and $\mathcal{P}_{(n)}$ denote the set of polytopes with at most n vertices and n facets, respectively. For a convex body C, i.e., a compact convex set with non-empty interior, we are interested in the asymptotic behavior as $n \to \infty$ of

$$\delta^S(C, \mathcal{P}_n) = \inf\{\delta^S(C, P) : P \in \mathcal{P}_n\}$$

and

$$\delta^S(C, \mathcal{P}_{(n)}) = \inf\{\delta^S(C, P) : P \in \mathcal{P}_{(n)}\}$$

where $\delta^S(.,.)$ is the symmetric difference metric, i.e., $\delta^S(C, D) = \operatorname{vol}(C \triangle D)$, the volume of the symmetric difference of C and D.

Before giving our results, we will describe some of the known results in this area, for more information we refer to the survey [8]. Let $\mathcal{P}_n^i$ be the set of polytopes having at most n vertices and being inscribed into C, i.e., their vertices are on the boundary of C, and let $\mathcal{P}_{(n)}^c$ be the set of polytopes having at most n facets and being circumscribed to C, i.e., each facet touches C. Define $\delta^S(C, \mathcal{P}_n^i)$ and $\delta^S(C, \mathcal{P}_{(n)}^c)$ as above. For a convex body C in Euclidean d-space $\mathbb{E}^d$ with boundary of differentiability class $\mathcal{C}^2$ and with positive Gaussian curvature κ_C, P.M. Gruber [9] proved that there are positive constants del_{d-1} and div_{d-1} (depending only on d) such that

$$\delta^S(C, \mathcal{P}_n^i) \sim \tfrac{1}{2} \operatorname{del}_{d-1} \left(\int_{\operatorname{bd} C} \kappa_C(x)^{1/(d+1)} d\sigma(x) \right)^{(d+1)/(d-1)} \frac{1}{n^{2/(d-1)}}$$

and

$$\delta^S(C, \mathcal{P}_{(n)}^c) \sim \tfrac{1}{2} \operatorname{div}_{d-1} \left(\int_{\operatorname{bd} C} \kappa_C(x)^{1/(d+1)} d\sigma(x) \right)^{(d+1)/(d-1)} \frac{1}{n^{2/(d-1)}}$$

as $n \to \infty$. Here σ is the surface area measure in $\mathbb{E}^d$. For the case $d = 2$ these results were stated by L. Fejes Tóth [4], p. 43, and proved by McClure and Vitale [11]. del_{d-1} and div_{d-1} are named after *Delone* triangulations and *Dirichlet-Voronoi* tilings, because these are used in the proofs of the asymptotic formulae. Only the values $\mathrm{del}_1 = 1/6$, $\mathrm{del}_2 = 1/(2\sqrt{3})$, $\mathrm{div}_1 = 1/12$ and $\mathrm{div}_2 = 5/(18\sqrt{3})$ (see [6] and [7] for the determination of del_2 and div_2, respectively) are known explicitly.

For approximation without restricting the polytopes to be inscribed or circumscribed the following is known. Let C be a convex body in $\mathbb{E}^d$ with boundary of differentiability class $\mathcal{C}^2$ and with positive Gaussian curvature κ_C. In the case $d = 2$, L. Fejes Tóth [4], p. 43, stated the following asymptotic formula,

$$\delta^S(C, \mathcal{P}_n) = \delta^S(C, \mathcal{P}_{(n)}) \sim \frac{1}{32} \left(\int_{\mathrm{bd}\, C} \kappa_C(x)^{1/3} d\sigma(x) \right)^3 \frac{1}{n^2} \tag{1.1}$$

as $n \to \infty$. For general d, Gruber and Kenderov [10] showed that there are positive constants α and β (depending only on d) such that

$$\frac{\alpha}{n^{2/(d-1)}} \leq \delta^S(C, \mathcal{P}_n) \leq \frac{\beta}{n^{2/(d-1)}} \tag{1.2}$$

for $n = d + 1, \ldots$. In our first theorem we give asymptotic results in the case of approximation by general polytopes.

Theorem 1 *Let C be a convex body in $\mathbb{E}^d$ with boundary of differentiability class $\mathcal{C}^2$ and with positive Gaussian curvature κ_C. Then there are positive constants* ldel_{d-1} *and* ldiv_{d-1}, *depending only on d, such that*

$$\delta^S(C, \mathcal{P}_n) \sim \tfrac{1}{2}\, \mathrm{ldel}_{d-1} \left(\int_{\mathrm{bd}\, C} \kappa_C(x)^{1/(d+1)} d\sigma(x) \right)^{(d+1)/(d-1)} \frac{1}{n^{2/(d-1)}} \tag{1.3}$$

and

$$\delta^S(C, \mathcal{P}_{(n)}) \sim \tfrac{1}{2}\, \mathrm{ldiv}_{d-1} \left(\int_{\mathrm{bd}\, C} \kappa_C(x)^{1/(d+1)} d\sigma(x) \right)^{(d+1)/(d-1)} \frac{1}{n^{2/(d-1)}} \tag{1.4}$$

as $n \to \infty$.

The constants ldel_{d-1} and ldiv_{d-1} are named after *Laguerre*, *Delone*, *Dirichlet* and *Voronoi*, because, instead of Voronoi tilings as in the case of circumscribed polytopes, Laguerre(-Delone-Dirichlet-Voronoi) tilings are used. It is easy to see that $\mathrm{ldel}_1 = \mathrm{ldiv}_1 = 1/16$ which proves (1.1). For the case $d = 3$, formula (1.3) was conjectured to hold in [6] with the constant $\mathrm{ldel}_2 = 1/(6\sqrt{3}) - 1/(8\pi)$. In a joint paper with K. Böröczky, Jr. [3] it is shown that this is the correct value and that $\mathrm{ldiv}_2 = 5/(18\sqrt{3}) - 1/(4\pi)$. For $d > 3$ it is probably difficult to determine the explicit values of ldel_{d-1} and ldiv_{d-1}.

$\int_{\mathrm{bd}\, C} \kappa_C(x)^{1/(d+1)} d\sigma(x)$ is called the affine surface area of C. The affine isoperimetric inequality (cf. [12], p. 419) states that among all convex bodies of given volume the affine surface area is maximal for ellipsoids. Thus (1.3) and (1.4) imply that among all convex bodies of given volume ellipsoids are asymptotically worst approximated by polytopes.

As a second notion of distance we use the L_1-distance of the support functions of the convex bodies, i.e.,

$$\delta_1(C, D) = \int_{S^{d-1}} |h_C(u) - h_D(u)| \, d\sigma(u)$$

where $h_C(u)$ is the support function of C (For notions of convex geometry not explained here, cf. [12].). For a convex body C in $\mathbb{E}^d$ with boundary of differentiability class $\mathcal{C}^2$ and with positive Gaussian curvature κ_C, Glasauer and Gruber [5] proved that

$$\delta_1(C, \mathcal{P}_n^i) \sim \tfrac{1}{2} \operatorname{div}_{d-1} \left(\int_{\mathrm{bd}\, C} \kappa_C(x)^{d/(d+1)} d\sigma(x) \right)^{(d+1)/(d-1)} \frac{1}{n^{2/(d-1)}}$$

and

$$\delta_1(C, \mathcal{P}_{(n)}^c) \sim \tfrac{1}{2} \operatorname{del}_{d-1} \left(\int_{\mathrm{bd}\, C} \kappa_C(x)^{d/(d+1)} d\sigma(x) \right)^{(d+1)/(d-1)} \frac{1}{n^{2/(d-1)}}$$

as $n \to \infty$. As before, the case $d = 2$ was treated by L. Fejes Tóth [4] and McClure and Vitale [11]. Our second theorem gives the respective results for approximation by general polytopes.

Theorem 2 *Let C be a convex body in $\mathbb{E}^d$ with boundary of differentiability class $\mathcal{C}^2$ and with positive Gaussian curvature κ_C. Then*

$$\delta_1(C, \mathcal{P}_n) \sim \tfrac{1}{2} \operatorname{ldiv}_{d-1} \left(\int_{\mathrm{bd}\, C} \kappa_C(x)^{d/(d+1)} d\sigma(x) \right)^{(d+1)/(d-1)} \frac{1}{n^{2/(d-1)}} \tag{1.5}$$

and

$$\delta_1(C, \mathcal{P}_{(n)}) \sim \tfrac{1}{2} \operatorname{ldel}_{d-1} \left(\int_{\mathrm{bd}\, C} \kappa_C(x)^{d/(d+1)} d\sigma(x) \right)^{(d+1)/(d-1)} \frac{1}{n^{2/(d-1)}} \tag{1.6}$$

as $n \to \infty$.

Both, Theorem 1 and Theorem 2 can be obtained from the following more general result. Let $w : \mathbb{E}^d \to \mathbb{R}$ be a continuous and positive function and define

$$\delta_w(C, D) = \int_{C \triangle D} w(x) \, dx$$

where $dx = dx^1 \ldots dx^d$ is the Lebesgue measure in $\mathbb{E}^d$.

Theorem 3 *Let C be a convex body in $\mathbb{E}^d$ with boundary of differentiability class $\mathcal{C}^2$ and with positive Gaussian curvature κ_C. Then*

$$\delta_w(C, \mathcal{P}_n) \sim \tfrac{1}{2} \operatorname{ldel}_{d-1} \left(\int_{\operatorname{bd} C} w(x)^{(d-1)/(d+1)} \kappa_C(x)^{1/(d+1)} d\sigma(x) \right)^{(d+1)/(d-1)} \frac{1}{n^{2/(d-1)}} \tag{1.7}$$

and

$$\delta_w(C, \mathcal{P}_{(n)}) \sim \tfrac{1}{2} \operatorname{ldiv}_{d-1} \left(\int_{\operatorname{bd} C} w(x)^{(d-1)/(d+1)} \kappa_C(x)^{1/(d+1)} d\sigma(x) \right)^{(d+1)/(d-1)} \frac{1}{n^{2/(d-1)}} \tag{1.8}$$

as $n \to \infty$.

Theorem 1 follows from this result by setting $w(x) = 1$. To obtain Theorem 2 we follow an idea of Glasauer described in [5] and repeat some of the arguments given there. We may assume that the origin $o \in \operatorname{int} C$ (where int stands for interior) and set $w(x) = \|x\|^{-(d+1)}$ where $\|x\|$ is the Euclidean norm of x. Then, by using polar coordinates we have

$$\begin{aligned} \delta_{\|x\|^{-(d+1)}}(C^*, D^*) &= \int_{C^* \triangle D^*} \|x\|^{-(d+1)}\, dx = \int_{S^{d-1}} \left| \frac{1}{\rho_{C^*}(u)} - \frac{1}{\rho_{D^*}(u)} \right| d\sigma(u) \\ &= \int_{S^{d-1}} |h_C(u) - h_D(u)|\, d\sigma(u) = \delta_1(C, D) \end{aligned}$$

where $C^* = \{x \in \mathbb{E}^d : \langle x, y \rangle \leq 1 \text{ for all } y \in C\}$ is the polar body of C and $\rho_{C^*}(u) = 1/\, h_C(u)$ is the radial function of C^*. If C is of class $\mathcal{C}^2$ and $\kappa_C > 0$ then C^* is of class $\mathcal{C}^2$ and $\kappa_{C^*} > 0$ (see [12], p. 111). Therefore, it follows from Theorem 3 that

$$\delta_1(C, \mathcal{P}_n) \sim \tfrac{1}{2} \operatorname{ldiv}_{d-1} \left(\int_{\operatorname{bd} C^*} \|x\|^{-(d-1)} \kappa_{C^*}(x)^{1/(d+1)} d\sigma(x) \right)^{(d+1)/(d-1)} \frac{1}{n^{2/(d-1)}}$$

as $n \to \infty$. Since

$$\int_{\operatorname{bd} C^*} \|x\|^{-(d-1)} \kappa_{C^*}(x)^{1/(d+1)} d\sigma(x) = \int_{\operatorname{bd} C} \kappa_C(x)^{d/(d+1)} d\sigma(x)$$

(see [5]), (1.5) is a consequence of (1.8) and similarly, (1.6) follows from (1.7). Thus, we only have to prove Theorem 3.

2 Approximation of paraboloids

As a first step in the proof of the asymptotic formulae (1.7) and (1.8) we consider the problem of approximating a paraboloid by a convex polyhedron and recall the connection of this question with Laguerre tilings. We denote by $|\ |$ volume in $\mathbb{E}^{d-1}$ and call balls and cubes in dimension $d-1$ circles and squares, respectively.

Let $q(u) = u^2 = (u^1)^2 + \ldots + (u^{d-1})^2$ where $u = (u^1, \ldots, u^{d-1}) \in \mathbb{E}^{d-1}$ and let $Q = \{(u, u^d) : u \in \mathbb{E}^{d-1}, u^d \geq q(u)\}$. For a given compact and Jordan measurable set $J \subset \mathbb{E}^{d-1}$ we want to find a convex polyhedron P with a given number of vertices or facets that minimizes $v_P(J) = \mathrm{vol}((P \triangle Q) \cap (J \times \mathbb{R}))$, i.e., the volume of that part of symmetric difference of Q and P which lies above J.

Let P be convex polyhedron with facets $F_1, \ldots, F_n$. P is the intersection of n half-spaces which — in the cases we are interested in — can be described as $\{(u, u^d) : u^d \geq l_i(u)\}$ where $l_i(u) = 2a_i u - a_i^2 + s_i$ with $a_i \in \mathbb{E}^{d-1}$ and $s_i \in \mathbb{R}$, i.e., $(a_i, q(a_i))$ is the point where the tangent plane to Q is parallel to F_i. Let $l(u) = \max_{i=1,\ldots,n} l_i(u)$, then $P = \{(u, u^d) : u^d \geq l(u)\}$ and

$$v_P(J) = \int_J |q(u) - l(u)|\, du.$$

Set $L = \{(a_1, s_1), \ldots, (a_n, s_n)\}$. $l(u)$ and P are determined by L and we will also write $v_L(J)$ for $v_P(J)$. We have

$$\begin{aligned} v_L(J) &= \int_J |q(u) - l(u)|\, du \\ &= \int_J |u^2 - \max_{i=1,\ldots,n} l_i(u)|\, du \\ &= \int_J |\min\{(u-a)^2 - s : (a, s) \in L\}|\, du. \end{aligned}$$

Define $V_i = \{u \in J : (u - a_i)^2 - s_i \leq (u - a_j)^2 - s_j \text{ for } j = 1, \ldots, n\}$. Then

$$v_L(J) = \sum_{i=1}^{n} \int_{V_i} |(u - a_i)^2 - s_i|\, du.$$

V_i is the intersection of J and the orthogonal projection of the facet F_i of P into $\mathbb{E}^{d-1}$. It is called *Laguerre* cell. The cells $V_1, \ldots, V_n$ form a tiling of J which is called *Laguerre tiling*, *Laguerre-Voronoi tiling*, *Dirichlet cell complex* or *power diagram* of L in J. In the case $s_1 = \ldots = s_n$ we obtain (ordinary) Dirichlet-Voronoi tilings. (For further information on Laguerre tilings, see, e.g., [1]).

First, we consider the problem of approximation by a polyhedron with a given number of facets. In this case it is easy to see that for a best approximating polyhedron every facet intersects Q which implies $s_i \geq 0$ and we set $s_i = r_i^2$. Define

$$\begin{aligned} v_{(n)}(J) &= \inf\{v_P(J) : P \in \mathcal{P}_{(n)}\} \\ &= \inf\{v_L(J) : L = \{(a_1, r_1^2), \ldots, (a_n, r_n^2)\}\}. \end{aligned} \tag{2.1}$$

Since $v_L(J)$ depends continuously on L, this infimum is attained for some L. We have $v_{(n)}(J) \to 0$ as $n \to \infty$. Therefore, if L_n are chosen such that $v_{L_n}(J) = v_{(n)}(J)$, then as $n \to \infty$

$$\max\{\mathrm{diam}\, V : V \text{ is Laguerre cell of } L_n\} \to 0 \tag{2.2}$$

where diam stands for diameter. Further we will use that for $\mu > 0$

$$v_{(n)}(\mu J) = \mu^{d+1} v_{(n)}(J) \tag{2.3}$$

which follows directly from the definition of $v_{(n)}(J)$.

We will now determine the asymptotic behavior of $v_{(n)}(J)$ as $n \to \infty$. The following lemma gives the definition of ldiv_{d-1}. See Lemma 2 of [9] for the respective result for Dirichlet-Voronoi tilings.

Lemma 1 *Let $I = \{u \in \mathbb{E}^{d-1} : 0 \le u^i \le 1\}$. For fixed d, there exists a positive constant ldiv_{d-1} such that*

$$v_{(n)}(I) \sim \frac{\mathrm{ldiv}_{d-1}}{n^{2/(d-1)}}$$

as $n \to \infty$.

Proof. For an $L = \{(a_1, r_1^2), \ldots, (a_n, r_n^2)\}$, denote by

$$B_i = \{u \in \mathbb{E}^{d-1} : (u - a_i)^2 \le r_i^2\}$$

the i-th Laguerre circle and by V_i the respective Laguerre cell. Define

$$v'_{(n)}(I) = \inf\{v_L(I) : L = \{(a_1, r_1^2), \ldots, (a_n, r_n^2)\} \text{ with } B_i \subset I \text{ for } i = 1, \ldots, n\}.$$

1. First, we show that there is a constant β such that

$$v'_{(n)}(I) \le \frac{\beta}{n^{2/(d-1)}} \tag{2.4}$$

for $n = 1, 2, \ldots$.

Let $k = 1, 2, \ldots$ and take the common tiling of I by k^{d-1} squares of side-length $1/k$. Let a_i, $i = 1, \ldots, k^{d-1}$, be the centers of these squares, set $r_1 = \ldots = r_{k^{d-1}} = 0$ and $L = \{(a_1, r_1), \ldots, (a_{k^{d-1}}, r_{k^{d-1}})\}$. Then

$$v'_{(k^{d-1})}(I) \le v_L(I) \le k^{d-1} \int_{-\frac{1}{2k}}^{\frac{1}{2k}} \cdots \int_{-\frac{1}{2k}}^{\frac{1}{2k}} ((u^1)^2 + \ldots + (u^{d-1})^2)\, du^1 \ldots du^{d-1} = \frac{d-1}{12\, k^2}.$$

For given n, choose k such that $k^{d-1} \le n \le (k+1)^{d-1}$. Then, there is a β such that

$$v'_{(n)}(I) n^{2/(d-1)} \le v'_{(k^{d-1})}(I)(k+1)^2 \le \frac{d-1}{12\, k^2}(k+1)^2 \le \beta$$

and (2.4) is proved.

2. The next step in our proof is to show that there is a positive constant α such that

$$v'_{(n)}(I) \ge \frac{\alpha}{n^{2/(d-1)}} \tag{2.5}$$

for $n = 1, 2, \ldots$.

To show this we use the following mean inequality

$$\sum_{i=1}^{n} \sigma_i^{(d+1)/(d-1)} n^{2/(d-1)} \geq (\sum_{i=1}^{n} \sigma_i)^{(d+1)/(d-1)} \tag{2.6}$$

where $\sigma_i \geq 0$, $i = 1, \ldots, n$, and the following inequality for the polar moment of inertia for a measurable set $D \subset \mathbb{E}^{d-1}$ (see, e.g., [2], p. 51)

$$\int_D u^2 \, du \geq \int_B u^2 \, du = \frac{d-1}{d+1} \kappa_{d-1} \Big(\frac{|D|}{\kappa_{d-1}} \Big)^{(d+1)/(d-1)} \tag{2.7}$$

where B is the $(d-1)$-dimensional circle of volume $|D|$ centered at the origin and κ_{d-1} is the volume of the $(d-1)$-dimensional unit circle.

Let L be chosen such that $v_L(I) = v'_{(n)}(I)$ and let $l(u)$ be the piecewise linear function given by L. Then, as $\varepsilon \to 0$

$$\int_I |q(u) - (l(u) + \varepsilon)| \, du = v_L(I) - \varepsilon \sum_{i=1}^{n} |V_i \backslash B_i| + \varepsilon \sum_{i=1}^{n} |B_i \cap V_i| + o(\varepsilon).$$

Since L is optimal, this implies

$$\sum_{i=1}^{n} |B_i \cap V_i| = \sum_{i=1}^{n} |V_i \backslash B_i|. \tag{2.8}$$

For every facet F_i of the polyhedron given by L, the cone with base $F_i \cap Q$ and apex at the point where the tangent plane of Q is parallel to F_i is contained in $Q \backslash P$ and has volume $r_i^2 |B_i \cap V_i|/d$. Moreover, these cones have pairwise disjoint interiors. Thus,

$$\sum_{i=1}^{n} r_i^2 |B_i \cap V_i| \leq d \sum_{i=1}^{n} \int_{B_i \cap V_i} (r_i^2 - (u - a_i)^2) \, du. \tag{2.9}$$

Consequently,

$$\begin{aligned} \sum_{i=1}^{n} \int_{B_i \cap V_i} (r_i^2 - (u - a_i)^2) \, du &= \sum_{i=1}^{n} r_i^2 |B_i \cap V_i| - \sum_{i=1}^{n} \int_{B_i \cap V_i} (u - a_i)^2 \, du \\ &\leq d \sum_{i=1}^{n} \int_{B_i \cap V_i} (r_i^2 - (u - a_i)^2) \, du - \sum_{i=1}^{n} \int_{B_i \cap V_i} (u - a_i)^2 \, du \end{aligned}$$

and by (2.7)

$$\begin{aligned} (d-1) \sum_{i=1}^{n} \int_{B_i \cap V_i} (r_i^2 - (u - a_i)^2) \, du &\geq \sum_{i=1}^{n} \int_{B_i \cap V_i} (u - a_i)^2 \, du \\ &\geq \frac{d-1}{d+1} \kappa_{d-1} \sum_{i=1}^{n} \Big(\frac{|B_i \cap V_i|}{\kappa_{d-1}} \Big)^{(d+1)/(d-1)}. \end{aligned}$$

Therefore, by (2.6) and (2.8)

$$\begin{aligned} v'_{(n)}(I)n^{2/(d-1)} &\geq \sum_{i=1}^{n} \int_{B_i \cap V_i} (r_i^2 - (u-a_i)^2)\, du\, n^{2/(d-1)} \\ &\geq \frac{1}{(d+1)\kappa_{d-1}^{2/(d-1)}} \sum_{i=1}^{n} |B_i \cap V_i|^{(d+1)/(d-1)} n^{2/(d-1)} \\ &\geq \frac{1}{(d+1)\kappa_{d-1}^{2/(d-1)}} \Big(\sum_{i=1}^{n} |B_i \cap V_i| \Big)^{(d+1)/(d-1)} \\ &\geq \frac{1}{(d+1)\kappa_{d-1}^{2/(d-1)}} \Big(\frac{1}{2} \Big)^{(d+1)/(d-1)} = \alpha \end{aligned}$$

and (2.5) is proved.
3. Define

$$\mathrm{ldiv}_{d-1} = \liminf_{n\to\infty} v'_{(n)}(I)n^{2/(d-1)}.$$

By (2.5) and (2.4) $0 < \mathrm{ldiv}_{d-1} < \infty$. We have to show that

$$\mathrm{ldiv}_{d-1} = \lim_{n\to\infty} v'_{(n)}(I)n^{2/(d-1)}. \tag{2.10}$$

To prove this it suffices to show the following for every $\lambda > 1$. If n_0 is chosen such that

$$v'_{(n_0)}(I)n_0^{2/(d-1)} \leq \lambda\, \mathrm{ldiv}_{d-1}, \tag{2.11}$$

then

$$v'_{(n)}(I)n^{2/(d-1)} \leq \lambda^2\, \mathrm{ldiv}_{d-1} \tag{2.12}$$

for all n sufficiently large.

Let $k = 1, 2, \ldots$ and take the common tiling of I by squares $I_1, \ldots, I_{k^{d-1}}$ of side-length $1/k$. Choose L_j such that $v'_{(n_0)}(I_j) = v_{L_j}(I_j)$. Then, since in the definition of $v'_{(n_0)}(I_j)$ we have $B = \{u \in \mathbb{E}^{d-1} : (u-a)^2 \leq r^2\} \subset I_j$ for every $(a, r^2) \in L_j$, we obtain

$$\int_{I_j} |\min\{(u-a)^2 - r^2 : (a, r^2) \in L_1 \cup \ldots \cup L_{k^{d-1}}\}|\, du \leq v_{L_j}(I_j),$$

and by (2.3)

$$v_{L_j}(I_j) = \frac{1}{k^{d+1}} v'_{(n_0)}(I)$$

for $j = 1, \ldots, k^{d-1}$. Therefore,

$$\begin{aligned} v'_{(n_0 k^{d-1})}(I) &\leq v_{L_1 \cup \ldots \cup L_{k^{d-1}}}(I_1 \cup \ldots \cup I_{k^{d-1}}) \\ &\leq v_{L_1}(I_1) + \ldots + v_{L_{k^{d-1}}}(I_{k^{d-1}}) \\ &\leq \frac{1}{k^2} v'_{(n_0)}(I) \end{aligned}$$

and

$$v'_{(n_0 k^{d-1})}(I)(n_0 k^{d-1})^{2/(d-1)} \leq v'_{(n_0)}(I) n_0^{2/(d-1)}. \tag{2.13}$$

Choose k_0 such that $((k+1)/k)^2 \leq \lambda$ for $k = k_0, k_0+1, \ldots$. For n sufficiently large we can find a $k \geq k_0$ such that $n_0 k^{d-1} \leq n \leq n_0(k+1)^{d-1}$. By (2.13) and (2.11) we have

$$\begin{aligned} v'_{(n)}(I) n^{2/(d-1)} &\leq v'_{(n_0 k^{d-1})}(I)(n_0(k+1)^{d-1})^{2/(d-1)} \\ &\leq v'_{(n_0)}(I) n_0^{2/(d-1)} (\frac{k+1}{k})^2 \leq \lambda^2 \mathrm{ldiv}_{d-1}. \end{aligned}$$

Thus, (2.12) and (2.10) are proved.
4. Obviously,

$$v_{(n)}(I) \leq v'_{(n)}(I). \tag{2.14}$$

We have to show that

$$\lim_{n\to\infty} v'_{(n)}(I) n^{2/(d-1)} \leq \liminf_{n\to\infty} v_{(n)}(I) n^{2/(d-1)}. \tag{2.15}$$

Let $\lambda > 1$ be chosen and let I' be a square concentric with I and with $|I'| = 1/\lambda$. Choose L_n such that $v_{(n)}(I) = v_{L_n}(I)$. (2.2) implies that for n sufficiently large

$$\bigcup\{V : V \text{ is Laguerre cell of } L_n \text{ with Laguerre circle } B \subset I\} \supset I' \tag{2.16}$$

and we denote by L'_n the subset of L_n defining such cells. It follows from (2.3) and (2.4) that we can choose $n' \leq (\lambda-1)n$ points $a_1, \ldots, a_{n'}$ lying in $I \backslash I'$ and define $L = \{(a_1, 0), \ldots, (a_{n'}, 0)\}$ such that

$$v_L(I\backslash I') \leq (1 - \frac{1}{\lambda})^{d+1} \frac{\beta}{((\lambda-1)n)^{2/(d-1)}}.$$

Combining this with (2.16) gives

$$v_{L'_n \cup L}(I) \leq v_{(n)}(I) + (1 - \frac{1}{\lambda})^{d+1} \frac{\beta}{((\lambda-1)n)^{2/(d-1)}}.$$

$L'_n \cup L$ has less than λn cells and for all of its Laguerre circles B we have $B \subset I$. Therefore,

$$v'_{(\lceil \lambda n \rceil)}(I) \leq v_{L'_n \cup L}(I)$$

and

$$\begin{aligned} \lim_{n\to\infty} v'_{(\lceil\lambda n\rceil)}(I)(\lceil \lambda n \rceil)^{2/(d-1)} &= \lim_{n\to\infty} v'_{(\lceil\lambda n\rceil)}(I)(\lambda n)^{2/(d-1)} \\ &\leq \lambda^{2/(d-1)} \liminf_{n\to\infty} v_{(n)}(I) n^{2/(d-1)} + (\frac{\lambda-1}{\lambda})^{(d^2-3)/(d-1)} \beta. \end{aligned}$$

Since $\lambda > 1$ was arbitrary, this shows (2.15). Combined with (2.10) this completes the proof of the lemma. □

Next, we extend Lemma 1 from the unit square to Jordan measurable sets and from $q(u) = u^2$ to general positive definite quadratic forms. For a convex function $f : \mathbb{E}^{d-1} \to \mathbb{R}$ define

$$v_P(J; f) = \int_J |f(u) - l(u)|\, du$$

where $l(u)$ describes the convex polyhedron P, and define $v_{(n)}(J; f)$ as above.

Lemma 2 *Let $J \subset \mathbb{E}^{d-1}$ be compact and Jordan measurable and q a positive definite quadratic form. Then*

$$v_{(n)}(J; q) \sim \mathrm{ldiv}_{d-1}(\det q)^{1/(d-1)}|J|^{(d+1)/(d-1)}\frac{1}{n^{2/(d-1)}}$$

as $n \to \infty$, where det *stands for determinant.*

Proof. We only have to prove the lemma for $q(u) = u^2$, the more general case can then be obtained by applying a linear transformation.

First, we show that

$$\limsup_{n\to\infty} v_{(n)}(J) n^{2/(d-1)} \le \mathrm{ldiv}_{d-1}|J|^{(d+1)/(d-1)}. \tag{2.17}$$

Let $\lambda > 1$ be chosen. Since J is Jordan measurable, we can find m squares $I_1, \dots, I_m$ of equal area $|I|$ such that

$$J \subset \bigcup_{l=1}^{m} I_l \tag{2.18}$$

and

$$m|I| \le \lambda |J|. \tag{2.19}$$

Thus, we have by (2.18), (2.10), (2.3), and (2.19)

$$\begin{aligned} v_{(km)}(J)(km)^{2/(d-1)} &\le \sum_{l=1}^{m} v'_{(k)}(I_l) k^{2/(d-1)} m^{2/(d-1)} \\ &\le \lambda\, \mathrm{ldiv}_{d-1}|I|^{(d+1)/(d-1)} m^{(d+1)/(d-1)} \qquad (2.20) \\ &\le \lambda^{(2d)/(d-1)}\mathrm{ldiv}_{d-1}|J|^{(d+1)/(d-1)} \end{aligned}$$

for k sufficiently large. Choose k_0 such that for $k = k_0, k_0 + 1, \dots$ (2.20) holds and $((k+1)/k)^{2/(d-1)} \le \lambda$. For sufficiently large n we can find a $k \ge k_0$ such that $mk \le n \le m(k+1)$. Therefore,

$$\begin{aligned} v_{(n)}(J) n^{2/(d-1)} &\le v_{(km)}(J)((k+1)m)^{2/(d-1)} \\ &\le \lambda^{(2d)/(d-1)}\mathrm{ldiv}_{d-1}|J|^{(d+1)/(d-1)}(\frac{k+1}{k})^{2/(d-1)} \\ &\le \lambda^{(3d-1)/(d-1)}\mathrm{ldiv}_{d-1}|J|^{(d+1)/(d-1)} \end{aligned}$$

and (2.17) is proved.

Second, we show that

$$\liminf_{n\to\infty} v_{(n)}(J) n^{2/(d-1)} \geq \mathrm{ldiv}_{d-1} |J|^{(d+1)/(d-1)}. \tag{2.21}$$

Let $\lambda > 1$ be chosen. Since J is Jordan measurable, there are pairwise disjoint squares $I_1, \ldots, I_m$ such that

$$\bigcup_{l=1}^{m} I_l \subset J \tag{2.22}$$

and

$$|J| \leq \lambda \sum_{l=1}^{m} |I_l|. \tag{2.23}$$

Choose L such that $v_{(n)}(J) = v_L(J)$ and denote by n_l the number of Laguerre cells of L which intersect I_l. Because of (2.2), we see that no Laguerre cell of L intersects two different squares for n sufficiently large. Therefore,

$$n_1 + \ldots + n_m \leq n. \tag{2.24}$$

By Lemma 1 and (2.3) there is a n_0 such that

$$v_{(k)}(I_l) \geq \frac{\mathrm{ldiv}_{d-1}}{\lambda} |I_l|^{(d+1)/(d-1)} \frac{1}{k^{2/(d-1)}} \tag{2.25}$$

for $k \geq n_0$ and $l = 1, \ldots, m$. Since $v_{(n)}(J) \to 0$ as $n \to \infty$,

$$n_l \geq n_0 \qquad \text{for } l = 1, \ldots, m$$

and n sufficiently large. Therefore, we have by (2.22) and (2.25)

$$v_{(n)}(J) \geq \sum_{l=1}^{m} v_{(n_l)}(I_l) \geq \frac{\mathrm{ldiv}_{d-1}}{\lambda} \sum_{l=1}^{m} \frac{|I_l|^{(d+1)/(d-1)}}{n_l^{2/(d-1)}}.$$

By using Hölder's inequality

$$\sum_{l=1}^{m} |I_l| = \sum_{l=1}^{m} \Big(\frac{|I_l|}{n_l^{2/(d+1)}}\Big) n_l^{2/(d+1)} \leq \Big(\sum_{l=1}^{m} \frac{|I_l|^{(d+1)/(d-1)}}{n_l^{2/(d-1)}}\Big)^{(d-1)/(d+1)} \Big(\sum_{l=1}^{m} n_l\Big)^{2/(d+1)},$$

(2.23) and (2.24), we obtain

$$\begin{aligned} v_{(n)}(J) &\geq \frac{\mathrm{ldiv}_{d-1}}{\lambda} \Big(\sum_{l=1}^{m} |I_l|\Big)^{(d+1)/(d-1)} \Big(\frac{1}{\sum_{l=1}^{m} n_l}\Big)^{2/(d-1)} \\ &\geq \frac{\mathrm{ldiv}_{d-1}}{\lambda^{(2d)/(d-1)}} |J|^{(d+1)/(d-1)} \frac{1}{n^{2/(d-1)}} \end{aligned}$$

for n sufficiently large and (2.21) is proved. □

As second case, we consider the problem of approximating a paraboloid by a convex polyhedron with n vertices. Define

$$\begin{aligned} v_n(J) &= \inf\{v_P(J) : P \in \mathcal{P}_n\} \\ &= \inf\{v_L(I) : L \text{ defines a Laguerre tiling of } I \text{ with at most } n \text{ vertices}\} \end{aligned}$$

and define $v_n(J; f)$ as above. Similar to the case of approximation by polyhedra with n facets we have the following results.

Lemma 3 *Let $I = \{u \in \mathbb{E}^{d-1} : 0 \leq u^i \leq 1\}$. Then, for fixed d, there exists a positive constant* ldel_{d-1} *such that*

$$v_n(I) \sim \frac{\mathrm{ldel}_{d-1}}{n^{2/(d-1)}}$$

as $n \to \infty$.

Lemma 4 *Let $J \subset \mathbb{E}^{d-1}$ be compact and Jordan measurable and q a positive definite quadratic form. Then*

$$v_n(J; q) \sim \mathrm{ldel}_{d-1} (\det q)^{1/(d-1)} |J|^{(d+1)/(d-1)} \frac{1}{n^{2/(d-1)}}$$

as $n \to \infty$.

We omit the proofs of these lemmata because they are similar to the proofs of Lemma 1 and Lemma 2. The upper and lower bounds for $v_n(I)$ needed in the proof of Lemma 3 can be obtained with the help of (1.2).

3 Proof of Theorem 3

We will only give the proof of (1.8). (1.7) can be obtained along similar lines.

3.1 We may assume that the origin $o \in \operatorname{int} C$. We use the following local representation of $\operatorname{bd} C$ (compare [9]): Given $p \in \operatorname{bd} C$, consider the ray R through p starting at o and let H be a hyperplane with $C \cap H = \emptyset$ which intersects R orthogonally. Choose a Cartesian coordinate system in H. Together with the normal unit vector of H which points to C it forms a Cartesian coordinate system in $\mathbb{E}^d$. We write " $'$ " for the orthogonal projection of $\mathbb{E}^d$ onto $H = \mathbb{E}^{d-1}$. Next, represent the lower side of $\operatorname{bd} C$ in the form $\{(u, f(u)) : u \in C'\}$. Then $f = f_p : C' \to \mathbb{R}$ is a convex function and $f|\operatorname{int} C'$ is of class $\mathcal{C}^2$. To each $u \in \operatorname{int} C'$ we let correspond the quadratic form $q_u = q_{p,u}$ defined by

$$q_u(s) = \sum_{i,j} f_{,ij}(u) s^i s^j \quad \text{for } s = (s^1, \ldots, s^{d-1}) \in H = \mathbb{E}^{d-1}$$

$(i, j = 1, \ldots, d-1)$. Here we denote by $f_{,j}$ the first partial derivatives of f. If $x = (u, f(u)) \in \operatorname{bd} C$, we write also $\kappa_C(u)$ for $\kappa_C(x)$ and have

$$\kappa_C(u) = \frac{\det q_u}{(1 + \sum_j f_{,j}(u)^2)^{(d+1)/2}} \tag{3.1}$$

for $u \in \operatorname{int} C'$.

3.2 Choose $\lambda > 1$. Since $\kappa_C > 0$, the quadratic forms q_u all are positive definite. Their coefficients are continuous. Therefore, we can choose an open convex neighbourhood U' of p' in $\operatorname{int} C'$ such that

$$\frac{1}{\lambda} q_{p'}(s) \leq q_u(s) \leq \lambda\, q_{p'}(s) \tag{3.2}$$

for $u \in U'$, $s \in \mathbb{E}^{d-1}$, and

$$\frac{1}{\lambda} \det q_{p'} \leq \det q_u \leq \lambda \det q_{p'}$$

for $u \in U'$. We denote by U the inverse image of U' with respect to the projection "$'$" on the lower side of $\operatorname{bd} C$.

The main step of the proof is to show the following result.

Proposition 1 *Let J be a compact Jordan measurable subset of U'. Then*

$$\frac{\operatorname{ldiv}_{d-1}}{2\lambda^{9d}} (\det q_{p'})^{1/(d-1)} |J|^{(d+1)/(d-1)} \frac{1}{k^{2/(d-1)}}$$
$$\leq v_{(k)}(J; f) \leq \frac{\lambda^{9d}}{2} \operatorname{ldiv}_{d-1} (\det q_{p'})^{1/(d-1)} |J|^{(d+1)/(d-1)} \frac{1}{k^{2/(d-1)}}$$

for k sufficiently large.

Proof. First, we establish the estimate from below for $v_{(k)}(J; f)$. There is a piecewise linear convex function $l(u) = \max_{i=1,\ldots,k} l_i(u)$ such that

$$v_{(k)}(J; f) = \int_J |f(u) - l(u)|\, du = \sum_{i=1}^{k} \int_{Q_i} |f(u) - l_i(u)|\, du$$

where $Q_i = \{u \in J : l_i(u) \geq l_j(u) \text{ for } j = 1, \ldots, k\}$. We can choose $a_i \in J$ such that the tangent plane to $f(u)$ at a_i is parallel to the plane determined by $l_i(u)$, i.e.,

$$l_i(u) = \sum_{l=1}^{d-1} f_{,l}(a_i)(u^l - a_i^l) + l_i(a_i).$$

As in the case of paraboloids, it is easy to see that every $l_i(u)$ intersects $f(u)$, and we set $r_i^2 = l(a_i) - f(a_i) \geq 0$. By Taylor's formula, there is, for fixed $u \in J$, a τ_i, $0 < \tau_i < 1$, such that

$$\begin{aligned} f(u) - l_i(u) &= f(a_i) + \sum_{l=1}^{d-1} f_{,l}(a_i)(u^l - a_i^l) + \tfrac{1}{2} q_{a_i + \tau_i(u - a_i)}(u - a_i) - l_i(u) \\ &= \tfrac{1}{2} q_{a_i + \tau_i(u - a_i)}(u - a_i) - r_i^2. \end{aligned} \tag{3.3}$$

Define

$$V_i = \{u \in J : \frac{1}{2\lambda} q(u - a_i) - r_i^2 \le \frac{1}{2\lambda} q(u - a_j) - r_j^2 \text{ for } j = 1, \dots, k\}$$

and

$$B_i = \{u \in \mathbb{E}^{d-1} : \frac{1}{2\lambda} q(u - a_i) \le r_i^2\},$$

where we write q for $q_{p'}$, and set $B = \bigcup_{i=1}^k B_i$.

For $u \in J$ and $u \notin B$, we have by (3.3) and (3.2)

$$f(u) - l_i(u) = \frac{1}{2} q_{a_i + \tau_i(u - a_i)}(u - a_i) - r_i^2 \ge \frac{1}{2\lambda} q(u - a_i) - r_i^2 \ge 0.$$

Thus,

$$\begin{aligned}
\int_{J \setminus B} |f(u) - l(u)|\, du &= \sum_{i=1}^k \int_{Q_i \setminus B} (f(u) - l_i(u))\, du \\
&\ge \sum_{i=1}^k \int_{Q_i \setminus B} (\frac{1}{2\lambda} q(u - a_i) - r_i^2)\, du \qquad (3.4) \\
&\ge \int_{J \setminus B} |\min\{\frac{1}{2\lambda} q(u - a_i) - r_i^2 : i = 1, \dots, k\}|\, du.
\end{aligned}$$

Next, we show that

$$\int_{J \cap B} |f(u) - l(u)|\, du \ge \frac{1}{\lambda^{8d}} \int_{J \cap B} |\min\{\frac{1}{2\lambda} q(u - a_i) - r_i^2 : i = 1, \dots, k\}|\, du. \quad (3.5)$$

Note that by (3.2) and (3.3)

$$\frac{1}{2\lambda} q(u - a_i) \le \frac{1}{2} q_{a_i + \tau_i(u - a_i)}(u - a_i) \le |f(u) - l_i(u)| + r_i^2. \quad (3.6)$$

We have

$$\int_{J \cap B} |f(u) - l(u)|\, du \ge \sum_{i=1}^k \int_{V_i \cap B_i} |f(u) - l_i(u)|\, du - \sum_{i=1}^k \int_{V_i \cap B_i} |l(u) - l_i(u)|\, du. \quad (3.7)$$

We need estimates for the right-hand side of (3.7). For the first expression we have by (3.3), (3.2) and (3.6)

$$\begin{aligned}
|f(u) - l_i(u)| &= |\frac{1}{2} q_{a_i + \tau_i(u - a_i)}(u - a_i) - r_i^2| \\
&\ge |\frac{1}{2\lambda} q(u - a_i) - r_i^2| - \frac{1}{2} |q_{a_i + \tau_i(u - a_i)}(u - a_i) - \frac{1}{\lambda} q(u - a_i)| \\
&\ge |\frac{1}{2\lambda} q(u - a_i) - r_i^2| - \frac{\lambda^2 - 1}{2\lambda} q(u - a_i) \\
&\ge |\frac{1}{2\lambda} q(u - a_i) - r_i^2| - (\lambda^2 - 1)|f(u) - l_i(u)| - (\lambda^2 - 1) r_i^2
\end{aligned}$$

and

$$|f(u) - l_i(u)| \geq \frac{1}{\lambda^2}|\frac{1}{2\lambda}q(u-a_i) - r_i^2| - \frac{\lambda^2-1}{\lambda^2}r_i^2.$$

Thus

$$\sum_{i=1}^{k}\int\limits_{V_i\cap B_i}|f(u)-l_i(u)|\,du \geq \frac{1}{\lambda^2}\sum_{i=1}^{k}\int\limits_{V_i\cap B_i}|\frac{1}{2\lambda}q(u-a_i)-r_i^2|\,du - \frac{\lambda^2-1}{\lambda^2}\sum_{i=1}^{k}r_i^2|V_i\cap B_i|. \tag{3.8}$$

As in (2.9), we have

$$\sum_{i=1}^{k} r_i^2|V_i\cap B_i| \leq d\sum_{i=1}^{k}\int_{V_i\cap B_i}|\frac{1}{2\lambda}q(u-a_i)-r_i^2|\,du. \tag{3.9}$$

Therefore, we get for (3.8)

$$\sum_{i=1}^{k}\int_{V_i\cap B_i}|f(u)-l_i(u)|\,du \geq \frac{1-(\lambda^2-1)d}{\lambda^2}\sum_{i=1}^{k}\int_{V_i\cap B_i}|\frac{1}{2\lambda}q(u-a_i)-r_i^2|\,du. \tag{3.10}$$

For the second expression on the right-hand side of (3.7) we obtain the following. For $u\in Q_j\cap V_i$, we have by (3.3)

$$\begin{aligned}
0\leq l(u)-l_i(u) &= l_j(u)-l_i(u)\\
&= (f(u)-l_i(u))-(f(u)-l_j(u))\\
&= \left(\frac{1}{2}q_{a_i+\tau_i(u-a_i)}(u-a_i)-r_i^2\right)-\left(\frac{1}{2}q_{a_j+\tau_j(u-a_j)}(u-a_j)-r_j^2\right)\\
&= \left(\frac{1}{2\lambda}q(u-a_i)-r_i^2\right)-\left(\frac{1}{2\lambda}q(u-a_j)-r_j^2\right)\\
&\quad+\left(\frac{1}{2}q_{a_i+\tau_i(u-a_i)}(u-a_i)-\frac{1}{2\lambda}q(u-a_i)\right)\\
&\quad-\left(\frac{1}{2}q_{a_j+\tau_j(u-a_j)}(u-a_j)-\frac{1}{2\lambda}q(u-a_j)\right).
\end{aligned}$$

By the definition of V_i, we have for $u\in V_i$

$$\frac{1}{2\lambda}q(u-a_i)-r_i^2 \leq \frac{1}{2\lambda}q(u-a_j)-r_j^2.$$

Thus, we obtain by (3.2)

$$\begin{aligned}
l(u)-l_i(u) &\leq \left(\frac{1}{2}q_{a_i+\tau_i(u-a_i)}(u-a_i)-\frac{1}{2\lambda}q(u-a_i)\right)\\
&\leq \frac{\lambda^2-1}{2\lambda}q(u-a_i)\\
&\leq (\lambda^2-1)\left(|\frac{1}{2\lambda}q(u-a_i)-r_i^2|+r_i^2\right)
\end{aligned}$$

which holds for all $u \in V_i$. Using (3.9) now gives

$$\begin{aligned}
\sum_{i=1}^{k} \int\limits_{V_i \cap B_i} & |l(u) - l_i(u)|\, du \\
& \leq (\lambda^2 - 1)\Big(\sum_{i=1}^{k} \int\limits_{V_i \cap B_i} |\frac{1}{2\lambda} q(u - a_i) - r_i^2|\, du + \sum_{i=1}^{k} r_i^2 |V_i \cap B_i| \Big) \\
& \leq (\lambda^2 - 1)(d+1) \sum_{i=1}^{k} \int_{V_i \cap B_i} |\frac{1}{2\lambda} q(u - a_i) - r_i^2|\, du.
\end{aligned}$$

Using this and (3.10), we obtain for (3.7)

$$\int_{J \cap B} |f(u) - l(u)|\, du \geq \frac{1}{\lambda^{8d}} \sum_{i=1}^{k} \int_{V_i \cap B_i} |\frac{1}{2\lambda} q(u - a_i) - r_i^2|\, du$$

for λ not too large, which proves (3.5).

Combining (3.4) and (3.5) and using Lemma 2 gives

$$\begin{aligned}
\int_J |f(u) - l(u)|\, du & \geq \frac{1}{\lambda^{8d}} v_{(k)}(J; \frac{1}{2\lambda} q) \\
& \geq \frac{1}{2\lambda^{8d+1}} \mathrm{ldiv}_{d-1} (\det q)^{1/(d-1)} |J|^{(d+1)/(d-1)} \frac{1}{k^{2/(d-1)}}
\end{aligned}$$

for k sufficiently large, i.e., the estimate from below is proved.

Second, we establish the estimate from above for $v_{(k)}(J; f)$. Choose $a_1, \ldots, a_k$ and $r_1, \ldots, r_k$ such that

$$v_{(k)}(J; \frac{\lambda}{2} q) = \sum_{i=1}^{k} \int_{V_i} |\frac{\lambda}{2} q(u - a_i) - r_i^2|\, du$$

where $V_i = \{u \in J : \lambda/2\, q(u - a_i) - r_i^2 \leq \lambda/2\, q(u - a_j) - r_j^2 \text{ for } j = 1, \ldots, k\}$. Define

$$l_i(u) = f(a_i) + \sum_{l=1}^{d-1} f_{,l}(a_i)(u^l - a_i^l) + r_i^2,$$

$Q_i = \{u \in J : l_i(u) \geq l_j(u) \text{ for } j = 1, \ldots, k\}$, and $D_i = \{u \in \mathbb{E}^{d-1} : f(u) \leq l_i(u)\}$ for $i = 1, \ldots, k$. Set $l(u) = \max_{i=1,\ldots,k} l_i(u)$ and $D = \bigcup_{i=1}^{k} D_i$.

For $u \notin D$, we have by (3.3) and (3.2)

$$0 \leq f(u) - l_i(u) \leq \frac{\lambda}{2} q(u - a_i) - r_i^2.$$

Thus, by the definition of $l(u)$ we have

$$\sum_{i=1}^{k} \int_{V_i \setminus D} |\frac{\lambda}{2} q(u - a_i) - r_i^2|\, du \geq \sum_{i=1}^{k} \int_{V_i \setminus D} (f(u) - l_i(u))\, du$$

$$\geq \sum_{i=1}^{k} \int_{V_i \setminus D} (f(u) - l(u))\, du \qquad (3.11)$$
$$= \int_{J \setminus D} |f(u) - l(u)|\, du.$$

Next, we show that

$$\sum_{i=1}^{k} \int_{V_i \cap D} |\frac{\lambda}{2} q(u-a_i) - r_i^2|\, du \geq \frac{1}{\lambda^{8d}} \int_{J \cap D} |f(u) - l(u)|\, du. \qquad (3.12)$$

Note that by the definition of the V_i's

$$\sum_{i=1}^{k} \int_{V_i \cap D} |\frac{\lambda}{2} q(u-a_i) - r_i^2|\, du = \int_{J \cap D} |\min\{\frac{\lambda}{2} q(u-a_j) - r_j^2 : j = 1, \ldots, k\}|\, du.$$

We have

$$|\min\{\frac{\lambda}{2} q(u-a_j) - r_j^2 : j = 1, \ldots, k\}| \geq$$
$$|\frac{\lambda}{2} q(u-a_i) - r_i^2| - |(\frac{\lambda}{2} q(u-a_i) - r_i^2) - \min\{\frac{\lambda}{2} q(u-a_j) - r_j^2 : j = 1, \ldots, k\}|. \qquad (3.13)$$

For the right-hand side of (3.13), we obtain the following. By (3.3) and (3.2),

$$|\frac{\lambda}{2} q(u-a_i) - r_i^2| \geq |f(u) - l_i(u)| - |\frac{\lambda}{2} q(u-a_i) - \frac{1}{2} q_{a_i + \tau_i(u-a_i)}(u-a_i)|$$
$$\geq |f(u) - l_i(u)| - \frac{\lambda^2 - 1}{2\lambda} q(u-a_i). \qquad (3.14)$$

For $u \in V_j \cap Q_i$, we see by the definition of Q_i and (3.2)

$$\begin{aligned}
&|(\frac{\lambda}{2} q(u-a_i) - r_i^2) - \min\{\frac{\lambda}{2} q(u-a_j) - r_j^2 : j = 1, \ldots, k\}| \\
&= (r_j^2 - \frac{\lambda}{2} q(u-a_j)) - (r_i^2 - \frac{\lambda}{2} q(u-a_i)) \\
&= (l_j(u) - f(u)) - (l_i(u) - f(u)) - (\frac{\lambda}{2} q(u-a_j) - \frac{1}{2} q_{a_j + \tau_j(u-a_j)}(u-a_j)) \\
&\quad + (\frac{\lambda}{2} q(u-a_i) - \frac{1}{2} q_{a_i + \tau_i(u-a_i)}(u-a_i)) \\
&\leq \frac{\lambda}{2} q(u-a_i) - \frac{1}{2} q_{a_i + \tau_i(u-a_i)}(u-a_i) \\
&\leq \frac{\lambda^2 - 1}{2\lambda} q(u-a_i) \\
&\leq (\lambda^2 - 1)\, |f(u) - l_i(u)| + (\lambda^2 - 1) r_i^2.
\end{aligned}$$

Combining this with (3.14) gives for (3.13)

$$|\min\{\frac{\lambda}{2} q(u-a_j) - r_j^2 : j = 1, \ldots, k\}|$$
$$\geq (1 - 2(\lambda^2 - 1))\, |f(u) - l_i(u)| - 2(\lambda^2 - 1) r_i^2.$$

Therefore,

$$\sum_{i=1}^{k} \int_{V_i \cap D} |\frac{\lambda}{2} q(u-a_i) - r_i^2|\, du \;\geq\; (1-2(\lambda^2-1)) \sum_{i=1}^{k} \int_{Q_i \cap D_i} |f(u) - l_i(u)|\, du$$
$$-2(\lambda^2-1) \sum_{i=1}^{k} r_i^2 |Q_i \cap D_i|.$$

Since we have as in (3.9)

$$\sum_{i=1}^{k} r_i^2 |Q_i \cap D_i| \leq d \sum_{i=1}^{k} \int_{Q_i \cap D_i} |f(u) - l_i(u)|\, du,$$

we obtain

$$\sum_{i=1}^{k} \int_{V_i \cap D} |\frac{\lambda}{2} q(u-a_i) - r_i^2|\, du \geq (1-2(d+1)(\lambda^2-1)) \int_{J \cap D} |f(u) - l(u)|\, du$$

and (3.12) follows. Combining (3.11) and (3.12) completes the proof of Proposition 1. □

3.3 Further we use the following decomposition of C, see [9].

Proposition 2 *There is a tiling of* $\mathbb{E}^d$ *with finitely many closed cones* $C_l, l = 1, \ldots, m$, *with common apex* o, *each of which has the following property: There is a point* $p_l \in C_l \cap \mathrm{bd}\, C = T_l$ *with corresponding* $R_l, H_l, U_l, U_l', f_l = f_{p_l}, q_u = q_{p_l,u}$, $q_l = q_{p_l,p_l'}$, *and* $w_l = w(p_l)$ *such that*

$$T_l \subset U_l,$$
$$\frac{1}{\lambda} q_l(s) \leq q_u(s) \leq \lambda\, q_l(s) \quad \textit{for } u \in U_l', s \in \mathbb{E}^{d-1},$$
$$\frac{1}{\lambda} \det q_l \leq \det q_u \leq \lambda \det q_l \quad \textit{for } u \in U_l', \tag{3.15}$$

and

$$\frac{1}{\lambda} w_l \leq w(x) \leq \lambda\, w_l \tag{3.16}$$

for $x \in C_l$ *and* x *at a distance less than* $\lambda - 1$ *from* T_l.

By our choice of T_l, (3.1), (3.15) and (3.16) we obtain the following.

$$\int_{\mathrm{bd}\, C} w(x)^{(d-1)/(d+1)} \kappa_C(x)^{1/(d+1)} d\sigma(x)$$
$$= \sum_{l=1}^{m} \int_{T_l'} w((u, f(u)))^{(d-1)/(d+1)} \frac{(\det q_u)^{1/(d+1)}}{(1+\sum_{k=1}^{d-1} f_{,k}(u)^2)^{1/2}} (1 + \sum_{k=1}^{d-1} f_{,k}(u)^2)^{1/2}\, du$$
$$\begin{cases} \leq & \lambda \sum_{l=1}^{m} w_l^{(d-1)/(d+1)} (\det q_l)^{1/(d+1)} |T_l'| \\ \geq & \frac{1}{\lambda} \sum_{l=1}^{m} w_l^{(d-1)/(d+1)} (\det q_l)^{1/(d+1)} |T_l'|. \end{cases} \tag{3.17}$$

3.4 Next, we give an estimate from below for $\delta_w(C, \mathcal{P}_{(n)})$.

Proposition 3 *For all n sufficiently large,*

$$\delta_w(C, \mathcal{P}_{(n)}) \geq \frac{1}{2\lambda^{12d}} \mathrm{ldiv}_{d-1} \Big(\int_{\mathrm{bd}\, C} w(x)^{(d-1)/(d+1)} \kappa_C(x)^{1/(d+1)} d\sigma(x) \Big)^{(d+1)/(d-1)} \frac{1}{n^{2/(d-1)}}.$$

Proof. There is a $P_n \in \mathcal{P}_{(n)}$ such that $\delta_w(C, P_n) = \delta_w(C, \mathcal{P}_{(n)})$. Denote by n_l the number of facets of P_n which lie entirely in C_l. Then

$$n_1 + \ldots + n_m \leq n. \tag{3.18}$$

Choose n_0 so large that for $k \geq n_0$ Proposition 1 holds for $f = f_l$ and $J = T_l'$ for $l = 1, \ldots, m$. $\kappa_C > 0$ implies that C is strictly convex. Since $P_n \overset{\delta_w}{\to} C$ as $n \to \infty$, this implies $\max\{\mathrm{diam}\, F : F \text{ is a facet of } P_n\} \to 0$ as $n \to \infty$. Therefore, we have

$$n_l \geq n_0$$

for n sufficiently large. We see by (3.16), by our choice of C_l, the definition of $v_{(k)}(T_l'; f_l)$ and Proposition 1 that

$$\begin{aligned} \delta_w(C, P_n) &\geq \frac{1}{\lambda} \sum_{l=1}^{m} w_l (\text{volume of the subset of } C \triangle P_n \text{ which lies in } C_l) \\ &\geq \frac{1}{\lambda^2} \sum_{l=1}^{m} w_l\, v_{(n_l)}(T_l'; f_l) \\ &\geq \frac{1}{2\lambda^{9d+2}} \mathrm{ldiv}_{d-1} \sum_{l=1}^{m} w_l (\det q_l)^{1/(d-1)} |T_l'|^{(d+1)/(d-1)} \frac{1}{n_l^{2/(d-1)}}. \end{aligned}$$

By Hölder's inequality

$$\sum_{l=1}^{m} w_l^{(d-1)/(d+1)} (\det q_l)^{1/(d+1)} |T_l'|$$
$$= \sum_{l=1}^{m} \Big(w_l^{(d-1)/(d+1)} (\det q_l)^{1/(d+1)} |T_l'| \frac{1}{n_l^{2/(d+1)}} \Big) n_l^{2/(d+1)}$$
$$\leq \Big(\sum_{l=1}^{m} w_l (\det q_l)^{1/(d-1)} |T_l'|^{(d+1)/(d-1)} \frac{1}{n_l^{2/(d-1)}} \Big)^{(d-1)/(d+1)} \Big(\sum_{l=1}^{m} n_l \Big)^{2/(d+1)}$$

we obtain by using (3.18) and (3.17)

$$\begin{aligned}
&\delta_w(C, P_n) \\
&\quad \geq \frac{\mathrm{ldiv}_{d-1}}{2\lambda^{9d+2}} \Big(\sum_{l=1}^{m} w_l^{(d-1)/(d+1)} (\det q_l)^{1/(d+1)} |T_l'| \Big)^{(d+1)/(d-1)} \frac{1}{(n_1 + \ldots + n_m)^{2/(d-1)}} \\
&\quad \geq \frac{\mathrm{ldiv}_{d-1}}{2\lambda^{9d+2}} \Big(\sum_{l=1}^{m} w_l^{(d-1)/(d+1)} (\det q_l)^{1/(d+1)} |T_l'| \Big)^{(d+1)/(d-1)} \frac{1}{n^{2/(d-1)}} \\
&\quad \geq \frac{\mathrm{ldiv}_{d-1}}{2\lambda^{9d+2+(d+1)/(d-1)}} \Big(\int_{\mathrm{bd}\, C} w(x)^{(d-1)/(d+1)} \kappa_C(x)^{1/(d+1)} d\sigma(x) \Big)^{(d+1)/(d-1)} \frac{1}{n^{2/(d-1)}}
\end{aligned}$$

for all n sufficiently large. Since $\delta_w(C, P_n) = \delta_w(C, \mathcal{P}_{(n)})$, the proof is complete. □

3.5 Next, we need an estimate from above for $\delta_w(C, \mathcal{P}_{(n)})$.

Proposition 4 *For all n sufficiently large,*

$$\delta_w(C, \mathcal{P}_{(n)}) \leq \frac{\lambda^{13d}}{2} \mathrm{ldiv}_{d-1} \left(\int_{\mathrm{bd}\, C} w(x)^{(d-1)/(d+1)} \kappa_C(x)^{1/(d+1)} d\sigma(x) \right)^{(d+1)/(d-1)} \frac{1}{n^{2/(d-1)}}.$$

Proof. To obtain this estimate we construct a polytope P with $\leq n$ facets in the following way: For $l = 1, \ldots, m$, choose polyhedra with n_l facets such that the infimum in $v_{(n_l)}(T_l'; f_l)$ is attained and define P as the intersection of these polyhedra. Then, P has at most $n_1 + \ldots + n_m$ facets and by (3.16)

$$\begin{aligned}
\delta_w(C, P) &\leq \lambda \sum_{l=1}^{m} w_l (\text{volume of the subset of } P \triangle C \text{ which lies in } C_l) \\
&\leq \lambda^2 \sum_{l=1}^{m} w_l \, v_{(n_l)}(T_l'; f_l)
\end{aligned}$$

for $n_1, \ldots, n_m$ sufficiently large. Define n_l as the largest integer such that

$$n_l \leq \frac{w_l^{(d-1)/(d+1)} (\det q_l)^{1/(d+1)} |T_l'|}{\sum_{j=1}^{m} w_j^{(d-1)/(d+1)} (\det q_j)^{1/(d+1)} |T_j'|} \, n \tag{3.19}$$

for $l = 1, \ldots, m$. Then,

$$n_1 + \ldots + n_m \leq n \tag{3.20}$$

and

$$\frac{1}{\lambda} \frac{w_l^{(d-1)/(d+1)} (\det q_l)^{1/(d+1)} |T_l'|}{\sum_{j=1}^{m} w_j^{(d-1)/(d+1)} (\det q_j)^{1/(d+1)} |T_j'|} \, n \leq n_l \tag{3.21}$$

for n sufficiently large. We have by (3.20), Proposition 1, (3.21) and (3.17)

$$
\begin{aligned}
&\delta_w(C, \mathcal{P}_{(n)}) \\
&\quad \le \lambda^2 \sum_{l=1}^{m} w_l\, v_{(n_l)}(T_l'; f_l) \\
&\quad \le \frac{\lambda^{9d+2}}{2} \mathrm{ldiv}_{d-1} \sum_{l=1}^{m} w_l (\det q_l)^{1/(d-1)} |T_l'|^{(d+1)/(d-1)} \frac{1}{n_l^{2/(d-1)}} \\
&\quad \le \frac{\lambda^{9d+2+2/(d-1)}}{2} \mathrm{ldiv}_{d-1} \Big(\sum_{j=1}^{m} w_j^{(d-1)/(d+1)} (\det q_j)^{1/(d+1)} |T_j'| \Big)^{2/(d-1)} \\
&\qquad \times \sum_{l=1}^{m} \Big(w_l^{1-2/(d+1)} (\det q_l)^{1/(d-1)-2/((d-1)(d+1))} |T_l'|^{(d+1)/(d-1)-2/(d-1)} \Big) \frac{1}{n^{2/(d-1)}} \\
&\quad \le \frac{\lambda^{9d+2+2/(d-1)}}{2} \mathrm{ldiv}_{d-1} \Big(\sum_{l=1}^{m} w_l^{(d-1)/(d+1)} (\det q_l)^{1/(d+1)} |T_l'| \Big)^{(d+1)/(d-1)} \frac{1}{n^{2/(d-1)}} \\
&\quad \le \frac{\lambda^{13d}}{2} \mathrm{ldiv}_{d-1} \Big(\int_{\mathrm{bd}\, C} w(x)^{(d-1)/(d+1)} \kappa_C(x)^{1/(d+1)} d\sigma(x) \Big)^{(d+1)/(d-1)} \frac{1}{n^{2/(d-1)}}
\end{aligned}
$$

for n sufficiently large. This concludes the proof of Proposition 4. □

3.5 Since $\lambda > 1$ was arbitrary, (1.8) follows from Propositions 3 and 4.

Acknowledgements

For several helpful discussions I would like to thank Prof. P.M. Gruber, Matthias Reitzner and Károly Böröczky, Jr.

References

[1] F. Aurenhammer, *Power diagrams: properties, algorithms and applications*, SIAM J. Comput. 16 (1987), 78–96.

[2] C. Bandle, *Isoperimetric inequalities and applications*, Pitman, Boston, 1980.

[3] K. Böröczky, Jr. and M. Ludwig, *Approximation of convex bodies and a momentum lemma for power diagrams*, Monatsh. Math. 127 (1999), 101–110.

[4] L. Fejes Tóth, *Lagerungen in der Ebene, auf der Kugel und im Raum*, 2nd ed., Springer, Berlin, 1972.

[5] S. Glasauer and P.M. Gruber, *Asymptotic estimates for best and stepwise approximation of convex bodies III*, Forum Math. 9 (1997), 383–404.

[6] P.M. Gruber, *Volume approximation of convex bodies by inscribed polytopes*, Math. Ann. 281 (1988), 229–245.

[7] P.M. Gruber, *Volume approximation of convex bodies by circumscribed polytopes*, Applied geometry and discrete mathematics, DIMACS Ser. Discrete Math. Theoret. Comput. Sci., 4, Amer. Math. Soc., Providence, RI, 1991, 309–317.

[8] P.M. Gruber, *Aspects of approximation of convex bodies*, Handbook of Convex Geometry A (P.M. Gruber and J. Wills, eds.), North-Holland, Amsterdam, 1993, 319–345.

[9] P.M. Gruber, *Asymptotic estimates for best and stepwise approximation of convex bodies II*, Forum Math. 5 (1993), 521–538.

[10] P.M. Gruber and P. Kenderov, *Approximation of convex bodies by polytopes*, Rend. Circ. Mat. Palermo 31 (1982), 195–225.

[11] D. McClure and R. Vitale, *Polygonal approximation of plane convex bodies*, J. Math. Anal. Appl. 51 (1975), 326–358.

[12] R. Schneider, *Convex bodies: the Brunn–Minkowski theory*, Cambridge Univ. Press, Cambridge, 1993.

Monika Ludwig
Abteilung für Analysis
Technische Universität Wien
Wiedner Hauptstraße 8-10/1142
A 1040 Vienna