

A NOTE ON STIRLING SERIES

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ABSTRACT. We study sums $S = S(d, n, k) = \sum_{j \geq 1} \frac{\begin{bmatrix} d \\ j \end{bmatrix}}{j^k \binom{n+j}{j} j!}$ with $d \in \mathbb{N} = \{1, 2, \dots\}$ and $n, k \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$ and relate them with (finite) multiple zeta functions. As a byproduct of our results we obtain asymptotic expansions of $\zeta(d+1) - H_n^{(d+1)}$ as n tends to infinity. Furthermore, we relate sums S to Nielsen's polylogarithm.

1. INTRODUCTION

The unsigned Stirling numbers of the first kind, also called Stirling cycle numbers, are defined by the recurrence relation

$$\begin{bmatrix} n \\ k \end{bmatrix} = (n-1) \begin{bmatrix} n-1 \\ k \end{bmatrix} + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}, \quad n \geq 1, \quad \text{with} \quad \begin{bmatrix} n \\ 0 \end{bmatrix} = \delta_{n,0}, \quad n \geq 0,$$

where $\delta_{i,j}$ denotes the Kronecker delta function. Throughout this work we use Knuth's notation $\begin{bmatrix} n \\ k \end{bmatrix}$. It is well known that Stirling numbers of the first kind are closely related to harmonic numbers, i.e. $\begin{bmatrix} n \\ 2 \end{bmatrix} = (n-1)!H_{n-1}$, $\begin{bmatrix} n \\ 3 \end{bmatrix} = (n-1)!(H_{n-1}^2 - H_n^{(2)})/2$, where for $s, n \in \mathbb{N}$ the values $H_n^{(s)} = \sum_{\ell=1}^n 1/\ell^s$ denote n -th harmonic numbers of order s , $H_n = H_n^{(1)}$. Furthermore, it is known (i.e. see Adamchik [1]) that Stirling numbers of the first kind are expressible in terms of (finite) multiple zeta functions defined by

$$\zeta_N(a_1, \dots, a_\ell) = \sum_{N \geq n_1 > n_2 > \dots > n_\ell \geq 1} \frac{1}{n_1^{a_1} n_2^{a_2} \dots n_\ell^{a_\ell}},$$

$$\zeta(a_1, \dots, a_\ell) = \sum_{n_1 > n_2 > \dots > n_\ell \geq 1} \frac{1}{n_1^{a_1} n_2^{a_2} \dots n_\ell^{a_\ell}},$$

by the following formula

$$\begin{bmatrix} n \\ k \end{bmatrix} = (n-1)! \zeta_{n-1}(\underbrace{1, \dots, 1}_{k-1}) = (n-1)! \cdot \zeta_{n-1}(\{1\}_{k-1}).$$

We use the shorthand notations $\zeta(\cup_{i=1}^r \{a_i\}) = \zeta(a_1, \dots, a_r)$, and $\zeta(\cup_{i=1}^r \{a\}) = \zeta(\{a\}_r)$. Note that for $n, s \in \mathbb{N}_0$ we have $\zeta_n(s) = H_n^{(s)}$. We are interested in evaluations of sums $S = \sum_{j \geq 1} \frac{\begin{bmatrix} d \\ j \end{bmatrix}}{j^k \binom{n+j}{j} j!}$ with $d \in \mathbb{N} = \{1, 2, \dots\}$ and $n, k \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$. We assume that

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n and k are chosen in way such that $n + k \geq 1$ in order to ensure that the sum converges. Special instances of this family of sums have been studied by Adamchik [1], and also by Choi and Srivastava [6].

2. EVALUATION OF SUM S

We obtain the following result.

Theorem 1. *The sum $S = S(d, n, k)$ with $d \in \mathbb{N} = \{1, 2, \dots\}$ and $n, k \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$ can be evaluated in terms of harmonic numbers and (finite) multiple zeta functions,*

$$S = \sum_{m=2}^{k+1} (-1)^{k+1-m} \zeta(m, \{1\}_{d-1}) \sum_{\sum_{i=1}^{k+1-m} i \cdot m_i = k+1-m} \prod_{r=1}^{k+1-m} \frac{(H_n^{(r)})_{m_r}}{r^{m_r} m_r!} \\ + (-1)^k \sum_{h=1}^k \sum_{1 \leq \ell_1 < \ell_2 < \dots < \ell_{h-1} < k} \zeta_n(\ell_1, \ell_2 - \ell_1, \dots, \ell_{h-1} - \ell_{h-2}, d + k - \ell_{h-1}),$$

subject to $\ell_0 := 0$. We have the short equivalent expression

$$S = (-1)^k \zeta_n^*(\{1\}_{k-1}, d+1) + \sum_{m=2}^{k+1} (-1)^{k+1-m} \zeta(m, \{1\}_{d-1}) \zeta_n^*(\{1\}_{k+1-m}).$$

Remark 1. The second expression for the sum S is given according to a variant of finite multiple zeta functions, $\zeta_N^*(a_1, \dots, a_k)$, which recently attracted some interest, [2, 12, 9, 7] where the summation indices satisfy $N \geq n_1 \geq n_2 \geq \dots \geq n_k \geq 1$ in contrast to $N \geq n_1 > n_2 > \dots > n_k > 1$, as in the usual definition (1),

$$\zeta_N^*(a_1, \dots, a_k) = \sum_{N \geq n_1 \geq n_2 \geq \dots \geq n_k \geq 1} \frac{1}{n_1^{a_1} n_2^{a_2} \dots n_k^{a_k}}.$$

The form stated above is due to the conversion formula below applied to $\zeta_n^*(\{1\}_{k-1}, d+1)$,

$$\zeta_N^*(a_1, \dots, a_k) = \sum_{h=1}^k \sum_{\substack{1 \leq \ell_1 < \ell_2 < \dots < \ell_{h-1} < k \\ \ell_0 = 0}} \zeta_N \left(\sum_{i_1=1}^{\ell_1} a_{i_1}, \sum_{i_2=\ell_1+1}^{\ell_2} a_{i_2}, \dots, \sum_{i_h=\ell_{h-1}+1}^k a_{i_h} \right).$$

Note that the first term $h = 1$ should be interpreted as $\zeta_N(\sum_{i_1=\ell_0+1}^k a_{i_1})$, subject to $\ell_0 = 0$. The notation $\zeta_N^*(a_1, \dots, a_k)$ is chosen in analogy with Aoki and Ohno [2] where infinite counterparts of $\zeta_N^*(a_1, \dots, a_k)$ have been treated; see also Ohno [12].

Remark 2. The sum $\zeta(m, \{1\}_{d-1})$ can be completely transformed into single zeta values. By results of Borwein, Bradley and Broadhoarst [3]

$$\zeta(2, \{1\}_d) = \zeta(d+2) \\ \zeta(3, \{1\}_d) = \frac{d+2}{2} \zeta(d+3) - \frac{1}{2} \sum_{\ell=1}^d \zeta(\ell+1) \zeta(d+2-\ell).$$

Furthermore, in the general case of $\zeta(m+2, \{1\}_d) = \zeta(d+2, \{1\}_m)$ one obtains products of up to $\min\{m+1, d+1\}$ zeta values, according to the generating function, see [3],

$$\sum_{m,n \geq 0} \zeta(m+2, \{1\}_n) x^{m+1} y^{n+1} = 1 - \exp\left(\sum_{k \geq 2} \frac{x^k + y^k - (x+y)^k}{k} \zeta(k)\right). \quad (1)$$

Below we state three specific evaluations of the sum S for special choices of d, n, k .

Corollary 1. *For $k = 0$ and arbitrary $n, d \in \mathbb{N}$ we get*

$$S(d, n, 0) = \sum_{j \geq 1} \frac{\begin{bmatrix} j \\ d \end{bmatrix}}{\binom{n+j}{j} j!} = \frac{1}{n^d}.$$

For $k = 1$ and arbitrary $n, d \in \mathbb{N}$ we get

$$S(d, n, 1) = \sum_{j \geq 1} \frac{\begin{bmatrix} j \\ d \end{bmatrix}}{j \binom{n+j}{j} j!} = \zeta(2, \{1\}_{d-1}) - \zeta_n(d+1) = \zeta(d+1) - H_n^{(d+1)}, \quad (2)$$

For $n = 0$ and arbitrary $d, k \in \mathbb{N}$ we get

$$S(d, 0, k) = \zeta(k+1, \{1\}_{d-1}).$$

In order to prove the results above we proceed as follows: Since

$$\frac{1}{\binom{n+j}{j}} = \frac{n!}{(n+j)^n} = \sum_{\ell=1}^n n \binom{n-1}{\ell-1} \frac{(-1)^{\ell-1}}{j+\ell},$$

we obtain

$$S = \sum_{j \geq 1} \frac{\begin{bmatrix} j \\ d \end{bmatrix}}{j^k \binom{n+j}{j} j!} = \sum_{\ell=1}^n n \binom{n-1}{\ell-1} (-1)^{\ell-1} \sum_{j \geq 1} \frac{\begin{bmatrix} j \\ d \end{bmatrix}}{j! j^k (j+\ell)}.$$

We use partial fraction decomposition and obtain

$$\frac{1}{j^k (j+\ell)} = \sum_{m=2}^k \frac{(-1)^{k-m}}{j^m \ell^{k+1-m}} + \frac{(-1)^{k+1}}{\ell^k} \left(\frac{1}{j} - \frac{1}{j+\ell} \right).$$

Consequently, we get by using the partial fraction decomposition above and the representation of Stirling numbers by finite multiple zeta functions

$$\begin{aligned} S &= \sum_{\ell=1}^n n \binom{n-1}{\ell-1} (-1)^{\ell-1} \sum_{m=2}^{k+1} \frac{(-1)^{k+1-m}}{\ell^{k+2-m}} \sum_{j \geq 1} \frac{\zeta_{j-1}(\{1\}_{d-1})}{j^m} \\ &+ \sum_{\ell=1}^n n \binom{n-1}{\ell-1} (-1)^{\ell-1} \frac{(-1)^k}{\ell^{k+1}} \sum_{j \geq 1} \zeta_{j-1}(\{1\}_{d-1}) \left(\frac{1}{j} - \frac{1}{j+\ell} \right) = S_1 + S_2. \end{aligned}$$

By definition of the multiple zeta function we get

$$\begin{aligned} S_1 &= \sum_{\ell=1}^n n \binom{n-1}{\ell-1} (-1)^{\ell-1} \sum_{m=2}^{k+1} \frac{(-1)^{k+1-m}}{\ell^{k+2-m}} \zeta(m, \{1\}_{d-1}) \\ &= \sum_{m=2}^{k+1} (-1)^{k+1-m} \zeta(m, \{1\}_{d-1}) \sum_{\ell=1}^n n \binom{n-1}{\ell-1} \frac{(-1)^{\ell-1}}{\ell^{k+2-m}}. \end{aligned}$$

We rewrite the inner sum as

$$\sum_{\ell=1}^n n \binom{n-1}{\ell-1} \frac{(-1)^{\ell-1}}{\ell^{k+2-m}} = \sum_{\ell=1}^n \binom{n}{\ell} \frac{(-1)^{\ell-1}}{\ell^{k+1-m}}.$$

This sum can be evaluated by using the following result of Flajolet and Sedgewick [8],

$$\sum_{\ell=1}^n \binom{n}{\ell} \frac{(-1)^{\ell-1}}{\ell^m} = \sum_{\sum_{i=1}^m i \cdot m_i = m} \prod_{r=1}^m \frac{(H_n^{(r)})^{m_r}}{r^{m_r} m_r!};$$

we recall that $H_n^{(s)} = \sum_{\ell=1}^n 1/\ell^s$ denotes the n -th harmonic number of order s ; in other words we have $H_n^{(s)} = \zeta_n(s)$, according to our previous definition of finite multiple zeta functions (1). Furthermore, it is well known that $\sum_{\ell=1}^n \binom{n}{\ell} \frac{(-1)^{\ell-1}}{\ell^m} = \zeta_n^*(\{1\}_m)$, which can immediately be deduced by repeated usage of the formula $\binom{n}{k} = \sum_{\ell=k}^n \binom{\ell-1}{k-1}$. The multiple zeta function $\zeta(m, \{1\}_d)$ is evaluated using a result of Borwein, Bradley and Broadhoarst [3], see Remark 2. Consequently, we can write sum S_1 as a finite sum involving higher order harmonic numbers and products of zeta functions and obtain the first part of our result. For the simplification of the inner sum

$$S_2 = \sum_{\ell=1}^n n \binom{n-1}{\ell-1} (-1)^{\ell-1} \frac{(-1)^k}{\ell^{k+1}} \sum_{j \geq 1} \zeta_{j-1}(\{1\}_{d-1}) \left(\frac{1}{j} - \frac{1}{j+\ell} \right),$$

we use the notation $T_{m,\ell} = \sum_{j \geq 1} \zeta_{j-1}(\{1\}_m) \left(\frac{1}{j} - \frac{1}{j+\ell} \right)$. Subsequently, we interchange summation, compare with Panholzer and Prodinger [11]. First we start with the simple case $m = 1$ and calculate $T_{1,\ell}$, since it is most instructive.

$$T_{1,\ell} = \sum_{j \geq 1} H_{j-1} \left(\frac{1}{j} - \frac{1}{j+\ell} \right) = \sum_{j \geq 1} H_j \left(\frac{1}{j+1} - \frac{1}{j+1+\ell} \right).$$

Since by definition $H_j = \sum_{h=1}^j 1/h$ we obtain after summation change (partial summation)

$$T_{1,\ell} = \sum_{h \geq 1} \frac{1}{h} \sum_{j \geq h} \left(\frac{1}{j+1} - \frac{1}{j+1+\ell} \right) = \sum_{h \geq 1} \frac{1}{h} \sum_{j=1}^{\ell} \frac{1}{j+h}.$$

By partial fraction decomposition we get

$$T_{1,\ell} = \sum_{j=1}^{\ell} \frac{1}{j} \sum_{h \geq 1} \left(\frac{1}{h} - \frac{1}{j+h} \right) = \sum_{j=1}^{\ell} \frac{H_j}{j} = \frac{H_{\ell}^2 + H_{\ell}^{(2)}}{2}.$$

Now we turn to the general case $T_{m,\ell}$. Shifting the index as before, and changing the order of summation leads to

$$T_{m,\ell} = \sum_{h \geq 1} \frac{\zeta_{h-1}(\{1\}_{m-1})}{h} \sum_{j \geq h} \left(\frac{1}{j+1} - \frac{1}{j+1+\ell} \right)$$

Consequently,

$$T_{m,\ell} = \sum_{j=1}^{\ell} \frac{1}{j} \sum_{h \geq 1} \zeta_{h-1}(\{1\}_{m-1}) \left(\frac{1}{h} - \frac{1}{h+j} \right) = \sum_{j=1}^{\ell} \frac{1}{j} T_{m-1,j}.$$

Hence, the value $T_{m,\ell}$ is a variant of the finite multiple zeta function $\zeta_{\ell}(\{1\}_{m+1})$, where the summation indices satisfy $N \geq n_1 \geq n_2 \geq \dots \geq n_m \geq n_{m+1} \geq 1$ instead of $N \geq n_1 > n_2 > \dots > n_m > n_{m+1} > 1$, see Remark 1, such that $T_{m,\ell} = \zeta_{\ell}^*(\{1\}_{m+1})$. We further obtain

$$T_{m,\ell} = \zeta_{\ell}^*(\{1\}_{m+1}) = \sum_{h=1}^{\ell} \binom{\ell}{h} \frac{(-1)^{h-1}}{h^{m+1}},$$

according to the well known formula $\binom{n}{k} = \sum_{\ell=k}^n \binom{\ell-1}{k-1}$. Consequently, the sum S_2 simplifies to

$$S_2 = (-1)^k \sum_{\ell=1}^n \binom{n}{\ell} \frac{(-1)^{\ell-1}}{\ell^k} \sum_{h=1}^{\ell} \binom{\ell}{h} \frac{(-1)^{h-1}}{h^d} = (-1)^k \sum_{h=1}^n \frac{(-1)^{h-1}}{h^d} \sum_{\ell=h}^n \binom{n}{\ell} \binom{\ell}{h} \frac{(-1)^{\ell-1}}{\ell^k},$$

or equivalently

$$S_2 = (-1)^k \sum_{\ell=1}^n \binom{n}{\ell} \frac{(-1)^{\ell-1}}{\ell^k} \zeta_{\ell}^*(\{1\}_d).$$

In order to obtain the final form of S_2 for $k \in \mathbb{N}$ we combine our previous considerations as follows:

$$S_2 = (-1)^k \sum_{h_1=1}^n \frac{1}{h_1} \sum_{h_2=1}^{h_1} \frac{1}{h_2} \dots \sum_{h_{k+1}=1}^{h_k} \binom{h_k}{h_{k+1}} (-1)^{h_{k+1}-1} \zeta_{h_{k+1}}^*(\{1\}_d).$$

We use the fact that $\sum_{\ell=h}^n \binom{n}{\ell} \binom{\ell}{h} (-1)^{\ell-1} = \delta_{h,n} (-1)^{n-1}$ and the sum S_2 simplifies to

$$S_2 = (-1)^k \zeta_n^*(\{1\}_{k-1}, d+1).$$

In the case $k = 0$ we use

$$S_2 = \sum_{h=1}^n \frac{(-1)^{h-1}}{h^d} \sum_{\ell=h}^n \binom{n}{\ell} \binom{\ell}{h} (-1)^{\ell-1} = \frac{1}{n^d}.$$

2.1. An application: asymptotic expansions. Following Romik [13] we note that the limit $\lim_{n \rightarrow \infty} \sum_{j \geq 1} \frac{\binom{j}{d}}{j^k \binom{n+j}{j} j!} = 0$ provides information about the convergence of the two sums appearing in the results for $S = S_1 + S_2$, stated in Theorem 1. This is of particular interest in the special case $k = 1$ and arbitrary $n, d \in \mathbb{N}$, where we have obtained $\zeta(d+1) - H_n^{(d+1)} = \sum_{j \geq 1} \frac{\binom{j}{d}}{j^k \binom{n+j}{j} j!}$, see Corollary 1.

Proposition 1. *We obtain the following asymptotic expansions for $n \rightarrow \infty$*

$$\begin{aligned} & (-1)^k \zeta_n^*(\{1\}_{k-1}, d+1) + \sum_{m=2}^{k+1} (-1)^{k+1-m} \zeta(m, \{1\}_{d-1}) \zeta_n^*(\{1\}_{k+1-m}) \\ &= \sum_{j=1}^n \frac{\binom{j}{d}}{j^k \binom{n+j}{j} j!} + \mathcal{O}\left(\frac{1}{\sqrt{n}2^n}\right), \end{aligned}$$

In the case of $S = S(d, n, 1)$ with $k = 1$ and arbitrary $n, d \in \mathbb{N}$ we obtain in particular:

$$\zeta(d+1) - H_n^{(d+1)} = \sum_{j=1}^n \frac{\binom{j}{d}}{j^k \binom{n+j}{j} j!} + \mathcal{O}\left(\frac{1}{\sqrt{n}2^n}\right), \quad \text{for } n \rightarrow \infty.$$

Proof. We can split the summation range $j \geq 1$ into $1 \leq j \leq N$ and $j \geq N+1 > k$. We get

$$\sum_{j \geq N+1} \frac{\binom{j}{d}}{j^k \binom{n+j}{j} j!} < \sum_{j \geq N+1} \frac{2}{j^k \binom{n+j}{j}} < \sum_{j \geq N+1} \frac{2}{k! \binom{j}{k} \binom{n+j}{j}} = \frac{(n+1+N)}{k!(k-1+N) \binom{n+1+N}{n} \binom{N+1}{k}}.$$

Consequently, we readily obtain, setting i.e. $N = n$ and using Stirling's formula,

$$n! = \frac{n^n}{e^n} \sqrt{2\pi n} \left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right),$$

the stated asymptotic expansions. □

3. RELATION TO NIELSEN'S POLYLOGARITHM

Nielsen's polylogarithm $L_{k,d}(z)$ is defined by

$$L_{k,d}(z) = \frac{(-1)^{k-1+d}}{(k-1)!d!} \int_0^1 \frac{\log^{k-1}(t) \log^d(1-zt)}{t} dt$$

By definition of the generating function of the Stirling cycle numbers

$$\sum_{n \geq k} \begin{bmatrix} n \\ k \end{bmatrix} \frac{z^n}{n!} = \frac{(-1)^k \log^k(1-z)}{k!},$$

it is evident that $L_{k,d}(z) = \sum_{j \geq 1} \frac{\binom{j}{d} z^j}{j^k j!}$. Hence, we obtain the following result.

Proposition 2. *The series $S(z) = S_{d,n,k}(z) = \sum_{j \geq 1} \frac{[d]_z^j}{j^k \binom{n+j}{j} j!}$ can be expressed by Nielsen's polylogarithm $L_{k,d}(z)$ in the following way.*

$$\sum_{j \geq 1} \frac{[d]_z^j}{j^k \binom{n+j}{j} j!} = \frac{n}{z} \int_0^z \left(1 - \frac{u}{z}\right)^{n-1} L_{k,d}(u) du.$$

Note that

$$\begin{aligned} S_{d,n,k}(z) &= \sum_{\ell=1}^n \ell (-1)^{\ell-1} \binom{n}{\ell} \frac{(-1)^{k-1}}{(k-1)! d!} \frac{1}{z^\ell} \int_0^z u^{\ell-1} \int_0^1 \frac{\log^{k-1}(t) \log^d(1-ut)}{t} dt du \\ &= \sum_{\ell=1}^n \ell (-1)^{\ell-1} \binom{n}{\ell} \frac{1}{z^\ell} \int_0^z u^{\ell-1} L_{k,d}(u) du. \end{aligned}$$

Interchanging summation and integration gives the desired result.

3.1. Generalized r -Stirling numbers of the first kind. In a recent work Mező [10] considered series involving so-called r -Stirling numbers of the first kind, see Broder [5]. For any positive integer $r \in \mathbb{N}$ the quantity $\left[\begin{smallmatrix} n \\ m \end{smallmatrix} \right]_r$ denotes the number of permutations of the set $\{1, \dots, n\}$ having m cycles such that the first r element are in distinct cycles. These numbers obey the recurrence relation

$$\left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right]_r = (n-1) \left[\begin{smallmatrix} n-1 \\ k \end{smallmatrix} \right]_r + \left[\begin{smallmatrix} n-1 \\ k-1 \end{smallmatrix} \right]_r, \quad n > r, \quad \left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right]_r = \delta_{k,r}, \quad n = r, \quad \left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right]_r = 0, \quad n < r.$$

For $r = 0$ and $r = 1$ these numbers coincide with the ordinary Stirling numbers of the first kind. We will consider the series

$$S^{(r)}(z) = S_{d,n,k,\ell}^{(r)}(z) = \sum_{j \geq 1} \frac{\left[\begin{smallmatrix} j+\ell+r \\ d+r \end{smallmatrix} \right]_r z^j}{j^k \binom{n+j}{j} j!},$$

which generalizes the series considered by Mező [10] (case $n = 0$) and our previously considered series S (case $\ell = r = 0$). Subsequently, we obtain representations of $S_{d,n,k,0}^{(r)}(z)$ and also of $S_{d,n,k,\ell}^{(r)}(z)$. We introduce the quantity $L_{n,k}^{(r)}(z)$, which generalizes Nielsen's polylogarithm

$$L_{k,d}^{(r)}(z) = \frac{(-1)^{k-1+d}}{(k-1)! d!} \int_0^1 \frac{\log^{k-1}(t) \log^d(1-zt)}{(1-zt)^r t} dt.$$

Proposition 3. *The series $S_{d,n,k,0}^{(r)}(z) = \sum_{j \geq 1} \frac{\left[\begin{smallmatrix} j+r \\ d+r \end{smallmatrix} \right]_r z^j}{j^k \binom{n+j}{j} j!}$ can be expressed by $L_{k,d}^{(r)}(z)$ in the following way.*

$$S_{d,n,k,0}^{(r)}(z) = \frac{n}{z} \int_0^z \left(1 - \frac{u}{z}\right)^{n-1} L_{k,d}^{(r)}(u) du.$$

The series $S_{d,n,k,\ell}^{(r)}(z)$ can be expression as a linear combination of the sums $S_{h,n,k,0}^{(r+\ell)}(z)$, with $0 \leq h \leq d$.

First we note that the r -Stirling numbers of the first kind have the generating function

$$\sum_{n \geq k} \left[\begin{matrix} n+r \\ k+r \end{matrix} \right]_r \frac{z^n}{n!} = \frac{(-1)^k \log^k(1-z)}{k!(1-z)^r}.$$

We observe that

$$L_{k,d}^{(r)}(z) = \sum_{j \geq 1} \frac{\left[\begin{matrix} j+r \\ d+r \end{matrix} \right]_r z^j}{j^k \binom{n+j}{j} j!} = S_{d,0,k,0}^{(r)}(z).$$

Consequently, we get

$$S_{d,n,k,0}^{(r)}(z) = \sum_{j \geq 1} \frac{\left[\begin{matrix} j+r \\ d+r \end{matrix} \right]_r z^j}{j^k \binom{n+j}{j} j!} = \int_0^z \frac{n(1-\frac{u}{z})^n}{(z-u)} L_{k,d}^{(r)}(u) du.$$

Next we turn to the general case $\ell \in \mathbb{N}$. Since

$$\sum_{n \geq k} \left[\begin{matrix} n+r \\ d+r \end{matrix} \right]_r \frac{z^n}{n!} = \frac{(-1)^d \log^d(1-z)}{d!(1-z)^r},$$

we obtain the exponential generating function of $\left[\begin{matrix} n+\ell+r \\ d+r \end{matrix} \right]_r$ by differentiating $\frac{(-1)^d \log^d(1-z)}{d!(1-z)^r}$ ℓ -times with respect to z and a subsequent shift of the index,

$$\frac{\partial^\ell}{\partial z^\ell} \frac{(-1)^d \log^d(1-z)}{d!(1-z)^r} = \sum_{n \geq d+\ell} \left[\begin{matrix} n+r \\ d+r \end{matrix} \right]_r \frac{z^{n-\ell}}{(n-\ell)!} = \sum_{n \geq \max\{d-\ell, 0\}} \left[\begin{matrix} n+\ell+r \\ d+r \end{matrix} \right]_r \frac{z^n}{n!}.$$

By Faà di Bruno's formula we get

$$\frac{\partial^\ell}{\partial z^\ell} \frac{(-1)^d \log^d(1-z)}{d!(1-z)^r} = \sum_{h=0}^{\ell} \frac{d^h (-1)^h \log^{d-h}(1-z)}{(1-z)^{r+\ell}} \sum_{i=h}^{\ell} r^{\ell-i} B_{i,h}(0!, 1!, 2!, \dots, (i-h)!),$$

where $B_{i,h}(x_1, x_2, \dots, x_{i-h+1})$ denote the Bell polynomials. Consequently, we can express the sum $S_{d,n,k,\ell}^{(r)}(z)$ as a linear combination of the sums $S_{h,n,k,0}^{(r)}(z)$, with $0 \leq h \leq d$, which proves the stated result.

Remark 3. Note that the sums $S_{d,n,k,\ell}^{(r)}(1) = \sum_{j \geq 1} \frac{\left[\begin{matrix} j+\ell+r \\ d+r \end{matrix} \right]_r z^j}{j^k \binom{n+j}{j} j!}$ can in principle also be treated using our previous approach; however, the expression become much more involved, therefore we refrain from going into this matter. Furthermore, one can evaluate sums of the form $\sum_{j \geq 1} \frac{\left[\begin{matrix} j \\ d \end{matrix} \right]_r}{j^k \binom{n+j}{j}^g j!}$, with $g \in \mathbb{N}$; however, the expressions get more and more involved.

4. A GENERALIZATION OF SERIES S

In the following we will briefly consider the more general series V defined by

$$V = V(a_1, \dots, a_r, n, k) = \sum_{j \geq 1} \frac{\zeta_{j-1}(a_1, \dots, a_r)}{j^{k+1} \binom{n+j}{j}},$$

with $a_i \in \mathbb{N}$ for $1 \leq i \leq r$, and $n, k \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$ such that $n+k \geq 1$. We reobtain our previous Stirling cycle number series S choosing $r = d-1$ and $a_i = 1$, $1 \leq i \leq d-1$.

Before we state our result for the series V we introduce one more series, namely a variant of the finite multiple zeta star function

$$A_N^*(a_1, \dots, a_r) = \sum_{N \geq n_1 \geq n_2 \geq \dots \geq n_r \geq 1} \binom{N}{n_1} \frac{(-1)^{a_1-1}}{n_1^{a_1} n_2^{a_2} \dots n_r^{a_r}} = \sum_{n_1=1}^N \binom{N}{n_1} \frac{(-1)^{a_1-1}}{n_1^{a_1}} \zeta_{n_1}^*(a_2, \dots, a_r),$$

which can be expressed in terms of $\zeta_N^*(a_1, \dots, a_r)$ by the relation

$$A_N^*(a_1, \{1\}_{b_1-1}, \cup_{i=2}^r \{a_i + 1, \{1\}_{b_i-1}\}) = \zeta_N^*(\cup_{i=1}^{r-1} \{\{1\}_{a_i-1}, b_j + 1\}, \{1\}_{a_r-1}, b_r),$$

which is due to Bradley [4].

Theorem 2. *The sum $V = V(a_1, \dots, a_r, n, k)$ with $a_i \in \mathbb{N}$ for $1 \leq i \leq r$, and $n, k \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$ such that $n + k \geq 1$ can be evaluated in terms of (finite) multiple zeta functions,*

$$\begin{aligned} V &= (-1)^k \sum_{g=1}^r (-1)^{\sum_{f=1}^{g-1} (a_f+1)} \sum_{m=2}^{a_g} (-1)^{a_g-m} \zeta(m, \cup_{i=g+1}^r \{a_i\}) A_n(k, \cup_{i=1}^{g-1} \{a_i\}, a_g + 1 - m) \\ &\quad + (-1)^{k+r+\sum_{f=1}^r a_f} A_n(k, \cup_{i=1}^r \{a_i\}, 1) + \sum_{m=2}^{k+1} (-1)^{k+1-m} \zeta(m, a_1, \dots, a_r) \zeta_n^*(\{1\}_{k+1-m}). \end{aligned}$$

Proof (Sketch). The proof is analogous to the proof of Theorem 1; therefore it is only sketched. We only elaborate on the main new difficulty—the evaluation of the sum $T_{a_1, \dots, a_r; \ell} = \sum_{j \geq 1} \zeta_{j-1}(a_1, \dots, a_r) (\frac{1}{j} - \frac{1}{j+\ell})$. Proceeding as before, i.e. interchanging summation and using partial fraction decomposition, we obtain the recurrence relation

$$T_{a_1, \dots, a_r; \ell} = \sum_{m=2}^{a_1} (-1)^{a_1-m} \zeta(m, a_2, \dots, a_r) \zeta_\ell^*(a_1 + 1 - m) + (-1)^{a_1+1} \sum_{i=1}^{\ell} \frac{1}{i^{a_1}} T_{a_2, \dots, a_r; i}.$$

One can show that

$$\begin{aligned} T_{a_1, \dots, a_r; \ell} &= \sum_{g=1}^r (-1)^{\sum_{f=1}^{g-1} (a_f+1)} \sum_{m=2}^{a_g} (-1)^{a_g-m} \zeta(m, \cup_{i=g+1}^r \{a_i\}) \zeta_\ell^*(\cup_{i=1}^{g-1} \{a_i\}, a_g + 1 - m) \\ &\quad + (-1)^{r+\sum_{f=1}^r a_f} \zeta_\ell^*(\cup_{i=1}^r \{a_i\}, 1), \end{aligned}$$

which implies the stated result for the series V . □

HISTORICAL REMARK

The author H.P. has found the formula (2) empirically in 2003. He contacted several specialists about it and got feedback from Christian Krattenthaler who provided a *hypergeometric* proof for it. Eventually it turned out that it was known already [6]. We are happy that in 2009 we could put new life into this project.

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