

Asymptotic Approximation of Convex Curves

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Abstract. L. Fejes Tóth gave asymptotic formulae as $n \rightarrow \infty$ for the distance between a smooth convex disc and its best approximating inscribed or circumscribed polygons with at most n vertices, where the distance is in the sense of the symmetric difference metric. In this paper these formulae are extended by specifying the second terms of the asymptotic expansions. Tools are from affine differential geometry.

1 Introduction

Let C be a closed convex curve in the Euclidean plane \mathbb{E}^2 and let $\mathcal{P}_n^i(C)$ be the set of all convex polygons with at most n vertices that are inscribed in C . We measure the distance of C and $P_n \in \mathcal{P}_n^i(C)$ by the symmetric difference metric δ^S and study the asymptotic behaviour of

$$\delta^S(C, \mathcal{P}_n^i) = \inf\{\delta^S(C, P_n) : P_n \in \mathcal{P}_n^i(C)\}$$

as $n \rightarrow \infty$. In case of circumscribed polygons the analogous notion is $\delta^S(C, \mathcal{P}_n^c)$.

For a $C \in \mathcal{C}^2$ with positive curvature function $\kappa(t)$, the following asymptotic formulae were given by L. Fejes Tóth [2], [3]

$$\delta^S(C, \mathcal{P}_n^i) \sim \frac{1}{12} \left(\int_0^l \kappa^{1/3}(t) dt \right)^3 \frac{1}{n^2} \quad \text{as } n \rightarrow \infty \quad (1)$$

and

$$\delta^S(C, \mathcal{P}_n^c) \sim \frac{1}{24} \left(\int_0^l \kappa^{1/3}(t) dt \right)^3 \frac{1}{n^2} \quad \text{as } n \rightarrow \infty, \quad (2)$$

where t is the arc length and l the length of C . Complete proofs of these results are due to D. E. McClure and R. A. Vitale [9].

In this article we extend these asymptotic formulae by deriving the second terms in the asymptotic expansions. For a $C \in \mathcal{C}^4$ these terms are of order $1/n^4$ and depend on the affine curvature of the curve.

Besides the above mentioned results several other asymptotic formulae for the distance of a convex body to its best approximating polytopes are known. In the case of the symmetric difference metric and inscribed and circumscribed polytopes these formulae were derived by P. M. Gruber in [5], [6], and [8] for any dimension; see also [10]. For detailed information see the surveys [4] and [7].

2 Some Tools from Affine Differential Geometry

$\delta^S(C, \mathcal{P}_n^i)$ and $\delta^S(C, \mathcal{P}_n^c)$ are invariant with respect to area-preserving affine transformations. Therefore, we choose the affine arc length to be the parameter of C .

Let $C \in \mathcal{C}^2$ and let the curvature of C be positive. Then the affine arc length is given by

$$s(t) = \int_0^t \kappa^{1/3}(\tau) d\tau, \quad 0 \leq t \leq l, \quad (3)$$

where t is the ordinary arc length and $\kappa(t)$ the curvature of C . The affine length λ of C is

$$\lambda = \int_0^l \kappa^{1/3}(\tau) d\tau.$$

In the following, let $x(s)$ be an affine arclength parametrization of C .

The affine curvature of C is then given by

$$k(s) = |x''(s), x'''(s)|, \quad (4)$$

where $'$ denotes differentiation with respect to affine arc length. It determines a curve up to an area-preserving affine transformation. Note that

$$|x'(s), x''(s)| = 1 \text{ and } |x'(s), x'''(s)| = 0. \quad (5)$$

See [1].

3 Asymptotic Expansion for $\delta^S(C, \mathcal{P}_n^i)$

Using the notions and notation from affine differential geometry just introduced, we are able to formulate our main result.

Theorem 1 *Let C be a closed convex curve in \mathbb{E}^2 of class \mathcal{C}^4 with positive ordinary curvature. Then*

$$\delta^S(C, \mathcal{P}_n^i) \sim \frac{1}{12} \frac{\lambda^3}{n^2} - \frac{1}{2} \frac{\lambda^4}{5!} \int_0^\lambda k(s) ds \frac{1}{n^4} + o\left(\frac{1}{n^4}\right)$$

as $n \rightarrow \infty$.

For the proof we need the following two lemmas.

Lemma 1 *For $0 \leq r \leq s \leq \lambda$, let $F(r, s)$ be the area of the piece between C and the line segment with end points $x(r)$ and $x(s)$. Then*

$$F(r, s) = \frac{1}{2} \left(\frac{(s-r)^3}{3!} - k(r) \frac{(s-r)^5}{5!} + o((s-r)^5) \right)$$

uniformly for all $0 \leq r \leq s \leq \lambda$ as $(s-r) \rightarrow 0$.

Proof. Without loss of generality, let $r = 0$. Since $C \in \mathcal{C}^4$, the affine arclength parametrization $x(s)$ of C is of class \mathcal{C}^3 . Let ω be a modulus of continuity function for $x'''(s)$; that is, ω is a continuous non-decreasing function on $[0, \infty]$ with $\omega(0) = 0$ and such that

$$\|x'''(s) - x'''(r)\| \leq \omega(|s - r|) \quad \text{for all } r, s \in [0, \lambda].$$

Taylor's formula yields

$$x''(s) = x''(0) + x'''(0)s + u_1(s), \quad (6)$$

where $\|u_1(s)\| \leq s\omega(s)$. Integrating (6) twice we see that

$$x'(s) = x'(0) + x''(0)s + x'''(0)\frac{s^2}{2} + u_2(s) \quad (7)$$

and

$$x(s) - x(0) = x'(0)s + x''(0)\frac{s^2}{2} + x'''(0)\frac{s^3}{3!} + u_3(s), \quad (8)$$

where $\|u_2(s)\| \leq \frac{s^2}{2}\omega(s)$ and $\|u_3(s)\| \leq \frac{s^3}{3!}\omega(s)$.

Since $x(s)$ is the affine arclength parametrization, (4), (5), (6), and (7) imply that

$$1 = |x'(s), x''(s)| = 1 + |x'(0), u_1(s)| + k(0)\frac{s^2}{2} + \alpha(s), \quad (9)$$

where

$$|\alpha(s)| \leq \mu_1 \frac{s^2}{2} \omega(s)$$

and μ_1 is a constant which depends only on C . Rewriting (9) in the form

$$|x'(0), u_1(s)| = -k(0)\frac{s^2}{2} - \alpha(s)$$

and integrating twice, we find that

$$|x'(0), u_2(s)| = -k(0)\frac{s^3}{3!} - \beta(s), \quad (10)$$

and

$$|x'(0), u_3(s)| = -k(0)\frac{s^4}{4!} - \gamma(s), \quad (11)$$

where $|\beta(s)| \leq \mu_1 \frac{s^3}{3!} \omega(s)$ and $|\gamma(s)| \leq \mu_1 \frac{s^4}{4!} \omega(s)$. Then combining (7), (8), (10), and (11), we obtain

$$|x(s) - x(0), x'(s)| = \frac{s^2}{2} - k(0)\frac{s^4}{4!} + \delta(s), \quad (12)$$

where

$$|\delta(s)| \leq \mu_2 s^4 \omega(s) \quad (13)$$

and μ_2 is a constant which depends only on C .

Finally, integrating (12) gives

$$F(0, s) = \frac{1}{2} \int_0^s |x(s) - x(0), x'(s)| ds = \frac{1}{2} \left(\frac{s^3}{3!} - k(0) \frac{s^5}{5!} + \int_0^s \delta(s) ds \right).$$

Now, noting that $[0, \lambda]$ is compact and μ_2 and ω depend only on C , the last equality together with (13) yields Lemma 1. \square

Lemma 2 *Let $P_n \in \mathcal{P}_n^i(C)$, $n = 3, 4, \dots$, be a sequence of best approximating polygons of C . Let $x(s_{ni}), i = 1, \dots, n$, be the vertices of P_n . Define $\lambda_{ni} = s_{ni} - s_{n,i-1}$. Then*

$$\lambda_{ni} = \frac{\lambda}{n} + o\left(\frac{1}{n}\right) \quad \text{uniformly in } i \text{ as } n \rightarrow \infty.$$

Proof. Since P_n is a best approximating, the line segment connecting $x(s_{n,i-1})$ and $x(s_{n,i+1})$ is parallel to the tangent at $x(s_{ni})$, i. e.

$$|x(s_{n,i+1}) - x(s_{n,i-1}), x'(s_{ni})| = 0 \quad \text{for } i = 1, \dots, n-1, \quad (14)$$

see [3].

We proceed by using Taylor's formula to derive expressions for the λ_{ni} 's from (14). Let ω be a modulus of continuity function for $x'''(s)$. Then for given n and i ,

$$x(s) = x(s_{ni}) + x'(s_{ni})(s - s_{ni}) + x''(s_{ni}) \frac{(s - s_{ni})^2}{2} + x'''(s_{ni}) \frac{(s - s_{ni})^3}{3!} + u_{ni}(s),$$

where

$$\|u_{ni}(s)\| \leq \frac{|s - s_{ni}|^3}{3!} \omega(s - s_{ni}). \quad (15)$$

Substituting $s_{n,i-1}$ and $s_{n,i+1}$ for s and subtracting, we find that

$$\begin{aligned} x(s_{n,i+1}) - x(s_{n,i-1}) &= x'(s_{ni})(\lambda_{ni} + \lambda_{n,i+1}) + \frac{x''(s_{ni})}{2}(\lambda_{n,i+1}^2 - \lambda_{ni}^2) + \\ &\quad + \frac{x'''(s_{ni})}{3!}(\lambda_{ni}^3 + \lambda_{n,i+1}^3) - u_{ni}(s_{n,i-1}) + u_{ni}(s_{n,i+1}). \end{aligned}$$

Combining this with (14), we obtain

$$|x(s_{n,i+1}) - x(s_{n,i-1}), x'(s_{ni})| =$$

$$= \frac{\lambda_{ni}^2}{2} - \frac{\lambda_{n,i+1}^2}{2} + |x'(s_{ni}), u_{ni}(s_{n,i-1})| - |x'(s_{ni}), u_{ni}(s_{n,i+1})| = 0.$$

This can be rewritten as

$$\lambda_{ni}^2 = \lambda_{n,i+1}^2 + 2|x'(s_{ni}), u_{ni}(s_{n,i+1}) - u_{ni}(s_{n,i-1})|$$

for $i = 1, \dots, n-1$. Thus

$$\lambda_{ni}^2 = \lambda_{nj}^2 + 2 \sum_{k=i}^{j-1} |x'(s_{nk}), u_{nk}(s_{n,k+1}) - u_{nk}(s_{n,k-1})| \quad (16)$$

for $i < j \leq n$ and

$$\lambda_{ni}^2 = \lambda_{nj}^2 - 2 \sum_{k=j}^{i-1} |x'(s_{nk}), u_{nk}(s_{n,k+1}) - u_{nk}(s_{n,k-1})| \quad (17)$$

for $1 \leq j < i$. Summing on j from $i+1$ to n in (16) and from 1 to $i-1$ in (17), we obtain

$$\begin{aligned} (n-i)\lambda_{ni}^2 &= \sum_{j=i+1}^n \lambda_{nj}^2 + 2 \sum_{j=i+1}^n \sum_{k=i}^{j-1} |x'(s_{nk}), u_{nk}(s_{n,k+1}) - u_{nk}(s_{n,k-1})| \\ (i-1)\lambda_{ni}^2 &= \sum_{j=1}^{i-1} \lambda_{nj}^2 - 2 \sum_{j=1}^{i-1} \sum_{k=j}^{i-1} |x'(s_{nk}), u_{nk}(s_{n,k+1}) - u_{nk}(s_{n,k-1})|. \end{aligned}$$

Adding λ_{ni}^2 to the sum of these equations gives

$$n\lambda_{ni}^2 = \sum_{j=1}^n \lambda_{nj}^2 + \alpha_{ni}, \quad (18)$$

where

$$\begin{aligned} \alpha_{ni} &= 2 \sum_{j=i+1}^n \sum_{k=i}^{j-1} |x'(s_{nk}), u_{nk}(s_{n,k+1}) - u_{nk}(s_{n,k-1})| - \\ &\quad - 2 \sum_{j=1}^{i-1} \sum_{k=j}^{i-1} |x'(s_{nk}), u_{nk}(s_{n,k+1}) - u_{nk}(s_{n,k-1})|. \end{aligned}$$

Multiplying (18) by n gives the desired expressions for the λ_{ni} 's:

$$n^2 \lambda_{ni}^2 = n \sum_{j=1}^n \lambda_{nj}^2 + n\alpha_{ni}, \quad (19)$$

We will study the behaviour of these equations as $n \rightarrow \infty$. For the right-hand side of (19) we will show that

$$\lim_{n \rightarrow \infty} n \sum_{i=1}^n \lambda_{ni}^2 = \lambda^2 \quad (20)$$

and

$$\lim_{n \rightarrow \infty} n \max_{i=1, \dots, n} |\alpha_{ni}| = 0. \quad (21)$$

In order to prove (20) we rewrite the inequalities

$$\frac{\lambda}{n} = \frac{1}{n} \sum_{i=1}^n \lambda_{ni} \leq \left(\frac{1}{n} \sum_{i=1}^n \lambda_{ni}^2 \right)^{1/2} \leq \left(\frac{1}{n} \sum_{i=1}^n \lambda_{ni}^3 \right)^{1/3},$$

in the form

$$\lambda^2 \leq n \sum_{i=1}^n \lambda_{ni}^2 \leq (n^2 \sum_{i=1}^n \lambda_{ni}^3)^{2/3}. \quad (22)$$

As a consequence of Lemma 1,

$$F_{ni} = \frac{\lambda_{ni}^3}{12} + o(\lambda_{ni}^3) \quad \text{uniformly as } \lambda_{ni} \rightarrow 0.$$

Hence, recalling the asymptotic formula (1)

$$\lim_{n \rightarrow \infty} n^2 \delta^S(C, \mathcal{P}_n^i) = \lim_{n \rightarrow \infty} n^2 \sum_{i=1}^n F_{ni} = \frac{\lambda^3}{12},$$

we obtain

$$\lim_{n \rightarrow \infty} n^2 \sum_{i=1}^n \lambda_{ni}^3 = \lambda^3, \quad (23)$$

and (20) follows from (22).

Next we establish (21). Define $\nu = \sup_{0 \leq s \leq \lambda} \|x'(s)\|$. Then (15) gives

$$\|x'(s_{ni}), u_{ni}(s_{n,i-1})\| \leq \frac{\nu}{3!} \lambda_{ni}^3 \omega(\lambda_{ni}). \quad (24)$$

The definition of the α_{ni} 's implies that

$$|\alpha_{ni}| \leq 2n \sum_{k=1}^n \|\|x'(s_{nk}), u_{nk}(s_{n,k+1}) - u_{nk}(s_{n,k-1})\|\|.$$

Combining this with (24), we have

$$n|\alpha_{ni}| \leq 4n^2 \sum_{k=1}^n \frac{\nu}{3!} \lambda_{ni}^3 \omega(\lambda_{ni}) \leq 4 \frac{\nu}{3!} \omega(\max_{i=1, \dots, n} \lambda_{ni}) n^2 \sum_{k=1}^n \lambda_{ni}^3. \quad (25)$$

Since $\delta^S(C, \mathcal{P}_n^i) \rightarrow 0$, $\lim_{n \rightarrow \infty} \max_{i=1, \dots, n} \lambda_{ni} = 0$. Hence (21) follows from (23) and (25).

By (20) and (21), it follows from (19) that

$$\lim_{n \rightarrow \infty} n^2 \max_{i=1, \dots, n} \lambda_{ni}^2 = \lim_{n \rightarrow \infty} n^2 \min_{i=1, \dots, n} \lambda_{ni}^2 = \lambda^2,$$

which proves the lemma. \square

Now we are able to prove the main theorem.

Proof of Theorem 1. First we choose for $n = 3, 4, \dots$ polygons Q_n with vertices at the points $x(i\frac{\lambda}{n}), i = 0, 1, \dots, n$, and denote the area of the i -th piece between C and P_n by F_{ni} . Let $\varepsilon > 0$ be chosen. Lemma 1 shows that there is an integer n_0 such that

$$F_{ni} \leq \frac{1}{2} \left(\frac{1}{3!} \frac{\lambda^3}{n^3} - \frac{1}{5!} k(i\frac{\lambda}{n}) \frac{\lambda^5}{n^5} + \varepsilon \frac{\lambda^5}{n^5} \right)$$

for all $n \geq n_0$. Hence

$$\begin{aligned} \delta^S(C, \mathcal{P}_n^i) - \frac{1}{12} \frac{\lambda^3}{n^2} &\leq \delta^S(C, P_n) - \frac{1}{12} \frac{\lambda^3}{n^2} \\ &\leq \sum_{i=1}^n F_{ni} - \frac{1}{12} \frac{\lambda^3}{n^2} \\ &\leq \frac{1}{2} \left(\sum_{i=1}^n \left(\frac{1}{3!} \frac{\lambda^3}{n^3} - \frac{1}{5!} k(i\frac{\lambda}{n}) \frac{\lambda^5}{n^5} + \varepsilon \frac{\lambda^5}{n^5} \right) - \frac{1}{3!} \frac{\lambda^3}{n^2} \right) = \\ &= \frac{1}{2} \left(\sum_{i=1}^n \left(-\frac{1}{5!} k(i\frac{\lambda}{n}) \frac{\lambda^5}{n^5} + \frac{\lambda^5}{n^5} \varepsilon \right) \right) = \\ &= -\frac{1}{2} \frac{\lambda^4}{5!} \frac{1}{n^4} \sum_{i=1}^n k(i\frac{\lambda}{n}) \frac{\lambda}{n} + \frac{1}{2} \frac{\lambda^5}{n^4} \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary,

$$\limsup_{n \rightarrow \infty} n^4 \left(\delta^S(C, \mathcal{P}_n^i) - \frac{1}{12} \frac{\lambda^3}{n^2} \right) \leq -\frac{1}{2} \frac{\lambda^4}{5!} \int_0^\lambda k(s) ds. \quad (26)$$

In order to show the opposite inequality, let P_n be a sequence of best approximating polygons. Let $x(s_{ni})$ be the vertices of P_n and let $\lambda_{ni} = s_{ni} - s_{n,i-1}$. Choose $\varepsilon > 0$. Then by Lemma 1, there is an integer n_0 such that

$$F_{ni} \geq \frac{1}{2} \left(\frac{\lambda_{ni}^3}{3!} - k(s_{n,i-1}) \frac{\lambda_{ni}^5}{5!} - \varepsilon \lambda_{ni}^5 \right)$$

for all $n \geq n_0$. Using the inequality

$$\frac{1}{n^2} \lambda^3 = \frac{1}{n^2} \left(\sum_{i=1}^n \lambda_{ni} \right)^3 \leq \sum_{i=1}^n \lambda_{ni}^3,$$

we see that

$$\begin{aligned} n^4 \left(\delta^S(C, \mathcal{P}_n^i) - \frac{1}{12} \frac{\lambda^3}{n^2} \right) &\geq n^4 \left(\sum_{i=1}^n F_{ni} - \frac{1}{12} \sum_{i=1}^n \lambda_{ni}^3 \right) \\ &\geq \frac{n^4}{2} \sum_{i=1}^n \left(\frac{\lambda_{ni}^3}{3!} - k(s_{n,i-1}) \frac{\lambda_{ni}^5}{5!} - \varepsilon \lambda_{ni}^5 - \frac{\lambda_{ni}^3}{3!} \right) = \\ &= \frac{n^4}{2} \sum_{i=1}^n \left(-k(s_{n,i-1}) \frac{\lambda_{ni}^5}{5!} - \varepsilon \lambda_{ni}^5 \right). \end{aligned}$$

By Lemma 2

$$\lambda_{ni} = \frac{\lambda}{n} + o\left(\frac{1}{n}\right) \quad \text{uniformly in } i.$$

Thus

$$\begin{aligned} \frac{n^4}{2} \sum_{i=1}^n \left(-k(s_{n,i-1}) \frac{\lambda_{ni}^5}{5!} - \varepsilon \lambda_{ni}^5 \right) &= \frac{n^4}{2} \sum_{i=1}^n \left(-\frac{k(s_{n,i-1}) \lambda^5}{5! n^5} + o\left(\frac{1}{n^5}\right) - \varepsilon \frac{\lambda^5}{n^5} \right) \\ &= -\frac{1}{2} \frac{\lambda^4}{5!} \sum_{i=1}^n \left(k(s_{n,i-1}) \frac{\lambda}{n} \right) - \varepsilon \frac{\lambda^5}{2} + o(1) = \\ &= -\frac{1}{2} \frac{\lambda^4}{5!} \sum_{i=1}^n k(s_{n,i-1}) \lambda_{ni} - \varepsilon \frac{\lambda^5}{2} + o(1). \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we have

$$\liminf_{n \rightarrow \infty} n^4 \left(\delta^S(C, \mathcal{P}_n^i) - \frac{1}{12} \frac{\lambda^3}{n^2} \right) \geq -\frac{1}{2} \frac{\lambda^4}{5!} \int_0^\lambda k(s) ds.$$

This together with (26) concludes the proof of Theorem 1. \square

4 Asymptotic Expansion for $\delta(C, \mathcal{P}_n^c)$

Theorem 2 *Let C be a closed convex curve in \mathbb{E}^2 of class \mathcal{C}^4 with positive ordinary curvature. Then*

$$\delta^S(C, \mathcal{P}_n^c) \sim \frac{1}{24} \frac{\lambda^3}{n^2} + \frac{1}{2} \frac{\lambda^4}{5!} \int_0^\lambda k(s) ds \frac{1}{n^4} + o\left(\frac{1}{n^4}\right)$$

as $n \rightarrow \infty$.

The proof is analogous to that of Theorem 1. As a tool we use the property, that the edges of best approximating circumscribed polygons of C touch C at their midpoints (cf. [3]). Thus instead of (14) we have the equality

$$\frac{|x(s_{n,i-1}) - x(s_{ni}), x'(s_{n,i-1})|}{|x'(s_{n,i-1}), x'(s_{ni})|} = \frac{|x(s_{n,i+1}) - x(s_{ni}), x'(s_{n,i+1})|}{|x'(s_{ni}), x'(s_{n,i+1})|}.$$

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