A Characterization of Affine Surface Area

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Abstract

We show that every upper semicontinuous and equi-affine invariant valuation on the space of d-dimensional convex bodies is a linear combination of affine surface area, volume and the Euler characteristic.

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1 Introduction and statement of result

Let K be a convex body, i.e. a compact convex set, in Euclidean d-space $\mathbb{E}^d$. For K with boundary $\operatorname{bd} K$ of differentiability class $\mathcal{C}^2$ the *affine surface area* Ω is defined as

$$\Omega(K) = \int\limits_{\operatorname{bd} K} \kappa(x)^{\frac{1}{d+1}}\, d\sigma(x),$$

where $\kappa(x)$ is the Gaussian curvature of $\operatorname{bd} K$ at x and σ is the $(d-1)$-dimensional Hausdorff measure. This notion was introduced in affine differential geometry (see Blaschke's monograph [4]). The reason for its importance in this field is that it is *equi-affine invariant*, i.e. invariant with respect to volume preserving affine transformations. Also outside affine differential geometry, affine surface area has important applications, for example, in problems of asymptotic approximation of convex bodies by polytopes (see [9], [10]) and in the theory of affine inequalities (see [21]).

Beginning with the work of Leichtweiß [16], several ways of defining affine surface area Ω for general (not necessarily smooth) convex bodies were proposed. Since it was shown that these definitions are all equivalent, we can speak of *the* affine surface area Ω of a general convex body. Here we describe briefly three definitions of Ω, from which the properties needed for our characterization can easily be deduced. For more detailed information, we refer to Leichtweiß' monograph [18] and for further ways of defining affine surface area also to [24] and [31].

A theorem of Aleksandrov (see subsection 2.2) says that with respect to the $(d-1)$-dimensional Hausdorff measure σ at almost every point $x \in \operatorname{bd} K$ there is a paraboloid osculating $\operatorname{bd} K$. The (generalized) Gaussian curvature $\kappa(x)$ of $\operatorname{bd} K$ at such an x is defined as the Gaussian curvature of this paraboloid at x, and the function $\kappa(x)$ is Lebesgue integrable. Hence affine surface area can be defined as

$$\Omega(K) = \int_{\operatorname{bd} K} \kappa(x)^{\frac{1}{d+1}} \, d\sigma(x). \tag{1}$$

This definition was given by Schütt and Werner [30]. Schütt [29] (or see [12]) showed that it is equivalent to the definition given by Leichtweiß [16].

Schütt and Werner [30] also showed the following. For $\delta > 0$ define the convex floating body K_δ of K as the intersection of all half-spaces whose complements intersect K in a set of volume δ. Generalizing results by Blaschke [4] and Leichtweiß [15], they proved that

$$\lim_{\delta \to 0} c_d \, \frac{V(K) - V(K_\delta)}{\delta^{\frac{2}{d+1}}} = \int_{\operatorname{bd} K} \kappa(x)^{\frac{1}{d+1}} \, d\sigma(x), \tag{2}$$

where c_d is a suitable constant and V stands for volume. Consequently, the left-hand side of (2) can also be used as a definition for Ω.

Lutwak [20] gave the following definition of affine surface area for general convex bodies. Let $\mathcal{S}_o^d$ be the set of starshaped bodies in $\mathbb{E}^d$ with non-empty interior and centroid at the origin. Define

$$\Omega(K) = \inf_{L \in \mathcal{S}_o^d} \{(d\,V(L))^{\frac{1}{d}} \int_{S^{d-1}} \frac{1}{\rho_L(u)} \, d\sigma_K(u)\}^{\frac{d}{d+1}}, \tag{3}$$

where S^{d-1} is the unit sphere centered at the origin, $\rho_L(u)$ is the radial function of L at $u \in S^{d-1}$, and σ_K is Aleksandrov's surface area measure of K. This definition is related to Petty's notion of geominimal surface area [25]. It was shown to be equivalent to the other definitions by Leichtweiß [17] and Dolzmann and Hug [6].

Let $\mathcal{K}^d$ be the space of convex bodies in $\mathbb{E}^d$ equipped with the usual topology induced by the Hausdorff metric (cf. [28]). Then Ω is a functional defined for every $K \in \mathcal{K}^d$ and has the following properties.

(i) It is *equi-affine invariant*, i.e. for every volume preserving affine transformation φ and every convex body K

$$\Omega(\varphi(K)) = \Omega(K)$$

holds. For general convex bodies, this follows, for example, from (2), since the volume of convex floating bodies is equi-affine invariant.

(ii) Ω is *upper semicontinuous*, i.e.

$$\Omega(K) \geq \limsup_{n\to\infty} \Omega(K_n)$$

for every $K \in \mathcal{K}^d$ and every sequence $K_n \in \mathcal{K}^d$ with $K_n \to K$. This was first – even for smooth bodies – proved by Lutwak [20]. The weak continuity of the surface area measure σ_K implies that the functionals over which the infimum is taken in (3) are continuous, and as an infimum of continuous functionals affine surface area is therefore upper semicontinuous. Since $\Omega(P) = 0$ for every polytope $P \in \mathcal{K}^d$, affine surface area is not continuous.

(iii) Ω is a *valuation*. Here a functional $\mu : \mathcal{K}^d \to \mathbb{R}$ is called a valuation, if for every $K, L \in \mathcal{K}^d$ with $K \cup L \in \mathcal{K}^d$

$$\mu(K \cup L) + \mu(K \cap L) = \mu(K) + \mu(L)$$

holds. That Ω has this property follows from (1) (see [29]).

Valuations play an important role in convex geometry (see [23], [22]) and have many applications in integral geometry (see [14]). One of the most important results in this field is the following characterization theorem by Hadwiger [11]:
A functional $\mu : \mathcal{K}^d \to \mathbb{R}$ is a continuous and rigid motion invariant valuation if and only if there are constants $c_0, c_1, \ldots, c_d$ such that

$$\mu(K) = c_0\, W_0(K) + \ldots + c_d\, W_d(K)$$

for every $K \in \mathcal{K}^d$.

Here $W_0(K), \ldots, W_d(K)$ are the quermassintegrals of K. In particular, $W_0(K)$ is equal to the volume $V(K)$ and $W_d(K)$ is a multiple of the Euler characteristic $\chi(K)$. For a short proof of this theorem, see Klain [13].

Prior to Hadwiger, Blaschke [5] indicated that every continuous and equi-affine invariant valuation on $\mathcal{K}^3$ is a linear combination of volume and the Euler characteristic. Our aim is to extend Blaschke's result in order to obtain also a characterization of affine surface area. Ω is an equi-affine invariant and upper semicontinuous valuation on $\mathcal{K}^d$. Other examples of such functionals are volume and the Euler characteristic. We show that these properties characterize affine surface area, volume and the Euler characteristic.

Theorem. *A functional $\mu : \mathcal{K}^d \to \mathbb{R}$ is an upper (or lower) semicontinuous and equi-affine invariant valuation if and only if there are constants c_0, c_1, and $c_2 \geq 0$ ($c_2 \leq 0$) such that*

$$\mu(K) = c_0\, \chi(K) + c_1\, V(K) + c_2\, \Omega(K)$$

for every $K \in \mathcal{K}^d$.

That such a linear combination of affine surface area, volume and the Euler characteristic is an upper semicontinuous and equi-affine invariant valuation follows from the properties of affine surface area described above. We show that also the converse holds.

In the planar case, this theorem was proved in [19] and independently by Tibor Ódor. Recently, he informed us that he has also obtained the above theorem.

2 Tools

1. We need the following result on valuations on the set of spherical polytopes. Let S^{d-1} denote the unit sphere in $\mathbb{E}^d$ and σ the $(d-1)$-dimensional Hausdorff measure. A $C \subset \mathbb{E}^d$ is a *polyhedral cone*, if it is the intersection of a finite family of closed half-spaces which have the origin in their boundaries, and a $P \subset S^{d-1}$ is called a *spherical polytope*, if there is a polyhedral cone C such that $P = S^{d-1} \cap C$. Let $\mathcal{P}(S^{d-1})$ be the set of spherical polytopes and let $\nu : \mathcal{P}(S^{d-1}) \to \mathbb{R}$ be a simple valuation, i.e. $\nu(P_1 \cup P_2) = \nu(P_1) + \nu(P_2)$ for every $P_1, P_2 \in \mathcal{P}(S^{d-1})$ with $P_1 \cup P_2 \in \mathcal{P}(S^{d-1})$ and where $P_1 \cap P_2$ is at most $(d-2)$-dimensional. Schneider [27] proved the following characterization theorem.

Let $\nu : \mathcal{P}(S^{d-1}) \to \mathbb{R}$ be a rotation invariant, non-negative and simple valuation. Then, there is a constant $c \geq 0$ such that $\nu(P) = c\,\sigma(P)$ for every $P \in \mathcal{P}(S^{d-1})$.

2. A convex function $f : \mathbb{E}^{d-1} \to \mathbb{R}$ is twice differentiable at a point x_0', if there exists a second order Taylor expansion at x_0', i.e. the gradient $\operatorname{grad} f$ at x_0' exists and there is a symmetric linear map $A_{f(x_0')}$ such that

$$f(x') = f(x_0') + \langle \operatorname{grad} f(x_0'), x' - x_0' \rangle + \frac{1}{2} \langle A_{f(x_0')}(x' - x_0'), x' - x_0' \rangle + o(|x' - x_0'|^2)$$

as $|x' - x_0'| \to 0$, where $\langle \cdot, \cdot \rangle$ denotes the inner product and $|\cdot|$ the norm in $\mathbb{E}^{d-1}$ (see [28]).

Let K be a convex body in $\mathbb{E}^d$ and $x_0 \in \operatorname{bd} K$. x_0 is called a *normal* point of $\operatorname{bd} K$, if $\operatorname{bd} K$ can be represented in a neighbourhood of x_0 by a convex function $f : \mathbb{E}^{d-1} \to \mathbb{R}$ such that $x = (x^1, \ldots, x^d) = (x', f(x'))$ for $x \in \operatorname{bd} K$ and such that f is twice differentiable at x_0'. In this case, choosing a suitable coordinate system in $\mathbb{E}^d$ makes it possible to write

$$f(x') = \frac{1}{2} \left(\kappa_1(x_0)\,(x^1 - x_0^1)^2 + \ldots + \kappa_{d-1}(x_0)\,(x^{d-1} - x_0^{d-1})^2 \right) + o(|x' - x_0'|^2)$$

as $|x' - x_0'| \to 0$. The coefficients $\kappa_1(x_0), \ldots, \kappa_{d-1}(x_0)$ are the (generalized) principal curvatures of $\operatorname{bd} K$ at x_0 and $\kappa(x_0) = \kappa_1(x_0) \cdots \kappa_{d-1}(x_0)$ is the (generalized) Gaussian curvature. The convexity of K implies that $\kappa_1, \ldots, \kappa_{d-1} \geq 0$. If all principal curvatures are positive, then K is osculated by suitable ellipsoids at x_0. If $\kappa(x_0) = 0$, then K is osculated by suitable cylinders at x_0.

Let $N \subset \operatorname{bd} K$ denote the set of normal points of $\operatorname{bd} K$. A classical theorem of Aleksandrov [2] (or see [3]) states that

$$\sigma(N) = \sigma(\operatorname{bd} K), \tag{4}$$

i.e. almost all points on $\operatorname{bd} K$ are normal.

3. We need the following results on packings. They follow immediately from the Euclidean results (see, for example, [8]), since spheres and cylinders are locally Euclidean. Let B^d be the closed unit ball in $\mathbb{E}^d$, and let $B(x,r)$ be a closed d-dimensional ball with center x and radius r. We say that the balls $B(x_1, r), \ldots, B(x_n, r)$ define a packing in S^{d-1}, if $x_i \in S^{d-1}$ for $i = 1, \ldots, n$ and if the sets $S^{d-1} \cap B(x_i, r)$ have pairwise disjoint relative interior in S^{d-1}. Let $m(r)$ be the maximum number of balls of radius r that define a packing in S^{d-1}. Then

$$\kappa_{d-1}\, m(r)\, r^{d-1} \to \delta_{d-1}\, d\, \kappa_d \tag{5}$$

as $r \to 0$, where $\delta_{d-1} > 0$ is the packing density of balls in $\mathbb{E}^{d-1}$ and κ_k is the volume of the k-dimensional unit ball.

A similar statement holds for cylinders. Let $I^k \subset \mathbb{E}^k$ denote the closed k-dimensional cube with side length 2 centered at the origin. Then $I^k \times B^{d-k}$, where B^{d-k} is the $(d-k)$-dimensional unit ball, is a cylinder. We say that the balls $B(x_1, r), \ldots, B(x_n, r)$ define a packing in the lateral surface $Z = I^k \times S^{d-k-1}$ of this cylinder, if $x_i \in Z$ for $i = 1, \ldots, n$ and if the sets $Z \cap B(x_i, r)$ have pairwise disjoint relative interior in Z. Let $m(r)$ be the maximum number of disjoint balls of radius r that define a packing in Z. Then

$$\kappa_{d-1}\, m(r)\, r^{d-1} \to \delta_{d-1}\, 2^k\, (d-k)\, \kappa_{d-k} \tag{6}$$

as $r \to 0$.

4. Finally, we make use of the following result from measure theory (see, for example, [7] and [26]). Let $N \subset \mathbb{E}^d$ be a set of finite $(d-1)$-dimensional Hausdorff measure $\sigma(N)$ and denote the diameter of set V by $\operatorname{diam} V$. We call a collection $\mathcal{V}$ of sets a *Vitali class* for N, if for every $x \in N$ and every $\delta > 0$, there is a $V \in \mathcal{V}$ such that $x \in V$, $0 < \operatorname{diam} V \leq \delta$, and

$$\frac{\sigma(V)}{(\operatorname{diam} V)^{d-1}} \geq q(x) > 0,$$

where $q(x)$ depends only on x. Then a version of Vitali's covering theorem states the following.

Let $\mathcal{V}$ be a Vitali class of closed sets for N. Then for every $\varepsilon > 0$ there are pairwise disjoint $V_1, \ldots, V_n \in \mathcal{V}$ such that

$$\sigma(N) \leq \sum_{i=1}^{n} \sigma(V_i) + \varepsilon.$$

3 Proof

The proof is organized in the following way. In the first part, we use induction on the dimension d to arrive at the characterization of affine surface area in Proposition 2. In the second part, it is shown that it suffices to consider ε-smooth convex bodies, i.e. to show Proposition 3, for which the proof is given in the last part.

1. If $-\mu$ is lower semicontinuous, then μ is upper semicontinuous. Thus it suffices to consider upper semicontinuous μ. Since μ is translation invariant, there is a constant c_0 such that

$$\mu(K) = c_0$$

for every singleton $K = \{x\}$. We define

$$\mu_0(K) = \mu(K) - c_0\,\chi(K)$$

for every $K \in \mathcal{K}^d$. Then μ_0 is an upper semicontinuous and equi-affine invariant valuation and it vanishes on singletons. Thus it suffices to show the following proposition to prove our theorem.

Proposition 1. *Let $\mu : \mathcal{K}^d \to \mathbb{R}$ be an upper semicontinuous and equi-affine invariant valuation which vanishes on singletons. Then there are constants c_1 and $c_2 \geq 0$ such that*

$$\mu(K) = c_1\,V(K) + c_2\,\Omega(K)$$

for every $K \in \mathcal{K}^d$.

To prove this proposition we use induction on the dimension d. Let $d = 1$. Then every convex body is a closed interval. Since μ vanishes on singletons and is translation invariant, this implies that $\mu(K)$ depends only on the length of the interval K. Thus we can define a function $f : [0, \infty) \to \mathbb{R}$ by

$$f(x) = \mu(K),$$

where $x = V_1(K)$ is the length of the interval K. Since μ is a valuation and vanishes on singletons,

$$f(x + y) = f(x) + f(y)$$

holds for every $x, y \in [0, \infty)$. Thus f is a solution of Cauchy's functional equation. Since μ is upper semicontinuous, also f has this property. By a well known property of solutions of Cauchy's functional equation (see, for example, [1]), this implies that there is a constant a such that

$$f(x) = a\,x$$

for every $x \in [0,\infty)$. Thus

$$\mu(K) = a\, V_1(K)$$

and Proposition 1 holds for $d = 1$.

So suppose that Proposition 1 holds in dimension $(d-1)$, i.e. for every upper semicontinuous and equi-affine invariant valuation $\nu : \mathcal{K}^{d-1} \to \mathbb{R}$ which vanishes on singletons, there are constants a_1 and a_2 such that

$$\nu(K) = a_1\, V_{d-1}(K) + a_2\, \Omega_{d-1}(K) \tag{7}$$

for every $K \in \mathcal{K}^{d-1}$, where V_{d-1} is the volume and Ω_{d-1} is the affine surface area in $\mathbb{E}^{d-1}$.

Now let $d \geq 2$ and let $\mu : \mathcal{K}^d \to \mathbb{R}$ be an upper semicontinuous and equi-affine invariant valuation which vanishes on singletons. Using (7), we show that μ is *simple*, i.e. $\mu(K) = 0$ for every at most $(d-1)$-dimensional $K \in \mathcal{K}^d$.

Lemma 1.1. *μ is simple.*

Proof. Let $H \subset \mathbb{E}^d$ be a hyperplane, let $\mathcal{K}(H)$ be the set of convex bodies $K \subset H$, and let ν be the restriction of μ to $\mathcal{K}(H)$. Then $\nu : \mathcal{K}(H) \to \mathbb{R}$ is an upper semicontinuous valuation which is invariant with respect to affine transformations (including dilations). If we identify $\mathcal{K}(H)$ with $\mathcal{K}^{d-1}$, we can apply our induction assumption (7) and obtain

$$\nu(K) = a_1\, V_{d-1}(K) + a_2\, \Omega_{d-1}(K)$$

for every $K \in \mathcal{K}(H)$ with suitable constants a_1, a_2. Since both V_{d-1} and Ω_{d-1} are homogeneous and neither V_{d-1} nor Ω_{d-1} is invariant with respect to dilations, we conclude that $a_1 = a_2 = 0$. □

Note that simple valuations have the following additivity property. If $P_1, \ldots, P_n$ are polytopes with pairwise disjoint interior and if K is a convex body with $K \subset P_1 \cup \ldots \cup P_n$, then

$$\mu(K) = \mu(K \cap P_1) + \ldots + \mu(K \cap P_n).$$

This follows easily from the valuation property by using induction on n (see, for example, [11], p. 81).

Next, we subtract a suitable multiple of volume from μ to obtain a valuation which vanishes on polytopes. Let S be a simplex, i.e. the convex hull of $d+1$ points in $\mathbb{E}^d$. Since μ is equi-affine invariant and simple, the value $\mu(S)$ depends only on the volume of S, i.e. there is a function $f : [0,\infty) \to \mathbb{R}$ such that

$$f(x) = \mu(S),$$

where $x = V(S)$. For given values $x_1, x_2 \geq 0$, we can find simplices S_1 and S_2 of volume x_1 and x_2, respectively, such that $S_1 \cup S_2$ is a simplex of volume $x_1 + x_2$. Therefore, taking into account that μ is simple valuation, we obtain

$$\mu(S_1 \cup S_2) = \mu(S_1) + \mu(S_2)$$

and

$$f(x_1 + x_2) = f(x_1) + f(x_2).$$

Thus f is a solution of Cauchy's functional equation and since it is upper semicontinuous, this implies that there is a constant c_1 such that

$$f(x) = c_1\, x$$

and consequently

$$\mu(S) = c_1\, V(S)$$

for every simplex S. Since every polytope P can be dissected into simplices and since μ is a simple valuation, this implies that

$$\mu(P) = c_1\, V(P)$$

for every polytope $P \in \mathcal{K}^d$.

We define

$$\mu_1(K) = \mu(K) - c_1\, V(K)$$

for every $K \in \mathcal{K}^d$ and obtain a functional $\mu_1 : \mathcal{K}^d \to \mathbb{R}$, which is an upper semicontinuous, equi-affine invariant and simple valuation with the property that $\mu_1(P) = 0$ for every polytope P. Since every convex body K can be approximated by polytopes P_n and since μ_1 vanishes on polytopes, the upper semicontinuity of μ_1 implies that

$$\mu_1(K) \geq \limsup_{n\to\infty} \mu_1(P_n) = 0, \tag{8}$$

i.e. μ_1 is non-negative.

Using our induction assumption (7), we now show that μ_1 vanishes on cylinders. A convex body Z is called a *cylinder*, if Z is the Minkowski sum of a $(d-1)$-dimensional convex body K and a closed line segment I, i.e. $Z = K + I$.

Lemma 1.2. *For every cylinder Z, $\mu_1(Z) = 0$.*

Proof. We choose a hyperplane H and a direction in $\mathbb{E}^d$ and consider only line segments I parallel to this direction. For a given $K \in \mathcal{K}(H)$, the translation invariance of μ_1 implies that $\mu_1(K + I)$ depends only on the length $x = V_1(I)$ of I, and we define

$$f_K(x) = \mu_1(K + I).$$

Since μ_1 is a translation invariant and simple valuation, we see that for line segments I_1, I_2

$$\mu_1(K + (I_1 + I_2)) = \mu_1(K + I_1) + \mu_1(K + I_2)$$

holds. Setting $x_1 = V_1(I_1)$ and $x_2 = V_1(I_2)$ we therefore obtain

$$f_K(x_1 + x_2) = f_K(x_1) + f_K(x_2),$$

which implies that f_K is a solution of Cauchy's functional equation. In addition, f_K is upper semicontinuous. Thus there is a $\nu(K)$ such that

$$f_K(x) = \nu(K)\, x$$

and

$$\mu_1(K + I) = \nu(K)\, V_1(I)$$

for every $K \in \mathcal{K}(H)$ and every line segment I.

ν is defined on $\mathcal{K}(H)$. Since μ_1 is an equi-affine invariant and upper semicontinuous valuation and vanishes on polytopes, also ν has these properties. We can therefore apply our assumption (7) for $\mathcal{K}^{d-1}$ and obtain

$$\nu(K) = a\, \Omega_{d-1}(K)$$

for every $K \in \mathcal{K}(H)$ with a suitable constant a, and consequently

$$\mu_1(K + I) = a\, \Omega_{d-1}(K)\, V_1(I) \tag{9}$$

for every $K \in \mathcal{K}(H)$ and every line segment I.

We now choose equi-affine invariant transformations φ_t in the following way. We dilate by a factor $t > 0$ in H and by a factor $\frac{1}{t^{d-1}}$ in the direction parallel to our line segments. Then $\varphi_t(K + I)$ is a translate of the cylinder $t\, K + \frac{1}{t^{d-1}} I$. ¿From the equi-affine invariance of μ_1 and from (9), we therefore obtain

$$\mu_1(\varphi_t(K + I)) = a\, \Omega_{d-1}(t\, K)\, V_1(\frac{1}{t^{d-1}} I) = \mu_1(K + I) = a\, \Omega_{d-1}(K)\, V_1(I).$$

If $a \neq 0$, then

$$\Omega_{d-1}(t\, K) = t^{d-1}\, \Omega_{d-1}(K)$$

for every $K \in \mathcal{K}(H)$ and every $t > 0$. But Ω_{d-1} is not homogeneous of degree $d - 1$. Thus $a = 0$ and $\mu_1(Z) = 0$. □

2. By the definition of μ_1, Lemma 1.1 and 1.2, it suffices to show the following result to complete our proof by induction of Proposition 1.

Proposition 2. *Let $\mu : \mathcal{K}^d \to [0, \infty)$ be an upper semicontinuous, equi-affine invariant and simple valuation, which vanishes on polytopes and on cylinders. Then there is a constant $c \geq 0$ such that*

$$\mu(K) = c\,\Omega(K)$$

for every $K \in \mathcal{K}^d$.

To prove this proposition we show that if we know the value of μ for the unit ball B^d, the value of μ for every $K \in \mathcal{K}^d$ is already uniquely determined. That this implies Proposition 2 can be seen in the following way. Let μ be defined as in Proposition 2. Then there is a $c \geq 0$ such that for the unit ball B^d

$$\mu(B^d) = c\,\sigma(S^{d-1}) = c\,\Omega(B^d). \tag{10}$$

Ω is also a valuation which fulfills the assumptions of Proposition 2. If for every $K \in \mathcal{K}^d$ the value of Ω as well as of μ is uniquely determined by the value for the unit ball, then (10) implies that $\mu(K) = c\,\Omega(K)$ for every $K \in \mathcal{K}^d$ and therefore Proposition 2 holds under this assumption.

Thus we have to show that μ is uniquely determined by the constant c chosen in (10). We already know the value of μ for polytopes and cylinders. As an application of Schneider's theorem cited in subsection 2.1, we now obtain the following result.

Lemma 2.1. *For every polytope P, $\mu(B^d \cap P) = c\,\sigma(S^{d-1} \cap P)$.*

Proof. First, we consider polyhedral cones. For every polyhedral cone C, $S^{d-1} \cap C$ is a spherical polytope. Set $\nu(S^{d-1} \cap C) = \mu(B^d \cap C)$. Then ν is defined on $\mathcal{P}(S^{d-1})$. Since μ is a rotation invariant and simple valuation, so is ν. In addition, μ and ν are non-negative by (8). Thus by Schneider's theorem there is a constant $a \geq 0$ such that $\nu(S^{d-1} \cap C) = a\,\sigma(S^{d-1} \cap C)$ for every polyhedral cone C. By (10), $\nu(S^{d-1}) = \mu(B^d) = c\,\sigma(S^{d-1})$. Thus $a = c$ and

$$\mu(B^d \cap C) = c\,\sigma(S^{d-1} \cap C). \tag{11}$$

Second, we prove the lemma in the case that P is the convex hull of a $(d-1)$-dimensional polytope F and the origin. Let C_F be the polyhedral cone generated by the ray starting from the origin and intersecting F, and let $\varepsilon > 0$ be chosen. We dissect F into polytopes $F_1, \dots, F_n$ and hence the cone C_F into cones $C_{F_1}, \dots, C_{F_n}$ such that for $i = 1, \dots, k$, $F_i \subset B^d$, for $i = m, \dots, n$, $F_i \cap B^d = \emptyset$, and such that

$$\sum_{i=k+1}^{m-1} \sigma(S^{d-1} \cap C_{F_i}) < \varepsilon. \tag{12}$$

Let P_i be the convex hull of F_i and the origin. Then

$$\mu(B^d \cap P_i) = 0 \tag{13}$$

for $i = 1, \ldots, k$, since μ vanishes on polytopes, and

$$\mu(B^d \cap P_i) = c\,\sigma(S^{d-1} \cap C_{F_i}) \tag{14}$$

for $i = m, \ldots, n$, since then $B^d \cap P_i = B^d \cap C_{F_i}$ and by (11). The polytopes P_i have pairwise disjoint interior and since μ is non-negative, this and (13) imply

$$\sum_{i=m}^{n} \mu(B^d \cap P_i) \leq \mu(B^d \cap P) = \sum_{i=k+1}^{n} \mu(B^d \cap P_i).$$

Since μ is non-negative, $\mu(B^d \cap P_i) \leq \mu(B^d \cap C_{F_i})$. Therefore we obtain by (14) and (11)

$$c \sum_{i=m}^{n} \sigma(S^{d-1} \cap C_{F_i}) \leq \mu(B^d \cap P) \leq c \sum_{i=k+1}^{n} \sigma(S^{d-1} \cap C_{F_i})$$

and thus by (12)

$$c\left(\sum_{i=k+1}^{n} \sigma(S^{d-1} \cap P_i) - \varepsilon\right) \leq \mu(B^d \cap P) \leq c\left(\sum_{i=k+1}^{n} \sigma(S^{d-1} \cap P_i) + \varepsilon\right).$$

Since $\sigma(S^{d-1} \cap P_i) = 0$ for $i = 1, \ldots, k$, and since $\varepsilon > 0$ was arbitrary, this proves

$$\mu(B^d \cap P) = c\,\sigma(S^{d-1} \cap P) \tag{15}$$

in the case that P is convex hull of a $(d-1)$-dimensional polytope and the origin.

Finally, for an arbitrary polytope P the lemma follows now easily from the observation that every polytope can be represented by the convex hulls of the $(d-1)$-dimensional facets $F_1, \ldots, F_n$ of P and the origin. The polytope P is the intersection of finitely many half-spaces $H_1^+, \ldots, H_n^+$ such that $F_i \subset \operatorname{bd} H_i^+$. Let H_i^+ for $i = 1, \ldots, m$ be the half-spaces which contain the origin. Denoting by P_i the convex hull of F_i and the origin, we thus obtain

$$\bigcup_{i=1}^{m} P_i = \bigcup_{i=m+1}^{n} P_i \cup P,$$

and the P_i's have for $i = 1, \ldots, m$ and for $i = m+1, \ldots, n$ pairwise disjoint interior. Hence

$$\mu(B^d \cap P) = \sum_{i=1}^{m} \mu(B^d \cap P_i) - \sum_{i=m+1}^{n} \mu(B^d \cap P_i).$$

Since by (15) our statement is already proved for the polytopes P_i, this completes the proof of the lemma. □

Therefore μ is uniquely determined by (10) for the intersection of the unit ball and a polytope, and since μ is equi-affine invariant, it is also determined on equi-affine images of such intersections. By Lemma 1.2, we know that μ vanishes on cylinders. Hence we have for the intersection of a polytope P and a cylinder Z

$$\mu(Z \cap P) = 0.$$

This can be seen by dissecting the cylinder Z by polytopes $P, P_1, \ldots, P_n$, since we have

$$0 = \mu(Z) = \mu(Z \cap P) + \mu(Z \cap P_1) + \ldots + \mu(Z \cap P_n)$$

and since μ is non-negative. We introduce the following family of convex bodies.

Definition. *Let $\mathcal{E}$ be the family of convex bodies E which can be represented as*

$$E = E_1 \cup \ldots \cup E_n,$$

where the E_i's have pairwise disjoint interior and every E_i is a polytope or an equi-affine image of the intersection of the unit ball or a cylinder with a polytope.

Having fixed $c \geq 0$ in (10) by the value of μ for the unit ball, we now know the value of μ for every $E \in \mathcal{E}$. Since the polytopes belong to $\mathcal{E}$, $\mathcal{E}$ is dense in $\mathcal{K}^d$ and we can approximate every $K \in \mathcal{K}^d$ by elements of $\mathcal{E}$. The upper semicontinuity of μ implies that

$$\mu(K) \geq \limsup_{n\to\infty} \mu(E_n) \tag{16}$$

for every sequence E_n with $E_n \to K$. We will prove that for every $K \in \mathcal{K}^d$ there is a sequence E_n such that we have equality in (16), i.e.

$$\mu(K) = \sup\{\limsup_{n\to\infty} \mu(E_n) : E_n \in \mathcal{E}, E_n \to K\}. \tag{17}$$

Showing this implies that μ is uniquely determined by (10) and therefore proves Proposition 2.

As a first step, we show that it suffices to prove (17) for ε-smooth bodies. Here a convex body K is *ε-smooth* if there is a convex body K_0 such that

$$K = K_0 + \varepsilon\, B^d.$$

We need the following lemma.

Lemma 2.2. *For every $K \in \mathcal{K}^d$ and every closed line segment I,*

$$\mu(K + I) = \mu(K).$$

Proof. We choose a direction in $\mathbb{E}^d$ and consider only line segments I parallel to this direction. Then, since μ is translation invariant, $\mu(K+I)$ depends only on the length $x = V_1(I)$, and we define

$$f_K(x) = \mu(K+I).$$

Let I_1, I_2 be line segments with disjoint relative interior such that the origin is contained in $I_1 \cap I_2$. Then

$$(K+I_1) \cup (K+I_2) = K + (I_1 + I_2)$$

and

$$(K+I_1) \cap (K+I_2) = K.$$

Since μ is a valuation, this implies that

$$\mu(K + (I_1 + I_2)) + \mu(K) = \mu(K+I_1) + \mu(K+I_2),$$

and setting $x_1 = V_1(I_1)$ and $x_2 = V_1(I_2)$, now shows that

$$f_K(x_1 + x_2) + f_K(0) = f_K(x_1) + f_K(x_2)$$

for every $x_1, x_2 \geq 0$. Thus $f_K(x) - f_K(0)$ is a solution of Cauchy's functional equation. Since it depends upper semicontinuously on x, there is a $\nu(K)$ such that

$$f_K(x) = \nu(K)\, x + f_K(0)$$

for every $x \geq 0$. Therefore

$$\mu(K+I) = \nu(K)\, V_1(I) + \mu(K) \tag{18}$$

holds for every $K \in \mathcal{K}^d$ and every line segment I.

For I sufficiently long (i.e. longer than the width of the shadow boundary of K in the direction of the line segment I), we can find closed half-spaces H_1^+ and H_2^- such that the part of $K+I$ not lying in one of these half-spaces is a non-degenerate cylinder Z. Set $I_1 = I \cap H_1^+$ and $I_2 = I \cap H_2^-$. Then

$$V_1(I) > V_1(I_1) + V_1(I_2). \tag{19}$$

Since μ vanishes on cylinders by Lemma 1.2 and since it is translation invariant, we obtain

$$\begin{aligned} \mu(K+I) &= \mu((K+I) \cap H_1^+) + \mu(Z) + \mu((K+I) \cap H_2^-) \\ &= \mu(K + I_1 + I_2). \end{aligned}$$

Consequently, (18) implies that

$$\nu(K)\, V_1(I) = \nu(K)\, V_1(I_1 + I_2).$$

Thus it follows from (19) that

$$\nu(K) = 0$$

and combining this with (18) shows that

$$\mu(K + I) = \mu(K)$$

holds for every $K \in \mathcal{K}^d$. □

Suppose that (17) does not hold for a $K \in \mathcal{K}^d$, i.e.

$$\mu(K) > \sup\{\limsup_{n\to\infty} \mu(E_n) : E_n \in \mathcal{E}, E_n \to K\}.$$

Then there is an $a > 0$ and a $\delta > 0$ such that

$$\mu(K) > \mu(E) + a\,\sigma(\mathrm{bd}\,K) \tag{20}$$

for every $E \in \mathcal{E}$ with $\delta^H(E, K) \leq \delta$, where δ^H stands for Hausdorff distance. We show that then there is also an ε-smooth convex body for which an inequality of this type holds.

The unit ball B^d can be approximated by zonotopes I_n, i.e. by Minkowski sums of finitely many line segments (cf. [28], Chapter 3.5). The upper semicontinuity of μ and Lemma 2.2 imply that

$$\mu(K + \varepsilon\, B^d) \geq \limsup_{n\to\infty} \mu(K + \varepsilon\, I_n) = \mu(K)$$

for every $\varepsilon > 0$, i.e. μ is larger for $K + \varepsilon\, B^d$ than for K. Thus for $\varepsilon \leq \frac{1}{2}\,\delta$, (20) implies that

$$\mu(K + \varepsilon\, B^d) > \mu(E) + a\,\sigma(\mathrm{bd}\,K)$$

for every $E \in \mathcal{E}$ with $\delta^H(K + \varepsilon\, B^d, E) \leq \frac{1}{2}\,\delta$, since for such an $E \in \mathcal{E}$

$$\delta^H(K, E) \leq \delta^H(K, K + \varepsilon\, B^d) + \delta^H(K + \varepsilon\, B^d, E) \leq \delta.$$

Since σ depends continuously on K, it now follows that

$$\mu(K + \varepsilon\, B^d) > \mu(E) + \frac{a}{2}\,\sigma(\mathrm{bd}(K + \varepsilon\, B^d))$$

for every $E \in \mathcal{E}$ with $\delta^H(K + \varepsilon\, B^d, E) \leq \frac{1}{2}\,\delta$ and $0 < \varepsilon \leq \frac{1}{2}\,\delta$ sufficiently small. This implies that if (17) does not hold for a $K \in \mathcal{K}^d$, then it does also not hold for $K + \varepsilon\, B^d$ for $\varepsilon > 0$ sufficiently small.

3. Thus it suffices to show the following proposition to prove (17) and thereby our theorem.

Proposition 3. *For every ε-smooth $K \in \mathcal{K}^d$, $\varepsilon > 0$, we have*

$$\mu(K) = \sup\{\limsup_{n\to\infty} \mu(E_n) : E_n \in \mathcal{E}, E_n \to K\}.$$

Let $K \in \mathcal{K}^d$ be ε-smooth, $\varepsilon > 0$, and let $P^i, P^c \in \mathcal{K}^d$ be polytopes such that $P^i \subset \operatorname{int} K \subset P^c$, where int stands for interior. We will prove that for every choice of such K, P^i, P^c, and every $a > 0$, an $E \in \mathcal{E}$ can be constructed such that $P^i \subset E \subset P^c$ and such that

$$\mu(K) \le \mu(E) + a\,\sigma(\operatorname{bd} K) \tag{21}$$

holds. This shows that there is always an $E \in \mathcal{E}$ arbitrarily close to K such that $\mu(E)$ is almost as large as $\mu(K)$ and therefore proves Proposition 3.

The proof that such an E can always be constructed is subdivided into four parts. In the first part, we show that for a normal point $x_0 \in \operatorname{bd} K$ with positive curvature an $E_r(x_0) \in \mathcal{E}$ and a small polytope $P_r(x_0)$ containing x_0 can be chosen such that $\mu(E_r(x_0) \cap P_r(x_0))$ is almost as large as $\mu(K \cap P_r(x_0))$. This is done by comparing K with a suitable unit ellipsoid close to an osculating ellipsoid of K. In the second part, a similar statement is proved for normal points with vanishing curvature. Here we compare K with suitable cylinders. In the third part, we use that K is ε-smooth to show that for every polytope P, $\mu(K \cap P)$ is bounded by $c(\varepsilon)\,\sigma(\operatorname{bd} K \cap P)$ with a suitable constant $c(\varepsilon)$, i.e. we prove a type of absolute continuity property of μ. Finally, using Aleksandrov's theorem and Vitali's covering theorem, we construct our $E \in \mathcal{E}$ and using the estimates from the preceding parts for normal points and the absolute continuity property, we show that (21) holds for our E.

3.1. Let $x_0 \in \operatorname{bd} K$ be a normal point with curvature $\kappa(x_0) > 0$. Let $R(x_0)$ be the ray starting at x_0, orthogonal to the tangent hyperplane of K at x_0 and intersecting K. Then there is a solid ellipsoid $E(x_0)$ of volume κ_d which osculates K at x_0 and whose center lies on $R(x_0)$. Every ellipsoid with center on $R(x_0)$ whose principal curvatures at x_0 in respective directions are larger than those of $E(x_0)$ lies locally inside of K and similarly, every ellipsoid with smaller principal curvatures lies locally outside of K. Thus for a given $t > 0$ there are solid ellipsoids $E_t^i(x_0)$ and $E_t^c(x_0)$ with centers on $R(x_0)$ of volume κ_d and $(1+t)^d\,\kappa_d$, respectively, such that $E_t^c(x_0)$ is homothetic to $E_t^i(x_0)$, $E_t^i(x_0)$ touches K at x_0 and lies locally inside of K, and $E_t^c(x_0)$ touches K at x_0 and lies locally outside of K.

For such an ellipsoid $E_t^i(x_0)$, let φ_t be the equi-affine map which transforms $E_t^i(x_0)$ into the unit ball. Denote by $\|\varphi_t\|$ the norm of φ_t. Then

$$|\varphi_t(x_1) - \varphi_t(x_2)| \le \|\varphi_t\|\,|x_1 - x_2|$$

for every $x_1, x_2 \in \mathbb{E}^d$, which implies that for every set S with finite $(d-1)$-dimensional Hausdorff measure, we have

$$\sigma(\varphi_t(S)) \leq \|\varphi_t\|^{d-1}\sigma(S).$$

This is a property of Lipschitz maps (see, for example, [7]). Since $\|\varphi_t\|$ depends continuously on t, for $t > 0$ bounded from above, say $t < \frac{1}{4}$, there is a $p(x_0) > 0$ (depending only on x_0) such that

$$\frac{\sigma(S)}{\sigma(\varphi_t(S))} \geq p(x_0) \tag{22}$$

for every S with finite $(d-1)$-dimensional Hausdorff measure. We now choose a $t > 0$ so small, that

$$4\sqrt{t} \leq 1 - \left(\frac{4\,c + 2\,a\,p(x_0)}{4\,c + a\,p(x_0)}\right)^{-\frac{1}{d}}, \tag{23}$$

where c is the constant chosen in (10), and that if $c \neq 0$, then also

$$4\sqrt{t} \leq \left(1 + \frac{a\,p(x_0)}{8\,c}\right)^{\frac{1}{d}} - 1 \tag{24}$$

holds. Denote by φ the equi-affine map belonging to this t. Note that (23) implies that $t < \frac{1}{4}$.

Let $y_0 = \varphi(x_0) \in S^{d-1}$ and $B_t^d = B(-t\,y_0, 1+t) = \varphi(E_t^c(x_0))$. We choose a polytope Q with the following properties:

$$Q \subset B^d \subset (1 + 2\sqrt{t})\,Q \tag{25}$$

and

$$S^{d-1} \cap r\,Q \ \text{ is a convex set on } S^{d-1} \tag{26}$$

for every $0 < r \leq 1$. Define

$$T_r = \varphi(\mathrm{bd}\,K) \cap (y_0 + r\,Q),$$

and

$$L_r = \varphi(K) \cap (y_0 + r\,Q).$$

Note that for $r > 0$ sufficiently small the set T_r is simply connected and that by the definition of $E_t^i(x_0)$, $E_t^c(x_0)$ and φ, T_r lies between B^d and B_t^d.

We show that $\mu(B^d \cap (y_0 + r\,Q))$ is almost as large as $\mu(L_r)$ for $r > 0$ sufficiently small. The basic idea is the following. By construction $\mathrm{bd}(\varphi(K))$ touches B^d at y_0 and lies locally outside of B^d and the boundary of the convex hull of L_r and B^d differs only around y_0 from S^{d-1}. We take a dense packing in S^{d-1} of rotated copies of that part of S^{d-1} not contained in the boundary of this convex hull and using the same rotations we construct a convex body $M(r)$ as the convex hull of rotated copies of L_r and B^d. Then $M(r) \to B^d$ as $r \to 0$ and since μ is upper semicontinuous, $\mu(M(r))$ cannot be much larger than $\mu(B^d)$ for $r > 0$ sufficiently small. From this we deduce that also $\mu(L_r)$ is not much larger than $\mu(B^d \cap (y_0 + r\,Q))$. Then, in 3.1.2., we replace $B^d \cap (y_0 + r\,Q)$ by an element E_r of $\mathcal{E}$ which consists of a relatively large piece of $B^d \cap (y_0 + r\,Q)$ contained in $\varphi(K)$ and suitable parts of cylinders and polytopes. This E_r is chosen in such a way that after transforming back by φ^{-1} we are able to build our $E \in \mathcal{E}$ using $\varphi^{-1}(E_r)$.

3.1.1. Suppose that for $r > 0$ arbitrarily small,

$$\mu(L_r) > \mu(B^d \cap (y_0 + r\,Q)) + \frac{a\,p(x_0)}{4}\,\sigma(S^{d-1} \cap (y_0 + r\,Q)) \tag{27}$$

holds.

To show that this leads to a contradiction the following technical claim will be useful. It states that the convex hull of T_r and B^d differs only in a small neighbourhood of y_0 from B^d.

CLAIM 1. For every r, $0 < r < 1$,

$$\operatorname{conv}\left(B^d \cup T_r\right) \setminus B^d \subset y_0 + (1 + 4\sqrt{t})\,r\,Q. \tag{28}$$

Proof. First we state some elementary facts which will be used throughout this section. Let $y \notin B^d$, denote by $\operatorname{dist}(y, B^d) = \inf\{|y - x| : x \in B^d\}$ the distance of y to B^d, and let $\bar{y} \in S^{d-1}$ be such that $\operatorname{aff}(y, \bar{y})$ is tangent to B^d. Here aff stands for affine hull. Then we have

$$|y - \bar{y}| = \sqrt{(\operatorname{dist}(y, B^d) + 1)^2 - 1} \le 2\sqrt{\operatorname{dist}(y, B^d)} \tag{29}$$

for $\operatorname{dist}(y, B^d) \le 1$. Note that this implies

$$\operatorname{conv}\left(B^d \cup y\right) \setminus B^d \subset y + 2\sqrt{\operatorname{dist}(y, B^d)}\,B^d. \tag{30}$$

Now, if $y \in B^d_t$ an elementary calculation shows that

$$\operatorname{dist}(y, B^d) \le t\,|y - y_0|^2. \tag{31}$$

Clearly, this is a two-dimensional problem. Denote by $f(s)$ and $f_t(s)$ the functions representing the circle with radius 1 and $1 + t$, respectively, and touching the s-axis at the origin from above. Then $\operatorname{dist}(y, B^d) < f(|y - y_0|) - f_t(|y - y_0|)$ and the Taylor expansions of f and f_t yield (31).

Since T_r lies in B^d_t, we now have by (31), (30), and (25)

$$\operatorname{conv}\left(B^d \cup y\right) \setminus B^d \subset y + 2\sqrt{t}\,(1 + 2\sqrt{t})\,r\,Q$$

for every $y \in T_r$. Since $y \in y_0 + r\,Q$ and $t < \frac{1}{4}$, this proves the claim. $\square$

We now construct the convex bodies $M(r)$. Let m_r be the maximum number of points $y_1, \dots, y_{m_r} \in S^{d-1}$ such that the sets

$$S^{d-1} \cap B(y_i, (1 + 4\sqrt{t})\,r)$$

for $i = 1, \dots, m_r$ form a packing in S^{d-1}. Then, since $m_r = m((1+4\sqrt{t})\,r)$, it follows from (5) that

$$\kappa_{d-1}\, m_r\, ((1+4\sqrt{t})\,r)^{d-1} \to \delta_{d-1}\, d\, \kappa_d \tag{32}$$

as $r \to 0$. We define

$$M(r) = \operatorname{conv}\left(B^d \cup \psi_{y_1}(T_r) \cup \ldots \cup \psi_{y_{m_r}}(T_r)\right),$$

where the ψ_{y_i}'s are rotations such that $\psi_{y_i}(y_0) = y_i$. This construction implies that

$$M(r) \to B^d \quad \text{as} \quad r \to 0,$$

and because of (28) that

$$\psi_{y_i}(T_r) \subset \operatorname{bd} M(r) \tag{33}$$

holds for $i = 1, \dots, m_r$ and $r > 0$ sufficiently small. We dissect

$$M(r) \backslash \bigcup_{i=1}^{m_r} \psi_{y_i}(L_r)$$

into convex polytopes $P_1, \dots, P_{k_r}$. It follows from (26) and (33) that the intersection of $\psi_{y_i}(L_r)$ and $\psi_{y_j}(L_r)$ for $i \neq j$ is empty and we obtain

$$\mu(M(r)) = \sum_{i=1}^{m_r} \mu(\psi_{y_i}(L_r)) + \sum_{j=1}^{k_r} \mu(M(r) \cap P_j). \tag{34}$$

Note that for a polytope P for which $P \cap \operatorname{bd} M(r) = P \cap S^{d-1}$ it follows from Lemma 2.1 that $\mu(M(r) \cap P) = c\,\sigma(S^{d-1} \cap P)$. Since μ is non-negative, dissecting $M(r) \cap P_j$ into small pieces therefore implies that

$$\sum_{j=1}^{k_r} \mu(M(r) \cap P_j) \geq c\,\sigma(S^{d-1} \cap \operatorname{bd} M(r)).$$

Using this and the rotation invariance of μ we obtain from (34)

$$\mu(M(r)) \geq m_r\, \mu(L_r) + c\,\sigma\left(S^{d-1} \cap \operatorname{bd} M(r)\right). \tag{35}$$

By the definition of $M(r)$ and (28), we have

$$\sigma(S^{d-1} \cap \operatorname{bd} M(r)) \geq \sigma(S^{d-1}) - m_r\, \sigma(S^{d-1} \cap (y_0 + (1 + 4\sqrt{t})\, r\, Q)),$$

and since σ is homogeneous of degree $(d-1)$ in the hyperplane tangent to S^{d-1} at y_0 and since S^{d-1} is locally Euclidean, the estimate

$$\sigma(S^{d-1} \cap (y_0 + (1 + 4\sqrt{t})\, r\, Q)) \leq (1 + 4\sqrt{t})^d\, \sigma(S^{d-1} \cap (y_0 + r\, Q))$$

holds for $r > 0$ sufficiently small.

Hence (35) and our assumption (27) now yield

$$\begin{aligned} \mu(M(r)) \;&\geq\; m_r \Big(\mu(B^d \cap (y_0 + r\, Q)) + \frac{a\, p(x_0)}{4}\, \sigma(S^{d-1} \cap (y_0 + r\, Q)) \Big) \\ &\qquad + c\, \Big(\sigma(S^{d-1}) - m_r\, \sigma(S^{d-1} \cap (y_0 + (1 + 4\sqrt{t})\, r\, Q)) \Big) \\ &\geq\; \mu(B^d) + m_r\, \sigma(S^{d-1} \cap (y_0 + r\, Q)) \Big(c + \frac{a\, p(x_0)}{4} - c\, (1 + 4\sqrt{t})^d \Big) \end{aligned}$$

for $r > 0$ sufficiently small. Since by (25) $B \subset (1 + 2\sqrt{t})\, Q$, comparison with the tangent hyperplane shows that the estimate

$$\sigma(S^{d-1} \cap (y_0 + r\, Q)) \geq \frac{r^{d-1} \kappa_{d-1}}{(1 + 4\sqrt{t})^{d-1}}$$

holds for $r > 0$ sufficiently small. Thus by our choice of t in (24), by (32), and since μ is upper semicontinuous, we obtain

$$\mu(B^d) \geq \limsup_{r \to 0} \mu(M(r)) \geq \mu(B^d) + \frac{a\, p(x_0)}{8}\, \frac{\delta_{d-1}\, d\, \kappa_d}{(1 + 4\sqrt{t})^{2(d-1)}}.$$

This is a contradiction, since $a > 0$. Therefore

$$\mu(L_r) \leq \mu(B^d \cap (y_0 + r\, Q)) + \frac{a\, p(x_0)}{4}\, \sigma(S^{d-1} \cap (y_0 + r\, Q)) \tag{36}$$

holds for $r > 0$ sufficiently small.

3.1.2. For $r > 0$, we construct a polyhedral cone C_r with

$$S^{d-1} \cap C_r \;\subset\; S^{d-1} \cap (y_0 + r\, Q)$$

such that C_r can be complemented by cylinders and polytopes to a suitable element E_r of $\mathcal{E}$ with the properties that for $r > 0$ sufficiently small

$$\mu(L_r) \leq \mu(E_r \cap (y_0 + r\, Q)) + \frac{a\, p(x_0)}{2}\, \sigma(E_r \cap (y_0 + r\, Q)), \tag{37}$$

$$\mu(E_r) = \mu(B^d \cap C_r), \tag{38}$$

and

$$\operatorname{bd} E_r \cap \varphi(K) \;\subset\; y_0 + r\, Q. \tag{39}$$

Since locally around y_0 B^d lies in L_r, the last property means that $\operatorname{bd} E_r$ intersects $\operatorname{bd} \varphi(K)$ before it intersects $y_0 + r \operatorname{bd} Q$.

For $y \in S^{d-1}$, let $H^+(y)$ be the closed half-space which contains B^d and is bounded by the tangent hyperplane to B^d at y, and for a convex set C, set

$$H^+(C) = \bigcap_{y \in S^{d-1} \cap C} H^+(y).$$

Note that for C a polyhedral cone, $H^+(C)$ is the union of $B^d \cap C$ and finitely many polyhedra and pieces of unbounded cylinders. Hence for a suitable polyhedral cone C_r we can set $E_r = H^+(C_r) \cap \varphi(P^c)$ and obtain an element of $\mathcal{E}$. Using this notation (39) can be written as

$$\operatorname{bd} H^+(C_r) \cap \varphi(K) \subset y_0 + r\,Q.$$

The following claim shows that it suffices to choose a polyhedral cone C_r with $S^{d-1} \cap C_r \subset y_0 + (1 - 3\sqrt{t})\,r\,Q$ to ensure that this condition holds.

CLAIM 2. For every r, $0 < r < 1$

$$\operatorname{bd} H^+(y_0 + (1 - 3\sqrt{t})\,r\,Q) \cap \varphi(K) \subset y_0 + r\,Q. \tag{40}$$

Proof. The proof of this claim is analogous to that of Claim 1 in 3.1.1. Let $y \in \varphi(K)$ and $\bar{y} \in S^{d-1} \cap (y_0 + (1 - 3\sqrt{t})r\,Q)$ be such that $\operatorname{aff}(y, \bar{y})$ is tangent to B^d. Then by (29) and (31), which can be used since $\operatorname{dist}(y, B^d) \leq 1$ and since $y \in B^d_t$,

$$|y - \bar{y}| < 2\sqrt{t}\,|y - y_0|.$$

By assumption, $|\bar{y} - y_0| \leq (1 - 3\sqrt{t})r$ and therefore this and the triangle inequality imply

$$|y - y_0| < \frac{1 - 3\sqrt{t}}{1 - 2\sqrt{t}}\,r.$$

Using (25) we therefore have

$$y \in \bar{y} + 2\sqrt{t}\,\frac{1 - 3\sqrt{t}}{1 - 2\sqrt{t}}\,(1 + 2\sqrt{t})\,r\,Q$$

Since $\bar{y} \in y_0 + (1 - 3\sqrt{t})r\,Q$, this proves the claim. □

By (26) $S^{d-1} \cap (y_0 + (1 - 3\sqrt{t})\,r\,Q)$ is a convex set on S^{d-1} and can therefore be approximated by polyhedral cones. We choose a polyhedral cone C_r such that

$$S^{d-1} \cap (y_0 + (1 - 4\sqrt{t})\,r\,Q) \subset S^{d-1} \cap C_r \subset S^{d-1} \cap (y_0 + (1 - 3\sqrt{t})\,r\,Q)$$

and define

$$E_r = H^+(S^{d-1} \cap C_r) \cap \varphi(P^c). \tag{41}$$

Then $E_r \in \mathcal{E}$ and since μ vanishes on polytopes and cylinders, we therefore have

$$\mu(E_r) = \mu(B^d \cap C_r) = c\,\sigma(S^{d-1} \cap C_r).$$

Thus (38) holds. By (40) and by the definition of C_r also (39) holds, and it remains to prove (37). Since σ is homogeneous of degree $(d-1)$ in the hyperplane tangent to S^{d-1} at y_0 and since S^{d-1} is locally Euclidean,

$$\sigma(S^{d-1} \cap (y_0 + r\,Q)) \le (1 - 4\sqrt{t})^{-d} \sigma(S^{d-1} \cap (y_0 + (1 - 4\sqrt{t})\,r\,Q)).$$

Consequently (23) implies that

$$\sigma(S^{d-1} \cap (y_0 + r\,Q)) \le \left(\frac{4\,c + 2\,a\,p(x_0)}{4\,c + a\,p(x_0)}\right) \sigma(S^{d-1} \cap (y_0 + (1 - 4\sqrt{t})\,r\,Q))$$

holds for $r > 0$ sufficiently small. Therefore we obtain by (36)

$$\mu(L_r) \le (c + \frac{a\,p(x_0)}{2})\,\sigma(S^{d-1} \cap (y_0 + (1 - 4\sqrt{t})\,r\,Q)).$$

Since by construction

$$\sigma(S^{d-1} \cap (y_0 + (1 - 4\sqrt{t})\,r\,Q)) \le \sigma(S^{d-1} \cap C_r) \le \sigma(E_r \cap (y_0 + r\,Q)),$$

we now have

$$\mu(L_r) \le c\,\sigma(S^{d-1} \cap C_r) + \frac{a\,p(x_0)}{2}\,\sigma(E_r \cap (y_0 + r\,Q)),$$

which combined with (38) implies (37).

3.1.3. We transform back and obtain the following. For $r > 0$ sufficiently small, there are polytopes

$$P_r(x_0) = \varphi^{-1}(y_0 + r\,Q) = x_0 + r\varphi^{-1}(Q) \tag{42}$$

and elements of $\mathcal{E}$

$$E_r(x_0) = \varphi^{-1}(E_r),$$

and a $q(x_0) > 0$ such that for every $r > 0$

$$\frac{\sigma(\operatorname{bd} K \cap P_r(x_0))}{\operatorname{diam}(\operatorname{bd} K \cap P_r(x_0))^{d-1}} \ge q(x_0). \tag{43}$$

Here (43) follows from (25), since φ depends only on x_0. $E_r(x_0)$ consists of a piece of a solid unit ellipsoid, which lies in K, and pieces of cylinders and polytopes and by (39) it has the property that

$$\operatorname{bd} E_r(x_0) \cap K \;\subset\; P_r(x_0) \tag{44}$$

for $r > 0$ sufficiently small. By (37) and by (22) we have

$$\mu(K \cap P_r(x_0)) \le \mu(E_r(x_0) \cap P_r(x_0)) + \frac{a}{2}\,\sigma(E_r(x_0) \cap P_r(x_0)). \tag{45}$$

Because of (41),

$$E_r(x_0) \subset P^c \tag{46}$$

and because of (42),

$$P^i \subset E_r(x_0) \tag{47}$$

for $r > 0$ sufficiently small, i.e. we can choose a $r(x_0)$ such that (45), (46), (47), and (44) hold for $0 < r \le r(x_0)$.

3.2. Let $x_0 \in \operatorname{bd} K$ be a normal point with curvature $\kappa(x_0) = 0$, i.e. there is a $k \ge 1$ such that, without loss of generality, $\kappa_1(x_0) = \ldots = \kappa_k(x_0) = 0$ and $\kappa_{k+1}(x_0), \ldots, \kappa_{d-1}(x_0) > 0$. Then there is an equi-affine map φ, such that for the principal curvatures of $\operatorname{bd}\varphi(K)$ at $\varphi(x_0)$ we have $\kappa_1(\varphi(x_0)) = \ldots = \kappa_k(\varphi(x_0)) = 0$ and $\kappa_{k+1}(\varphi(x_0)) = \ldots = \kappa_{d-1}(\varphi(x_0)) = 1$, and as in the case $\kappa(x_0) > 0$, there is a constant $p(x_0) > 0$ such that

$$\frac{\sigma(S)}{\sigma(\varphi(S))} \ge p(x_0) \tag{48}$$

for every set S with finite $(d-1)$-dimensional Hausdorff measure.

Let E_t, $t > 0$, be the solid ellipsoid with the equation

$$t\,(x^1)^2 + \ldots + t\,(x^k)^2 + (1+t)(x^{k+1})^2 + \ldots + (1+t)(x^d)^2 \le 1.$$

Let $I^k \subset \mathbb{E}^k$ be the k-dimensional cube centered at the origin, with side length 2, and edges parallel to the coordinate axes. Then $E_t \cap I^d$ tends to the cylinder $I^k \times B^{d-k}$ as $t \to 0$ and $\operatorname{bd} E_t \cap I^d$ tends to the lateral surface $Z = I^k \times S^{d-k-1}$ of this cylinder. This implies the following. For every $z_1, z_2 \in Z$ denote by y_1 and y_2 the nearest point on $\operatorname{bd} E_t$ to z_1 and z_2, respectively. Then the ratio of $|z_1 - z_2|$ to $|y_1 - y_2|$ tends to 1 as $t \to 0$. In particular, we have

$$|y_1 - y_2| \ge \frac{1}{2}\,|z_1 - z_2| \tag{49}$$

for $t > 0$ sufficiently small. Since μ is upper semicontinuous and vanishes on cylinders, $\mu(E_t \cap I^d)$ is arbitrarily small for $t > 0$ sufficiently small. Consequently

$$\mu(E_t \cap I^d) < \frac{a\,p(x_0)}{4}\,\frac{\delta_{d-1}\,2^k\,(d-k)\,\kappa_{d-k}}{2^{d-1}\,(8\,d)^{d-1}} \tag{50}$$

holds for $t > 0$ sufficiently small. We assume that t, $0 < t < \frac{1}{4\,d}$, is chosen so small that (49) and (50) are satisfied.

We now choose a polytope Q with the property

$$Q \subset B^d \subset 2\,Q. \tag{51}$$

Define

$$T_r = \varphi(\operatorname{bd} K) \cap (\varphi(x_0) + r\,Q),$$

and

$$L_r = \varphi(K) \cap (\varphi(x_0) + r\,Q).$$

We show that $\mu(L_r)$ is not much larger than $\mu(E_t \cap (\varphi(x_0) + r\,Q))$ for $r > 0$ sufficiently small.

3.2.1. Suppose that for $r > 0$ arbitrarily small

$$\mu(L_r) > \frac{a\,p(x_0)}{2}\,\sigma\,(H(\varphi(x_0)) \cap (\varphi(x_0) + r\,Q)) \tag{52}$$

holds, where $H(\varphi(x_0))$ is the hyperplane tangent to $\varphi(K)$ at $\varphi(x_0)$.

For a $y \in \operatorname{bd} E_t \cap I^d$, let ψ_y be a rigid motion such that $\psi_y(T_r)$ touches E_t at y in such a way that the corresponding principal directions of $\psi_y(T_r)$ and $\operatorname{bd} E_t$ at y coincide. We show that $\psi_y(T_r)$ lies locally outside of E_t and that the convex hull of $\psi_y(T_r)$ and E_t differs only in a small neighbourhood of y from E_t.

CLAIM. There exists a $r_1 > 0$ such that $\psi_{y_0}(T_r) \cap E_t = \{y_0\}$ and

$$\operatorname{conv}(E_t \cup \psi_{y_0}(T_r)) \setminus E_t \;\subset\; y_0 + 4\,d\,r\,Q \tag{53}$$

for $0 < r \le r_1$ and for every $y_0 \in \operatorname{bd} E_t \cap I^d$.

Proof. To simplify the notion we identify the hyperplane tangent to E_t at y_0 with $\mathbb{E}^{d-1}$. Choosing a suitable coordinate system in $\mathbb{E}^d$, we can represent $\operatorname{bd} E_t$ in a neighbourhood of $y_0 = (y_0', 0)$ by a convex function $f(x')$ (x', $y_0' \in \mathbb{E}^{d-1}$) for which

$$f(x') = \frac{1}{2}\sum_{i=1}^{d-1} \kappa_i(y_0)\,(x^i - y_0^i)^2 \;+ o(|x' - y_0'|^2)$$

as $|x' - y_0'| \to 0$. Here the coefficients $\kappa_1(y_0), \dots, \kappa_{d-1}(y_0)$ are the principal curvatures of $\operatorname{bd} E_t$ at y_0 (see 2.2). An elementary calculation, using the rotational symmetry of E_t and Taylor expansions, shows that for every $y_0 \in \operatorname{bd} E_t \cap I^d$

$$\frac{t}{\sqrt{1+t}} \le \;\; \kappa_i(y_0) \;\; \le \frac{t}{\sqrt{(1+t)\,(1-k\,t)^3}} \qquad \text{for } i = 1, \dots, k,$$

$$\sqrt{1+t} \le \;\; \kappa_i(y_0) \;\; \le \frac{\sqrt{1+t}}{\sqrt{1-k\,t}} \qquad \text{for } i = k+1, \dots, d-1.$$

Since $t < \frac{1}{4d}$ there exists a $r_2 > 0$ such that

$$\frac{t}{4}\sum_{i=1}^{k}(x^i-y_0^i)^2+\frac{1}{2}\sum_{i=k+1}^{d-1}(x^i-y_0^i)^2\le f(x')\le\frac{3t}{2}\sum_{i=1}^{k}(x^i-y_0^i)^2+\frac{1+dt}{2}\sum_{i=k+1}^{d-1}(x^i-y_0^i)^2 \tag{54}$$

for $|x'-y_0'|\le r_2$. Note that r_2 can be chosen independently of y_0 since the third order terms of the Taylor expansion of $\operatorname{bd} E_t\cap I^d$ are uniformly bounded.

Denote by $\operatorname{dist}(y,E_t)=\inf\{|y-x| : x\in E_t\}$ the distance of $y=(y',0)$ to E_t ($y'\in\mathbb{E}^{d-1}$). Since f is a convex function

$$\operatorname{dist}(y,E_t)\ge\operatorname{dist}(y,H(y',f(y'))),$$

where $H(y',f(y'))$ is the hyperplane tangent to $\operatorname{bd} E_t$ at $(y',f(y'))$. Since f is twice differentiable at y_0' the angle between $\mathbb{E}^{d-1}$ and $H(y',f(y'))$ tends to 0 as $|y'-y_0'|\to 0$. This implies that

$$\operatorname{dist}(y,H(y',f(y')))\ge\frac{1}{2}f(y')$$

for $|y'-y_0'|$ sufficiently small. Hence

$$\operatorname{dist}(y,E_t)\;\ge\;\frac{t}{8}\sum_{i=1}^{k}(y^i-y_0^i)^2+\frac{1}{4}\sum_{i=k+1}^{d-1}(y^i-y_0^i)^2\ge\frac{t}{8}|y-y_0|^2 \tag{55}$$

holds for $|y'-y_0'|$ sufficiently small.

By the definition of φ and since $\psi_{y_0}(T_r)$ touches E_t at y_0 in such a way that the corresponding principal directions of $\psi_{y_0}(T_r)$ and $\operatorname{bd} E_t$ coincide, we can represent $\psi_{y_0}(\varphi(K))$ in a neighbourhood of y_0 by a convex function $g(x')$, ($x'\in\mathbb{E}^{d-1}$), with

$$g(x')=\frac{1}{2}\sum_{i=k+1}^{d-1}(x^i-y_0^i)^2\;+o(|x'-y_0'|^2)$$

as $|x'-y_0'|\to 0$. Since g is convex, there exists an $r_3>0$ such that

$$g(x')\ge\frac{1-t}{2}\sum_{i=k+1}^{d-1}(x^i-y_0^i)^2 \tag{56}$$

for $|x'-y_0'|\le r_3$. The proof now proceeds like the proof of Claim 1 in 3.1.1. Let $y=(y',g(y'))\in\psi_{y_0}(T_r)$ and let $\bar{y}=(\bar{y}',f(\bar{y}'))\in\operatorname{bd} E_t\cap I^d$ be such that $\operatorname{aff}(y,\bar{y})$ is tangent to E_t. By (54), (56) and the obvious inequality $\operatorname{dist}(y,E_t)\le f(y')-g(y')$, we have

$$\begin{aligned}\operatorname{dist}(y,E_t)\;&\le\;\frac{3t}{2}\sum_{i=1}^{k}(y^i-y_0^i)^2+\frac{(d+1)t}{2}\sum_{i=k+1}^{d-1}(y^i-y_0^i)^2\\&\le\;\frac{(d+1)t}{2}\,|y-y_0|^2.\end{aligned}$$

On the other hand, since $\text{aff}(y, \bar{y})$ is tangent to E_t, we can apply (55) to $\bar{y}$ instead of y_0 and obtain

$$\text{dist}(y, E_t) \geq \frac{t}{8}\, |y - \bar{y}|^2.$$

Combined these inequalities yield by the definition of T_r

$$|y - \bar{y}|^2 \leq 4\,(d+1)\,|y - y_0|^2 \leq 4\,(d+1)\,r^2.$$

By (51) this proves the claim, since $y \in y_0 + r\,Q$. □

Let m_r be the maximum number of points $z_1, \ldots, z_{m_r} \in Z$ such that the sets

$$B(z_i, 8\,d\,r) \cap Z \tag{57}$$

for $i = 1, \ldots, m_r$ form a packing in Z. Then, since $m_r = m(8\,d\,r)$, it follows from (6) that

$$\kappa_{d-1}\, m_r\, r^{d-1} \to \frac{\delta_{d-1}\, 2^k\,(d-k)\,\kappa_{d-k}}{(8\,d)^{d-1}} \tag{58}$$

as $r \to 0$. Let y_i be the nearest point to z_i on $\text{bd}\, E_t$. Then it follows from (57) and (49) that the sets

$$B(y_i, 4\,d\,r) \cap \text{bd}\, E_t \cap I^d \tag{59}$$

for $i = 1, \ldots, m_r$ form a packing in $\text{bd}\, E_t \cap I^d$.

We define

$$M(r) = \text{conv}\left((E_t \cap I^d) \cup \psi_{y_1}(T_r) \cup \ldots \cup \psi_{y_{m_r}}(T_r)\right).$$

This construction implies that

$$M(r) \to E_t \cap I^d \quad \text{as} \quad r \to 0,$$

and by (59) and (53) that

$$\psi_{y_i}(T_r) \subset \text{bd}\, M(r)$$

holds for $i = 1, \ldots, m_r$ and $r > 0$ sufficiently small. Therefore the intersection of $\psi_{y_i}(L_r)$ and $\psi_{y_j}(L_r)$ for $i \neq j$ is empty or a convex polytope. Since μ is non-negative and rigid motion invariant and vanishes on polytopes, we therefore obtain

$$\mu(M(r)) \geq m_r\, \mu(L_r).$$

¿From this it follows by (51) and by our assumption (52) that

$$\mu(M(r)) \geq \frac{a\,p(x_0)}{2}\, m_r\, \sigma\Big(H(\varphi(x_0)) \cap (\varphi(x_0) + \frac{r}{2}\, B^d)\Big).$$

Since μ is upper semicontinuous, we now obtain by (58)

$$\mu(E_t \cap I^d) \geq \limsup_{r \to 0} M(r) \geq \frac{a\,p(x_0)}{2}\, \frac{\delta_{d-1}\, 2^k\,(d-k)\,\kappa_{d-k}}{2^{d-1}\,(8\,d)^{d-1}}.$$

Because of our upper bound for $\mu(E_t \cap I^d)$ in (50), this is a contradiction. Thus

$$\mu(L_r) \leq \frac{a\,p(x_0)}{2}\,\sigma(H(\varphi(x_0)) \cap (\varphi(x_0) + r\,Q)) \tag{60}$$

holds for $r > 0$ sufficiently small.

3.2.2. We transform back and obtain the following. For $r > 0$ sufficiently small, there are polytopes

$$P_r(x_0) = \varphi^{-1}(\varphi(x_0) + r\,Q) = x_0 + r\,\varphi^{-1}(Q) \tag{61}$$

and elements of $\mathcal{E}$

$$E_r(x_0) = H^+(x_0) \cap P^c, \tag{62}$$

where $H^+(x_0)$ is the closed half-space which contains K and is bounded by the tangent hyperplane to K at x_0, and by (51) there is a $q(x_0) > 0$ such that for every $r > 0$

$$\frac{\sigma(\operatorname{bd} K \cap P_r(x_0))}{\operatorname{diam}(\operatorname{bd} K \cap P_r(x_0))^{d-1}} \geq q(x_0). \tag{63}$$

In addition

$$P^i \subset E_r(x_0) \subset P^c \tag{64}$$

holds for every $r > 0$ and by (60) combined with (48)

$$\mu(K \cap P_r(x_0)) \leq \frac{a}{2}\,\sigma(E_r(x_0) \cap P_r(x_0)) \tag{65}$$

for $r > 0$ sufficiently small, i.e. we can choose a $r(x_0) > 0$ such that (65) holds for $0 < r \leq r(x_0)$.

3.3. Using that K is ε smooth, we now prove the following absolute continuity property. There is a $c(\varepsilon)$ such that

$$\mu(K \cap P) \leq c(\varepsilon)\,\sigma(\operatorname{bd} K \cap P) \tag{66}$$

for every polytope P.

First, we show that an inequality of this type holds if P is a suitable cube. There is a $c(\varepsilon)$ such that

$$\begin{aligned} \mu(K \cap (x_0 + r\,I)) &\leq \frac{2^{d-2}\,c(\varepsilon)}{(1 + 2\sqrt{d})^{d-1}}\,r^{d-1} \\ &\leq \frac{c(\varepsilon)}{2\,(1 + 2\sqrt{d})^{d-1}}\,\sigma(\operatorname{bd} K \cap (x_0 + r\,I)) \end{aligned} \tag{67}$$

for every $x_0 \in \operatorname{bd} K$ and every closed cube I of side length 2 with center at the origin such that one of its facets is parallel to the tangent hyperplane $H(x_0)$ to K at x_0. The following proof of (67) will be almost the same as the proof of (36).

Since K is ε-smooth there is a ball of radius ε touching K at x_0 from the interior. For $y_0 \in \varepsilon S^{d-1}$, there is a rigid motion ψ_{y_0} which maps this ball to εB^d and x_0 to y_0. Similar to Claim 1 of 3.1.1, we show that the convex hull of $\psi_{y_0}(K) \cap (y_0 + r\,\psi_{y_0}(I))$ and εB^d differs only in a small neighbourhood from εB^d:

$$\operatorname{conv}\left(\varepsilon B^d \cup (\psi_{y_0}(\operatorname{bd} K) \cap (y_0 + r\,\psi_{y_0}(I)))\right) \setminus (\varepsilon B^d) \subset y_0 + 2\sqrt{d}\, r\, \psi_{y_0}(I) \quad (68)$$

for $r > 0$ sufficiently small. This is proved by stating (30) and (31) for the present situation. Let $y \in \psi_{y_0}(\operatorname{bd} K) \cap (y_0 + r\,\psi_{y_0}(I))$ and let $\bar{y} \in \varepsilon S^{d-1}$ be such that $\operatorname{aff}(y, \bar{y})$ is tangent to εB^d. Then

$$\operatorname{conv}\left(\varepsilon B^d \cup y\right) \setminus \varepsilon B^d \subset y + 2\sqrt{\operatorname{dist}(y, \varepsilon B^d)}\sqrt{\varepsilon} B^d, \quad (69)$$

and since y lies between the tangent hyperplane and εB^d

$$\operatorname{dist}(y, \varepsilon B^d) \le \frac{1}{\varepsilon}\, |y - y_0|^2, \quad (70)$$

which implies (68) since $y \in y_0 + r\,\psi_{y_0}(I)$.

We construct from suitable $\psi_y(K) \cap (y + r\,\psi_y(I))$ with $y \in \varepsilon S^{d-1}$ a convex body $M(r)$ in the following way. Let m_r be the maximum number of points $y_1, \ldots, y_{m_r} \in \varepsilon S^{d-1}$ such that the sets

$$\varepsilon S^{d-1} \cap B(y_i, 2\sqrt{d}\, r)$$

for $i = 1, \ldots, m_r$ form a packing in εS^{d-1}. Then, since $m_r = m(2\sqrt{d}\, r)$, it follows from (5) that there is an $r_0(\varepsilon) > 0$ such that

$$\kappa_{d-1}\, m_r\, (2\sqrt{d}\, r)^{d-1} \ge \frac{1}{2}\, \delta_{d-1}\, d\, \kappa_d\, \varepsilon^{d-1} > 0 \quad (71)$$

for $r \le r_0(\varepsilon)$. We define

$$M(r) = \operatorname{conv}\left((\varepsilon B^d) \cup \bigcup_{i=1}^{m_r} (\psi_{y_i}(K) \cap (y_i + r\,\psi_{y_i}(I))) \right).$$

Then $M(r) \to \varepsilon B^d$ as $r \to 0$. Since μ is upper semicontinuous, this implies that

$$\mu(\varepsilon B^d) \ge \frac{1}{2}\, \mu(M(r)) \quad (72)$$

for $0 < r \le r_1(\varepsilon)$ with a suitable $r_1(\varepsilon) > 0$. By (68) and our construction of $M(r)$

$$\psi_{y_i}(\operatorname{bd} K) \cap (y_i + \psi_{y_i}(I)) \subset \operatorname{bd} M(r)$$

for $i = 1, \ldots, m_r$. Therefore the intersection of $\psi_{y_i}(K) \cap (y_i + \psi_{y_i}(I))$ and $\psi_{y_i}(K) \cap (y_i + \psi_{y_i}(I))$ for $i \neq j$ is either empty or a convex polytope. Since μ is non-negative, rigid motion invariant and vanishes on polytopes, this and the definition of $M(r)$ imply that

$$\mu(M(r)) \geq \sum_{i=1}^{m_r} \mu(\psi_{y_i}(K) \cap (y_i + r\,\psi_{y_i}(I))) = m_r\,\mu(K \cap (x_0 + rI)).$$

¿From this combined with (72) and (71) it follows that

$$\begin{aligned} \mu(K \cap (x_0 + rI)) &\leq 2\,\mu(\varepsilon\, B^d)\, m_r^{-1} \\ &\leq 2\,\mu(\varepsilon\, B^d) \frac{\kappa_{d-1}\left(2\sqrt{d}\, r\right)^{d-1}}{\frac{1}{2}\delta_{d-1}\, d\, \kappa_d\, \varepsilon^{d-1}} \end{aligned} \tag{73}$$

for $0 < r \leq \min\{r_0(\varepsilon), r_1(\varepsilon)\}$. Since a facet of I is parallel to the tangent hyperplane $H(x_0)$,

$$\sigma(\mathrm{bd}\, K \cap (x_0 + r\, I)) \geq \sigma(H(x_0) \cap (x_0 + r\, I)) \geq (2\, r)^{d-1}, \tag{74}$$

and this combined with (73) implies that (67) holds with a suitable $c(\varepsilon)$.

Now let P be an arbitrary polytope and let U be a relatively open set in $\mathrm{bd}\, K$ such that

$$\mathrm{bd}\, K \cap P \subset U \tag{75}$$

and

$$\sigma(U) \leq 2\,\sigma(\mathrm{bd}\, K \cap P). \tag{76}$$

Let $\mathcal{J}$ be the family of all closed cubes $I = I(x, r)$ with center $x \in \mathrm{bd}\, K \cap P$ and side length $2\, r$ such that one facet of I is parallel to the tangent hyperplane to K at x, $0 < r \leq \min\{r_0(\varepsilon), r_1(\varepsilon)\}$, and $\mathrm{bd}\, K \cap I \subset U$. Then the relative interior of $\mathrm{bd}\, K \cap I$ for $I \in \mathcal{J}$ form an open covering of $\mathrm{bd}\, K \cap P$. Since $\mathrm{bd}\, K \cap P$ is compact, we can choose a finite subcovering and denote by $\mathcal{I} \subset \mathcal{J}$ the set of closed cubes corresponding to this subcovering. A standard argument known as Vitali's lemma (see, for example, [7]) shows that we can choose from $\mathcal{I}$ pairwise disjoint cubes $I(x_1, r_1), \ldots, I(x_n, r_n)$ such that

$$\mathrm{bd}\, K \cap P \subset \bigcup_{i=1}^{n} I\left(x_i, (1 + 2\sqrt{d})\, r_i\right).$$

Since μ is non-negative, this implies that

$$\mu(K \cap P) \leq \sum_{i=1}^{n} \mu\left(K \cap I\left(x_i, (1 + 2\sqrt{d})\, r_i\right)\right),$$

and applying the estimate (67) now shows that

$$\mu(K \cap P) \leq \frac{2^{d-2}\, c(\varepsilon)}{(1+2\sqrt{d})^{d-1}} \sum_{i=1}^{n} ((1+2\sqrt{d})\, r_i)^{d-1}. \tag{77}$$

Since the $I(x_i, r_i)$ are pairwise disjoint, we obtain by (74), (75), and (76)

$$\begin{aligned} \sum_{i=1}^{n} r_i^{d-1} &\leq 2^{-(d-1)} \sum_{i=1}^{n} \sigma(\operatorname{bd} K \cap I(x_i, r_i)) \\ &\leq 2^{-(d-1)} \sigma(U) \leq 2^{-(d-2)} \sigma(\operatorname{bd} K \cap P). \end{aligned}$$

Combined with (77), this proves (66).

3.4. Using the results from 3.1.3, 3.2.2 and (66), we now construct our $E \in \mathcal{E}$. Let $N \subset \operatorname{bd} K$ be the set of normal points of $\operatorname{bd} K$ and let $\mathcal{V}$ be the collection of sets

$$V_r(x) = \operatorname{bd} K \cap P_r(x)$$

for $x \in N$ and $0 < r \leq r(x)$, where $P_r(x)$ and $r(x)$ are defined above in 3.1.3 and 3.2.2. (42), (43), (61), and (63) imply that $\mathcal{V}$ is a Vitali class for N. Set

$$\eta = \frac{a\, \sigma(\operatorname{bd} K)}{4\, c(\varepsilon)}. \tag{78}$$

Then by Vitali's covering theorem (see subsection 2.4), there are pairwise disjoint $V_{r_1}(x_1), \ldots, V_{r_m}(x_m) \in \mathcal{V}$ such that

$$\sigma(N) - \sum_{i=1}^{m} \sigma(V_{r_i}(x_i)) \leq \eta. \tag{79}$$

Let $P_{r_1}(x_1), \ldots, P_{r_m}(x_m)$ be the polytopes and $E_{r_1}(x_1), \ldots, E_{r_m}(x_m)$ the elements of $\mathcal{E}$ corresponding to these $V_{r_1}(x_1), \ldots, V_{r_m}(x_m)$ as defined in 3.1.3 and 3.2.2. By (44) and (62) we have that for $i \neq j$, $\operatorname{bd} E_{r_i}(x_i)$ does not intersect $\operatorname{bd} E_{r_j}(x_j)$ within K. We can therefore choose a polytope P such that $K \subset P$ and such that for every i, j, $i \neq j$, $\operatorname{bd} E_{r_i}(x_i)$ intersects $\operatorname{bd} P$ before intersecting $\operatorname{bd} E_{r_j}(x_j)$ and define

$$E = \bigcap_{i=1}^{m} E_{r_i}(x_i) \cap P.$$

Then we have $E \in \mathcal{E}$. Since σ depends continuously on E, P can be chosen such that also

$$\sigma(\operatorname{bd} E) \leq \frac{3}{2}\, \sigma(\operatorname{bd} K) \tag{80}$$

holds.

Next, we dissect $P\setminus\bigcup_{i=1}^m P_{r_i}(x_i)$ into polytopes $P_1,\dots,P_k$ and have

$$K\subset\bigcup_{i=1}^m P_{r_i}(x_i)\cup\bigcup_{j=1}^k P_j.$$

Since the $V_{r_i}(x_i)$'s are disjoint, our definition of $P_r(x)$ implies that for $i\neq j$ the intersection of $K\cap P_{r_i}(x_i)$ and $K\cap P_{r_j}(x_j)$ is empty or a polytope which is contained in the interior of K. Since μ vanishes on polytopes, we therefore obtain

$$\mu(K)=\sum_{i=1}^m \mu(K\cap P_{r_i}(x_i))+\sum_{j=1}^k \mu(K\cap P_j). \tag{81}$$

Our definition of $E_r(x)$ implies that for a normal point x with positive curvature, $E_r(x)$ consists of a piece of an ellipsoid, which lies in K, and pieces of cylinders and polytopes. Since for a normal point with vanishing curvature $E_r(x)$ is a polytope and since μ vanishes on cylinders and polytopes, we therefore have

$$\mu(E)=\sum_{i=1}^m \mu(E_{r_i}(x_i)\cap P_{r_i}(x_i)).$$

Using this, (45), (65), and (80) we obtain

$$\begin{aligned}\sum_{i=1}^m \mu(K\cap P_{r_i}(x_i)) &\leq \sum_{i=1}^m\Big(\mu\left(E_{r_i}(x_i)\cap P_{r_i}(x_i)\right)+\frac{a}{2}\,\sigma(E_{r_i}(x_i)\cap P_{r_i}(x_i))\Big)\\ &\leq \mu(E)+\frac{a}{2}\,\sigma(\mathrm{bd}\,E)\\ &\leq \mu(E)+\frac{3\,a}{4}\,\sigma(\mathrm{bd}\,K).\end{aligned} \tag{82}$$

Applying (66) for $P_1,\dots,P_k$ shows that

$$\sum_{j=1}^k \mu(K\cap P_j)\leq c(\varepsilon)\sum_{j=1}^k \sigma(\mathrm{bd}\,K\cap P_j).$$

Since by Aleksandrov's theorem (4) $\sigma(\mathrm{bd}\,K)=\sigma(N)$, our choice of the P_j's and (79) imply that

$$\sum_{j=1}^k \sigma(\mathrm{bd}\,K\cap P_j)\leq\eta.$$

Consequently, we have by our definition of η in (78)

$$\sum_{j=1}^k \mu(K\cap P_j)\leq\frac{a}{4}\,\sigma(\mathrm{bd}\,K). \tag{83}$$

By (81), (82), and (83) we now obtain

$$\mu(K) \leq \mu(E) + a\,\sigma(\mathrm{bd}\,K). \tag{84}$$

Therefore (21) holds, since (47), (46) and (64) imply that $P^i \subset E \subset P^c$. Thus Proposition 3 is proved and the proof of the theorem is complete.

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References

[1] J. Aczel, "Lectures on functional equations and their applications," Academic Press, New York, 1966.

[2] A.D. Aleksandrov, Almost everywhere existence of the second differential of a convex function and some properties of convex surfaces connected with it (in Russian), *Uchenye Zapiski Leningrad. Gos. Univ., Math. Ser.* **6** (1939), 3–35.

[3] V. Bangert, Analytische Eigenschaften konvexer Funktionen auf Riemannschen Mannigfaltigkeiten, *J. reine angew. Math.* **307/308** (1979), 309–324.

[4] W. Blaschke, "Differentialgeometrie II: Affine Differentialgeometrie," Springer, Berlin, 1923.

[5] W. Blaschke, "Vorlesungen über Integralgeometrie. H. 2.," Teubner, Berlin, 1937.

[6] G. Dolzmann and D. Hug, Equality of two representations of extended affine surface area, *Arch. Math.* **65** (1995), 352–356.

[7] K. Falconer, "The geometry of fractal sets," Cambridge University Press, Cambridge, 1985.

[8] G. Fejes Tóth, New results in the theory of packing and covering, *in* "Convexity and its applications" (P.M. Gruber and J. Wills, eds.), pp. 318–359, Birkhäuser, 1983.

[9] P.M. Gruber, Aspects of approximation of convex bodies, *in* "Handbook of Convex Geometry A" (P.M. Gruber and J. Wills, eds.), pp. 319–345, North-Holland, Amsterdam, 1993.

[10] P.M. Gruber, Comparisons of best and random approximation of convex bodies by polytopes, *Suppl. Rend. Circ. Mat. Palermo* **50** (1997), 189 –216.

[11] H. Hadwiger, "Vorlesungen über Inhalt, Oberfläche und Isoperimetrie", Springer, Berlin, 1957.

[12] D. Hug, Contributions to affine surface area, *Manuscripta Math.* **91** (1996), 283–301.

[13] D. Klain, A short proof of Hadwiger's characterization theorem, *Mathematika* **42** (1995), 329–339.

[14] D. Klain and G. Rota, "Introduction to geometric probability," Cambridge University Press, Cambridge, 1997.

[15] K. Leichtweiß, Über eine Formel Blaschkes zur Affinoberfläche, *Stud. Sci. Math. Hungarica* **21** (1986), 453–474.

[16] K. Leichtweiß, Zur Affinoberfläche konvexer Körper, *Manuscripta Math.* **56** (1986), 429–464.

[17] K. Leichtweiß, Bemerkungen zur Definition einer erweiterten Affinoberfläche von E. Lutwak, *Manuscripta Math.* **65** (1989), 181–197.

[18] K. Leichtweiß, "Affine geometry of convex bodies," Hüthig Verlag (Johann Ambrosius Barth), Heidelberg, 1998.

[19] M. Ludwig, A characterization of affine length and asymptotic approximation of convex discs, *Abh. Math. Semin. Univ. Hamb.*, to appear.

[20] E. Lutwak, Extended affine surface area, *Adv. Math.* **85** (1991), 39–68.

[21] E. Lutwak, Selected affine isoperimetric inequalities, *in* "Handbook of Convex Geometry A" (P.M. Gruber and J. Wills, eds.), pp. 151–176, North-Holland, Amsterdam, 1993.

[22] P. McMullen, Valuations and dissections, *in* "Handbook of Convex Geometry B" (P.M. Gruber and J. Wills, eds.), pp. 933–990, North-Holland, Amsterdam, 1993.

[23] P. McMullen and R. Schneider, Valuations on convex bodies, *in* "Convexity and its applications" (P.M. Gruber and J. Wills, eds.), pp. 170–247, Birkhäuser, 1983.

[24] M. Meyer and E. Werner, The Santalo-regions of a convex body, *Trans. Am. Math. Soc.* **350** (1998), 4569-4591.

[25] C. Petty, Geominimal surface area, *Geom. Dedicata* **3** (1974), 77–97.

[26] S. Saks, "Theory of the integral," Dover, New York, 1964.

[27] R. Schneider, Curvature measures of convex bodies, *Ann. Mat. Pura Appl.* **11** (1978), 101–134.

[28] R. Schneider, "Convex bodies: the Brunn–Minkowski theory," Cambridge Univ. Press, Cambridge, 1993.

[29] C. Schütt, On the affine surface area, *Proc. Am. Math. Soc.* **118** (1993), 1213–1218.

[30] C. Schütt and E. Werner, The convex floating body, *Math. Scand.* **66** (1990), 275–290.

[31] E. Werner, A general geometric construction for affine surface area, preprint (1998).