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Controlling cardinal characteristics without adding reals

Martin Goldstern[*,¶], Jakob Kellner[*,‖], Diego A. Mejía[†,**]
and Saharon Shelah[‡,§,††]

[*]*TU Wien, Institute of Discrete Mathematics and Geometry*
Wiedner Hauptstrasse 8-10/104, 1040 Wien, Austria

[†]*Creative Science Course (Mathematics), Faculty of Science*
Shizuoka University, Ohya 836, Suruga-ku
Shizuoka-shi 422-8529, Japan

[‡]*Einstein Institute of Mathematics*
The Hebrew University of Jerusalem, Edmond J. Safra Campus
Givat Ram, Jerusalem 91904, Israel
[§]*Department of Mathematics, Rutgers University*
New Brunswick, NJ 08854, USA
[¶]*goldstern@tuwien.ac.at*
[‖]*jakob.kellner@tuwien.ac.at*
[**]*diego.mejia@shizuoka.ac.jp*
[††]*shlhetal@math.huji.ac.il*

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We investigate the behavior of cardinal characteristics of the reals under extensions that do not add new $<\kappa$-sequences (for some regular κ). As an application, we show that consistently the following cardinal characteristics can be different: The ("independent") characteristics in Cichoń's diagram, plus $\aleph_1 < \mathfrak{m} < \mathfrak{p} < \mathfrak{h} < \mathrm{add}(\mathcal{N})$. (So we get thirteen different values, including $\aleph_1$ and continuum). We also give constructions to alternatively separate other MA-numbers (instead of $\mathfrak{m}$), namely: MA for k-Knaster from MA for $k+1$-Knaster; and MA for the union of all k-Knaster forcings from MA for precaliber.

Keywords: Cardinal characteristics of the continuum; forcing extensions without new reals; Cichoń's diagram.

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1. Introduction

In this work, we investigate how to preserve and how to change certain cardinal characteristics of the continuum in No New Reals (NNR) extensions, i.e. extensions

[**]Corresponding author.

that do not add reals; or more generally that do not add $<\kappa$-sequences of ordinals for some regular κ. It is known that the "Blass-uniform" characteristics (see Definition 2.1) tend to keep their values in such extensions (cf. Mildenberger's [34, Proposition 2.1]), and we give some explicit results in that direction. Other cardinal characteristics tend to keep a value θ only if $\theta < \kappa$. We will use this effect to combine various forcing notions (most of them already known) to get models with many simultaneously different "classical" characteristics.

In particular, we look at the entries of Cichoń's diagram, which we call *Cichoń-characteristics* (see Fig. 1, we assume that the reader is familiar with this diagram), and the following characteristics.

Definition 1.1. Let $\mathcal{P}$ be a class of posets.

(1) $\mathfrak{m}(\mathcal{P})$ denotes the minimal cardinal where Martin's axiom for the posets in $\mathcal{P}$ fails. More explicitly, it is the minimal κ such that, for some poset $Q \in \mathcal{P}$, there is a collection $\mathcal{D}$ of size κ of dense subsets of Q such that there is no filter in Q intersecting all the members of $\mathcal{D}$.
(2) $\mathfrak{m} := \mathfrak{m}(\text{ccc})$.
(3) Write $a \subseteq^* b$ iff $a \smallsetminus b$ is finite. Say that $a \in [\omega]^{\aleph_0}$ is a *pseudo-intersection* of $F \subseteq [\omega]^\omega$ if $a \subseteq^* b$ for all $b \in F$.
(4) The *pseudo-intersection number* $\mathfrak{p}$ is the smallest size of a filter base of a free filter on ω that has no pseudo-intersection in $[\omega]^{\aleph_0}$.
(5) The *tower number* $\mathfrak{t}$ is the smallest order type of a $\subseteq^*$-decreasing sequence in $[\omega]^{\aleph_0}$ without pseudo-intersection.
(6) The *distributivity number* $\mathfrak{h}$ is the smallest size of a collection of dense subsets of $([\omega]^{\aleph_0}, \subseteq^*)$ whose intersection is empty.
(7) A family $D \subseteq [\omega]^{\aleph_0}$ is *groupwise dense* if

    (i) $a \subseteq^* b$ and $b \in D$ implies $a \in D$ and
    (ii) whenever $(I_n : n < \omega)$ is an interval partition of ω, there is some $a \in [\omega]^{\aleph_0}$ such that $\bigcup_{n\in a} I_n \in D$.

    The *groupwise density number* $\mathfrak{g}$ is the smallest size of a collection of groupwise dense sets whose intersection is empty.

$$\begin{tikzcd}
 & \mathrm{cov}(\mathcal{N}) \arrow[r] & \mathrm{non}(\mathcal{M}) \arrow[r] & \boxed{} \arrow[r] & \mathrm{cof}(\mathcal{N}) \arrow[r] & 2^{\aleph_0} \\
 & & \mathfrak{b} \arrow[u] \arrow[r] & \mathfrak{d} \arrow[u] & & \\
\aleph_1 \arrow[r] & \mathrm{add}(\mathcal{N}) \arrow[uu] \arrow[r] & \boxed{} \arrow[u] \arrow[r] & \mathrm{cov}(\mathcal{M}) \arrow[u] \arrow[r] & \mathrm{non}(\mathcal{N}) \arrow[uu] &
\end{tikzcd}$$

Fig. 1. Cichoń's diagram with the two "dependent" values removed, which are $\mathrm{add}(\mathcal{M}) = \min(\mathfrak{b}, \mathrm{cov}(\mathcal{M}))$ and $\mathrm{cof}(\mathcal{M}) = \max(\mathrm{non}(\mathcal{M}), \mathfrak{d})$. An arrow $\mathfrak{x} \to \mathfrak{y}$ means that ZFC proves $\mathfrak{x} \leq \mathfrak{y}$.

We are aware of the following ZFC provable relations between these cardinals:

$$\mathfrak{m} \leq \mathfrak{p} = \mathfrak{t} \leq \mathfrak{h} \leq \mathfrak{g}, \quad \mathfrak{m} \leq \mathrm{add}(\mathcal{N}), \quad \mathfrak{t} \leq \mathrm{add}(\mathcal{M}), \quad \mathfrak{h} \leq \mathfrak{b}, \quad \mathfrak{g} \leq \mathrm{cof}(\mathfrak{d}). \tag{1.1}$$

Also, with the exception of $\mathfrak{m}$ and $\mathfrak{d}$, all the cardinals in (1.1) are known to be regular (and uncountable), $2^{<\mathfrak{t}} = \mathfrak{c}$ and $\mathfrak{g} \leq \mathrm{cof}(\mathfrak{c})$. For details see, e.g. Blass [7]; but for $\mathfrak{p} = \mathfrak{t}$ see [35] with Malliaris,[a] and $\mathfrak{g} \leq \mathrm{cof}(\mathfrak{d})$ follows from the fact that $\mathrm{cof}((\omega, <)^\omega/\mathcal{U}) = \mathrm{cof}(\mathfrak{d})$ for some ultrafilter $\mathcal{U}$, due to Canjar [11], and $\mathfrak{g} \leq \mathrm{cof}((\omega, <)^\omega/\mathcal{U})$ for any ultrafilter $\mathcal{U}$, due to Blass and Mildenberger [8].

Recently [20] constructed, assuming four strongly compact cardinals, a ZFC model where the ten (non-dependent) Cichoń-characteristics are pairwise different. This orders the characteristics as shown in Fig. 2. In [19], we give a construction that does not require large cardinals.

To continue with this line of work, we ask whether other classical cardinal characteristics of the continuum can be included and forced to be pairwise different. Our main result is that we can additionally force that $\aleph_1 < \mathfrak{m} < \mathfrak{p} < \mathfrak{h} = \mathfrak{g} < \mathrm{add}(\mathcal{N})$, thus yielding a model where 13 classical cardinal characteristics are pairwise different.

We now give an outline of this paper.

S. 2, p. 4: Preliminaries. We review some aspects of the Cichoń's Maximum construction (the construction from [19] that gives 10 different values in Cichoń's diagram). In particular, we mention Blass-uniform characteristics and the Linear Cofinally Unbounded (LCU) and Cone of Bounds (COB) properties.

S. 3, p. 8: NNR extensions. We define some classes of cardinal characteristics and show how they are affected (or unaffected) by extensions that do not add new $<\kappa$-sequences for some regular κ; in particular: under $<\kappa$-distributive forcing extensions; and when intersecting the poset with some $<\kappa$-closed elementary submodel.

S. 4, p. 11: $\mathfrak{m}$. Using classical methods of Barnett and Todorčević [2, 39, 40], we modify the Cichoń's Maximum construction to additionally force $\mathfrak{m} = \lambda_\mathfrak{m}$ for any given regular value $\lambda_\mathfrak{m}$ between $\aleph_1$ and $\mathrm{add}(\mathcal{N})$.

$$\begin{tikzcd}
 & & & & \mathrm{cov}(\mathcal{N}) \arrow[ddr] & \mathrm{non}(\mathcal{M}) \arrow[ddr] & \boxed{} \arrow[ddr] & \mathrm{cof}(\mathcal{N}) \arrow[r] & 2^{\aleph_0} \\
 & & & & & \mathfrak{b} \arrow[u] \arrow[d, equal] & \mathfrak{d} \arrow[u, equal] & & \\
\aleph_1 \arrow[r] & \mathfrak{m} \arrow[r] & \mathfrak{p} \arrow[r] & \mathfrak{h} \arrow[r] & \mathrm{add}(\mathcal{N}) \arrow[uu] & \boxed{} & \mathrm{cov}(\mathcal{M}) \arrow[u] & \mathrm{non}(\mathcal{N}) \arrow[uu] &
\end{tikzcd}$$

Fig. 2. The model we construct in this paper; here $\mathfrak{x} \to \mathfrak{y}$ means that $\mathfrak{x} < \mathfrak{y}$. Any number of the $<$ signs can be replaced by $=$ as desired.

[a]However, only the trivial inequality $\mathfrak{p} \leq \mathfrak{t}$ is used in this text.

In addition to $\mathfrak{m}$, we can control the Knaster-numbers $\mathfrak{m}(k\text{-Knaster})$ as well. But this does not give a larger number of simultaneously different characteristics (as all Knaster numbers bigger than $\aleph_1$ have the same value, which is also the value of $\mathfrak{m}(\text{precaliber})$). We give models for all possible constellations (at least for regular λ): All Knaster numbers (and $\mathfrak{m}(\text{precaliber})$) can be $\aleph_1$. There can be a $k \geq 1$ such that $\mathfrak{m}(\ell\text{-Knaster}) = \aleph_1$ for all $1 \leq \ell < k$ and $\mathfrak{m}(\ell\text{-Knaster}) = \lambda$ for $\ell \geq k$ (for notational convenience, we identify 1-Knaster with ccc).

S. 5, p. 18: $\mathfrak{m}$(precaliber). We deal with a case that was left open in the previous section: We construct a model where all Knaster numbers are $\aleph_1$, and the precaliber number is some regular $\lambda > \aleph_1$.

S. 6, p. 21: $\mathfrak{h}$. Given a poset P, we show how to obtain a complete subposet P' of P forcing smaller values to $\mathfrak{g}$ (and thus $\mathfrak{h} \leq \mathfrak{g}$), while preserving certain other values for cardinal characteristics already forced by P. This method allows us to get $\mathfrak{p} = \mathfrak{h} = \mathfrak{g}$.

S. 7, p. 24: $\mathfrak{p}$. Based on a result with Dow [15], we show that the product of a ξ-cc poset P with the poset $\xi^{<\xi}$ may add a tower of length ξ, while preserving the cardinal $\mathfrak{h}$ above ξ and the values for the Cichoń-characteristics that were already forced by P.

This allows us to prove the main theorem, thirteen pairwise different characteristics.

S. 8, p. 25: Extensions. We remark on alternative initial forcings (i.e. forcings for the left-hand side of Cichoń's diagram) and an alternative order.

Notation. When we are investigating a characteristic $\mathfrak{x}$ and plan to force a specific value to it, we will usually call this value $\lambda_{\mathfrak{x}}$. Let us stress that calling a cardinal $\lambda_{\mathfrak{x}}$ is *not* an implicit *assumption* that $P \Vdash \mathfrak{x} = \lambda_{\mathfrak{x}}$ for the P under investigation; it is just an (implicit) declaration of intent.

2. Preliminaries

We mention some of the required definitions and constructions from [19, 20]. We will not give all required proofs and not even the complete construction, as it is rather involved. We will have to assume that the reader either knows this construction, or is willing to accept it as a blackbox.

2.1. *LCU and COB, the initial forcing P^{pre} for the left side*

Definition 2.1. A *Blass-uniform cardinal characteristic* is a characteristic of the form

$$\mathfrak{d}_R := \min\{|D| : D \subseteq \omega^\omega \text{ and } (\forall\, x \in \omega^\omega)\, (\exists y \in D)\ xRy\}$$

for some Borel[b] R. To avoid trivialities, we will only consider relations R for which $\mathfrak{d}_R$ (and the dual $\mathfrak{b}_R$ in what follows) are well defined.[c]

Such characteristics have been studied systematically since at least the 1980s by many authors, including Fremlin [16], Blass [6, 7] and Vojtáš [42].

Note that its dual cardinal

$$\mathfrak{b}_R := \min\{|F| : F \subseteq \omega^\omega \text{ and } (\forall\, y \in \omega^\omega)\, (\exists x \in F)\ \neg xRy\}$$

is also Blass-uniform because $\mathfrak{b}_R = \mathfrak{d}_{R^\perp}$, where $xR^\perp y$ if and only if $\neg(yRx)$.

Remark. All Blass-uniform characteristics in this paper, and many others, such as those in Blass' survey [7] or those in [22], are in fact of the form $\mathfrak{b}_R$ or $\mathfrak{d}_R$ for some Σ^0_2 relation R which is invariant under finite modifications of its arguments. When we restrict to such relations, there is no ambiguity as to which Blass-uniform cardinal characteristics are of the form $\mathfrak{b}_R$ and which are of the form $\mathfrak{d}_R$. It was shown by Blass [6] that for such relations R we must have $\mathfrak{b}_R \leq \mathrm{non}(\mathcal{M})$ and $\mathfrak{d}_R \geq \mathrm{cov}(\mathcal{M})$, thus $\mathfrak{b}_R$ is always on the left side of Cichoń's diagram, and $\mathfrak{d}_R$ is on the right side.

Remark 2.2. It can be more practical to consider more generally relations on $X \times Y$ for some Polish spaces X, Y other than ω^ω, in particular as many examples of Blass-uniform cardinals are naturally defined in such spaces.

To cover such cases, one can either modify the definition, or use a Borel isomorphisms to translate the relation to ω^ω.

The Cichoń-characteristics are all Blass-uniform, defined by natural[d] relations. Accordingly, they come in pairs $(\mathfrak{b}_R, \mathfrak{d}_R)$ for the according Borel relation R: $(\mathrm{add}(\mathcal{N}), \mathrm{cof}(\mathcal{N}))$, $(\mathrm{cov}(\mathcal{N}), \mathrm{non}(\mathcal{N}))$, $(\mathrm{add}(\mathcal{M}), \mathrm{cof}(\mathcal{M}))$, $(\mathrm{non}(\mathcal{M}), \mathrm{cov}(\mathcal{M}))$ and $(\mathfrak{b}, \mathfrak{d})$ (the last pair, for example, is defined by eventual domination $\leq^*$).

Another example for a Blass-uniform pair is $(\mathfrak{s}, \mathfrak{r}) = (\mathfrak{b}_R, \mathfrak{d}_R)$ where $\mathfrak{s}$ is splitting number and $\mathfrak{r}$ the reaping number and R is the relation on $[\omega]^{\aleph_0}$ that states xRy if and only if "x does not split y".

We will often have a situation where $(\mathfrak{b}_R, \mathfrak{d}_R) = (\lambda, \mu)$ is "strongly witnessed", as follows.

[b] We could just as well assume that R is analytic or co-analytic. More specifically, for all results in this paper, it is enough to assume that R is absolute between the extensions we consider; in our case between extensions that do not add new reals. So even projective relations would be fine. However, all concrete relations that we will actually use are Borel, even of very low rank. Regarding "on ω^ω", see Remark 2.2.

[c] i.e. $(\forall\, x \in \omega^\omega)\, (\exists y, z \in \omega^\omega)\, xRy \wedge \neg zRx$.

[d] The relations R used to define the following characteristics are "natural", but not entirely "canonical". For example, a different choice of a natural relation R such that $\mathfrak{b}_R = \mathfrak{s}$ leads to a different dual $\mathfrak{d}_R = \mathfrak{r}_\sigma$. See [7, Example 4.6].

Definition 2.3. Fix a Borel relation R, λ a regular cardinal and μ an arbitrary cardinal. We define two properties[e]:

Linearly cofinally unbounded: $\mathrm{LCU}_R(\lambda)$ means: There is a family $\bar{f} = (f_\alpha : \alpha < \lambda)$ of reals such that

$$(\forall\, g \in \omega^\omega)\,(\exists \alpha \in \lambda)\,(\forall\, \beta \in \lambda \setminus \alpha)\ \neg f_\beta R g. \tag{2.1}$$

Cone of bounds: For $\lambda \leq \mu$, $\mathrm{COB}_R(\lambda, \mu)$ means[f]: There is a $<\lambda$-directed partial order $\trianglelefteq$ on μ,[g] and a family $\bar{g} = (g_s : s \in \mu)$ of reals such that

$$(\forall\, f \in \omega^\omega)\,(\exists s \in \mu)\,(\forall\, t \trianglerighteq s)\ f R g_t. \tag{2.2}$$

Fact 2.4. $\mathrm{LCU}_R(\lambda)$ implies $\mathfrak{b}_R \leq \lambda \leq \mathfrak{d}_R$.

$\mathrm{COB}_R(\lambda, \mu)$ implies $\mathfrak{b}_R \geq \lambda$ and $\mathfrak{d}_R \leq \mu$.

Remark 2.5. $\mathrm{COB}_R(\lambda, \mu)$ clearly implies $\mathrm{COB}_R(\lambda', \mu)$ whenever $\lambda' \leq \lambda$. The property $\mathrm{COB}_R(2, \mu)$, the weakest of these notions, just says that there is a witness for $\mathfrak{d}_R \leq \mu$, or in other words: there is an R-dominating[h] family of size μ.

Also, $\mathrm{COB}_R(\lambda, \mu)$ implies $\mathrm{COB}_R(\lambda, \mu')$ whenever $\mu' \geq \mu$.

Informally, we call the objects $\bar{f}$ in the definition of LCU and $(\trianglelefteq, \bar{g})$ for COB "strong witnesses", and say that the corresponding cardinal inequalities (or equalities) are "strongly witnessed".

In [20] (building on [21]) the following is shown.

Lemma 2.6. *Assume GCH and $\aleph_1 < \nu_1 < \nu_2 < \nu_3 < \nu_4 < \theta_\infty$ are all successors of regular cardinals. Then there is a* ccc *countable support iteration P^{pre} of length $\theta_\infty + \theta_\infty$ forcing that*

$$\aleph_1 < \mathrm{add}(\mathcal{N}) = \nu_1 < \mathrm{cov}(\mathcal{N}) = \nu_2 < \mathfrak{b} = \nu_3 < \mathrm{non}(\mathcal{M}) = \nu_4 < \mathfrak{c} = \theta_\infty.$$

Moreover, all the equalities are strongly witnessed; all iterands in P are (σ, k)-linked (see Definition 4.1) for all k; and in the first θ_∞ many steps we add Cohen reals.

In this work, we will modify this construction P^{pre} to get similar iterations P that allow us to add additional characteristics. We claim that these modifications will not change the fact that the characteristics in Lemma 2.6 are strongly witnessed. A reader who does not know the proof of Lemma 2.6 will hopefully trust us on this; for the others we give the (simple) argument:

- We get the required COB properties simply by bookkeeping, when forcing with "partial random", or "partial eventually different", etc., forcings. This will not

[e] In [10] (and in other related work), a family with $\mathrm{LCU}_R(\lambda)$ is said to be *strongly $\lambda - R$-unbounded of size λ*, while a family with $\mathrm{COB}_R(\lambda, \mu)$ is said to be *strongly $\lambda - R$-dominating of size μ*.

[f] Note that $\mathrm{COB}_R(\lambda, \mu)$ for $\lambda > \mu$ would violate our assumption that $\mathfrak{b}_R$ is well-defined: In that case, $\mathrm{COB}_R(\lambda, \mu)$ would imply that μ has a top element with respect to the order $\trianglelefteq$, so there is an $x \in \omega^\omega$ with yRx for all y.

[g] i.e. every subset of μ of cardinality $<\lambda$ has a $\trianglelefteq$-upper bound

[h] Formally: $D \subseteq \omega^\omega$ is R-dominating iff $(\forall\, x \in \omega^\omega)\,(\exists y \in D)\ xRy$.

change when we add additional iterands (as long as, cofinally often, we choose the iterands as in the original construction).

- Fix a (left-hand) Cichoń-characteristic $\mathfrak{x}$ other than $\mathfrak{b}$. We get the strong witness $\mathrm{LCU}_R(\nu)$ (for R a relation connected to $\mathfrak{x}$ and ν the according ν_i) because all the iterands are "(ν, R)-good".

  Any forcing of size $<\nu$ is automatically good, so adding small iterands will not be a problem.

  Also, σ-centered forcings are always good for the characteristics $\mathrm{add}(\mathcal{N})$ and $\mathrm{cov}(\mathcal{N})$.
- For $\mathfrak{b}$, it is more cumbersome to prove $\mathrm{LCU}_R(\nu_3)$, but at least it is clear that adding additional iterands of size $<\nu_3$ will not interfere with the proof.

So we can summarize:

Claim 2.7. *We can add to P^{pre} arbitrary iterands that all are*

- *either of size $<\nu_1$,*
- *or σ-centered and of size $<\nu_3$,*

and still force strong witnesses for the Cichoń-characteristics of Lemma 2.6.

(Of course these new iterands have to be added in a way so that we still use the old iterands unboundedly often; we cannot just add new iterands at the end.)

Remark. Instead of the construction of [20], one can use alternative constructions that require weaker assumptions, cf. Sec. 8.3.

2.2. *The Cichoń's maximum construction*

As before, we will not require or describe the construction in detail, but only present the basic structure and certain properties.

The following is the main [19, Theorem 3.1]. As we will use the assumptions of the theorem repeatedly, we make them explicit.

Assumption 2.8. Assume GCH, and that

$$\aleph_1 \leq \kappa \leq \lambda_{\mathrm{add}(\mathcal{N})} \leq \lambda_{\mathrm{cov}(\mathcal{N})} \leq \lambda_{\mathfrak{b}} \leq \lambda_{\mathrm{non}(\mathcal{M})}$$
$$\leq \lambda_{\mathrm{cov}(\mathcal{M})} \leq \lambda_{\mathfrak{d}} \leq \lambda_{\mathrm{non}(\mathcal{N})} \leq \lambda_{\mathrm{cof}(\mathcal{N})} \leq \lambda_{\infty}$$

are regular cardinals, with the possible exception of λ_∞, for which we only require $\lambda_\infty^{<\kappa} = \lambda_\infty$.

Theorem 2.9. *Under these assumptions, there is a* ccc *poset P^{fin} forcing strong witnesses for*

$$\aleph_1 \leq \mathrm{add}(\mathcal{N}) = \lambda_{\mathrm{add}(\mathcal{N})} \leq \mathrm{cov}(\mathcal{N}) = \lambda_{\mathrm{cov}(\mathcal{N})} \leq \mathfrak{b} = \lambda_{\mathfrak{b}} \leq \mathrm{non}(\mathcal{M}) = \lambda_{\mathrm{non}(\mathcal{M})}$$
$$\leq \mathrm{cov}(\mathcal{M}) = \lambda_{\mathrm{cov}(\mathcal{M})} \leq \mathfrak{d} = \lambda_{\mathfrak{d}} \leq \mathrm{non}(\mathcal{N}) = \lambda_{\mathrm{non}(\mathcal{N})} \leq \mathrm{cof}(\mathcal{N}) = \lambda_{\mathrm{cof}(\mathcal{N})}$$
$$\leq \mathfrak{c} = \lambda_{\infty}.$$

Note that κ does not make much sense in this theorem, as you can just set $\kappa = \aleph_1$ (resulting in the weakest requirement $\lambda_\infty^{\aleph_0} = \lambda_\infty$). Indeed this is what is done in [19] (where κ is not mentioned at all); but mentioning κ explicitly here will be useful in Lemma 2.10.

The construction in [19] is as follows:

(A) Pick a sequence of successors of regular cardinals (strictly) above λ_∞:

$$\xi_1 < \nu_1 < \xi_2 < \nu_2 < \xi_3 < \nu_3 < \xi_4 < \nu_4 < \theta_\infty.$$

(B) Start with any initial κ-cc poset P^{pre} for the "left-hand side", which forces "strong witnesses" for

$$\mathrm{add}(\mathcal{N}) = \nu_1 < \mathrm{cov}(\mathcal{N}) = \nu_2 < \mathfrak{b} = \nu_3 < \mathrm{non}(\mathcal{M}) = \nu_4 < \mathfrak{c} = \theta_\infty$$

(so we can use the forcing of Lemma 2.6, or any modification satisfying Claim 2.7).

The proof in [19] can then be formulated as the following:

Lemma 2.10. *Under Assumption* 2.8, *and given a forcing* P^{pre} *as in* (*A*) *and* (*B*), *there is a* $<\kappa$*-closed*[i] *elementary submodel* N^* *of* $H(\chi)$ *such that* $P^{\mathrm{fin}} := P^{\mathrm{pre}} \cap N^*$ *witnesses Theorem* 2.9

(as usual, χ is a sufficiently large, regular cardinal).

2.3. *History*

We briefly remark on the history of the result of this section.

A (by now) classical series of results by various authors [1, 4, 13, 25–27, 32, 33, 36] (summarized by Bartoszyński and Judah [3]) shows that any assignments of $\{\aleph_1, \aleph_2\}$ to the Cichoń-characteristics that satisfy the well known ZFC restrictions is consistent. This leaves the questions how to show that many values can be simultaneously different. The "left-hand side" part was done in [21] and uses eventually different forcing $\mathbb{E}$ to ensure $\mathrm{non}(\mathcal{M}) \geq \lambda_{\mathrm{non}(\mathcal{M})}$ and ultrafilter-limits of $\mathbb{E}$ to show that $\mathfrak{b}$ remains small. It relies heavily on the notion of goodness, introduced in [25] (with Judah) and by Brendle [9], and summarized in, e.g. [21] or [12] (with Cardona).

Based on this construction, [20] uses Boolean ultrapowers to get simultaneously different values for all (independent) Cichoń-characteristics, modulo four strongly compact cardinals.

For this, the construction for the left-hand side first has to be modified to get a ccc forcing starting with a ground model satisfying GCH.

Then Boolean ultrapowers are applied to separate the cardinals on the right side. Paper [29] (with Tănasie and Tonti) gives an introduction to the Boolean

[i] Paper [19] uses the case $\kappa = \aleph_1$, so we get only a countably closed N^*. But the proof there works for any uncountable regular κ, with only the trivial change: We let N_8 be a $<\kappa$-closed model of size λ_∞, and note that then N^* is $<\kappa$-closed as well.

ultrapower construction. Such Boolean ultrapowers are applied four times, once for each pair of cardinals on the right side that are separated.

For this it is required that there is a strongly compact cardinal between two values corresponding to adjacent cardinals characteristics on the left side, so the cardinals on this side are necessarily very far apart. Paper [10] improves the left-hand side construction of [21] to include $\mathrm{cov}(\mathcal{M}) < \mathfrak{d} = \mathrm{non}(\mathcal{N}) = \mathfrak{c}$. This is achieved by using matrix iterations of partial Frechet-linked posets (the latter concept is originally from [31]). Then the same method of Boolean ultrapowers as before can be applied, in the same way, to force different values for all Cichoń-characteristics, modulo three strongly compact cardinals.

Finally, in [19], we can get the result without assuming large cardinals; this is the construction we use in this paper.

3. Cardinal Characteristics in Extensions Without New $<\kappa$-sequences

Let us consider $<\kappa$-distributive forcing extensions for some regular κ (in particular these extensions are NNR, i.e. do not add new reals). For such extensions, we can also preserve strong witnesses in some cases.

Lemma 3.1. *Assume that Q is θ-cc and $<\kappa$-distributive for κ regular uncountable, and let λ be a regular cardinal and R a Borel relation.*

(1) *If* $\mathrm{LCU}_R(\lambda)$, *then* $Q \Vdash \mathrm{LCU}_R(\mathrm{cof}(\lambda))$.
So if additionally $\lambda \leq \kappa$ or $\theta \leq \lambda$, then $Q \Vdash \mathrm{LCU}_R(\lambda)$.
(2) *If* $\mathrm{COB}_R(\lambda, \mu)$ *and either $\lambda \leq \kappa$ or $\theta \leq \lambda$, then* $Q \Vdash \mathrm{COB}_R(\lambda, |\mu|)$.
So for any λ, $\mathrm{COB}_R(\lambda, \mu)$ *implies* $Q \Vdash \mathrm{COB}_R(\min(|\lambda|, \kappa), |\mu|)$.

Proof. For (1) it is enough to assume that Q does not add reals: Take a strong witness for $\mathrm{LCU}_R(\lambda)$. This object still satisfies (2.1) in the Q-extension (as there are no new reals), but the index set will generally not be regular any more; we can just take a cofinal subset of order type $\mathrm{cof}(\lambda)$ which will still satisfy (2.1).

Similarly, a strong witness for $\mathrm{COB}_R(\lambda, \mu)$ still satisfies (2.2) in the Q extension. However, the index set is generally not $<\lambda$-directed any more, unless we either assume $\lambda \leq \kappa$ (as in that case there are no new small subsets of the partial order) or Q is λ-cc (as then every small set in the extension is covered by a small set from the ground model). $\square$

If P forces strong witnesses, then any complete subforcing that includes names for all witnesses also forces strong witnesses.

Lemma 3.2. *Assume that R is a Borel relation, P' is a complete subforcing of P, λ regular and μ is a cardinal, both preserved in the P-extension.*

(a) *If* $P \Vdash \mathrm{LCU}_R(\lambda)$ *witnessed by some $\dot{\bar{f}}$ and $\dot{\bar{f}}$ is actually a P'-name, then* $P' \Vdash \mathrm{LCU}_R(\lambda)$.

(b) *If* $P \Vdash \mathrm{COB}_R(\lambda,\mu)$ *witnessed by some* $(\dot\trianglelefteq, \dot{\bar g})$, *and* $(\dot\trianglelefteq, \dot{\bar g})$ *is actually a* P'*-name, then* $P' \Vdash \mathrm{COB}_R(\lambda,\mu)$.

Proof. Let V_2 be the P-extension and V_1 the intermediate P'-extension. For LCU: (2.1) holds in V_2, $V_1 \subseteq V_2$ and $(f_i)_{i<\lambda} \in V_1$, and R is absolute between V_1 and V_2, so (2.1) holds in V_1. The argument for COB is similar. $\square$

We now define three properties of cardinal characteristics (more general than Blass-uniform) that have implications for their behavior in extensions without new $<\kappa$-sequences. We call these properties, e.g. $\mathfrak{t}$-like to refer to the "typical" representative $\mathfrak{t}$. But note that this is very superficial: There is no deep connection or similarity to $\mathfrak{t}$ for all $\mathfrak{t}$-like characteristics, it is just that $\mathfrak{t}$ is a well-known example for this property, and "$\mathfrak{t}$-like" seems easier to memorize than other names we came up with.

Definition 3.3. Let $\mathfrak{x}$ be a cardinal characteristic.

(1) $\mathfrak{x}$ is $\mathfrak{t}$-*like*, if it has the following form: There is a formula $\psi(x)$ (possibly with, e.g. real parameters) absolute between universe extensions that do not add reals,[j] such that $\mathfrak{x}$ is the smallest cardinality λ of a set A of reals such that $\psi(A)$.

All Blass-uniform characteristics are $\mathfrak{t}$-like; other examples are $\mathfrak{t}$, $\mathfrak{u}$, $\mathfrak{a}$ and $\mathfrak{i}$.

(2) $\mathfrak{x}$ is called $\mathfrak{h}$-*like*, if it satisfies the same, but with A being a family of sets of reals (instead of just a set of reals).

Note that $\mathfrak{t}$-like implies $\mathfrak{h}$-like, as we can include "the family of sets of reals is a family of singletons" in ψ. Examples are $\mathfrak{h}$ and $\mathfrak{g}$.

(3) $\mathfrak{x}$ is called $\mathfrak{m}$-*like*, if it has the following form: There is a formula φ (possibly with, e.g. real parameters) such that $\mathfrak{x}$ is the smallest cardinality λ such that $H(\leq\lambda) \vDash \varphi$.

Any infinite $\mathfrak{t}$-like characteristic is $\mathfrak{m}$-like: If ψ witnesses $\mathfrak{t}$-like, then we can use $\varphi = (\exists A)\,[\psi(A)\&(\forall a \in A)\ a$ is a real$]$ to get $\mathfrak{m}$-like (since $H(\leq\lambda)$ contains all reals). Examples are[k] $\mathfrak{m}$, $\mathfrak{m}$(Knaster), etc.

(Actually, we do not know anything about $\mathfrak{t}$-like characteristics in general, apart from the fact that they are both $\mathfrak{m}$-like and $\mathfrak{h}$-like).

Lemma 3.4. *Let* $V_1 \subseteq V_2$ *be models* (*possibly classes*) *of set theory* (*or a sufficient fragment*), V_2 *transitive and* V_1 *is either transitive or an elementary submodel of* $H^{V_2}(\chi)$ *for some large enough regular* χ, *such that* $V_1 \cap \omega^\omega = V_2 \cap \omega^\omega$.

(a) *If* $\mathfrak{x}$ *is* $\mathfrak{h}$*-like, then* $V_1 \vDash \mathfrak{x} = \lambda$ *implies* $V_2 \vDash \mathfrak{x} \leq |\lambda|$.

[j]Concretely, if $M_1 \subseteq M_2$ are transitive (possibly class) models of a fixed, large fragment of ZFC, with the same reals, then ψ is absolute between M_1 and M_2.
[k]$\mathfrak{m}$ can be characterized as the smallest λ such that there is in $H(\leq\lambda)$ a ccc forcing Q and a family $\bar D$ of dense subsets of Q such that "there is no filter $F \subseteq Q$ meeting all D_i" holds.

In addition, whenever κ is uncountable regular in V_1 and $V_1^{<\kappa} \cap V_2 \subseteq V_1$:

(b) *If $\mathfrak{x}$ is $\mathfrak{m}$-like, then $V_1 \vDash \mathfrak{x} \geq \kappa$ iff $V_2 \vDash \mathfrak{x} \geq \kappa$.*
(c) *If $\mathfrak{x}$ is $\mathfrak{m}$-like and $\lambda < \kappa$, then $V_1 \vDash \mathfrak{x} = \lambda$ iff $V_2 \vDash \mathfrak{x} = \lambda$.*
(d) *If $\mathfrak{x}$ is $\mathfrak{t}$-like and $\lambda = \kappa$, then $V_1 \vDash \mathfrak{x} = \lambda$ implies $V_2 \vDash \mathfrak{x} = \lambda$.*

Proof. First note that (d) follows by (a) and (b) because any $\mathfrak{t}$-like characteristic is both $\mathfrak{m}$-like and $\mathfrak{h}$-like.

Assume V_1 is transitive. For (a), if ψ witnesses that $\mathfrak{x}$ is $\mathfrak{h}$-like, $A \in V_1$ and V_1 satisfies $\psi(A)$, then the same holds in V_2. For (b) and (c), note that $H^{V_1}(\leq \mu) = H^{V_2}(\leq \mu)$ for all $\mu < \kappa$ (easily shown by $\in$-induction).

The case $V_1 = N \preceq H^{V_2}(\chi)$ is similar. Note that $H^{V_2}(\chi)$ is a transitive subset of V_2, so (a) follows by the previous case. For (b) and (c), work inside V_2. Note that $\kappa \subseteq N$ (by induction). Whenever $\mu < \kappa$, μ is regular iff $N \models$"μ regular", and $H(\leq\mu) \subseteq N$. So $N \models$"$H(\leq\mu) \models \phi$" iff $H(\leq\mu) \models \phi$.

Alternatively, the case $V_1 \preceq H^{V_2}(\chi)$ is a consequence of the first case. Work in V_2. Let $\pi : V_1 \to \bar{V}_1$ be the transitive collapse of V_1. Note that $\pi(x) = x$ for any $x \in \omega^\omega \cap V_1$, so $\omega^\omega \cap \bar{V}_1 = \omega^\omega \cap V_1 = \omega^\omega$. To see (a), $V_1 \vDash \mathfrak{x} = \lambda$ implies $\bar{V}_1 \vDash \mathfrak{x} = \pi(\lambda)$, so $\mathfrak{x} \leq |\pi(\lambda)| \leq |\lambda|$ by the transitive case.

Now assume $V_1^{<\kappa} \subseteq V_1$ (still inside V_2), so we also have $\bar{V}_1^{<\kappa} \subseteq \bar{V}_1$. To see (b), $V_1 \models \mathfrak{x} \geq \kappa$ iff $\bar{V}_1 \models \mathfrak{x} \geq \pi(\kappa) = \kappa$, iff $V_2 \models \mathfrak{x} \geq \kappa$ by the transitive case. Property (c) follows similarly by using $\pi(\lambda) = \lambda$ (when $\lambda < \kappa$). $\square$

We apply this to three situations: Boolean ultrapowers (which we will not apply in this paper), extensions by distributive forcings and complete subforcings.

Corollary 3.5. *Assume that κ is uncountable regular, $P \Vdash \mathfrak{x} = \lambda$, and*

(i) *<u>either</u> Q is a P-name for a $<\kappa$-distributive forcing, and we set $P^+ := P * Q$ and $j(\lambda) := \lambda$;*
(ii) *<u>or</u> P is ν-cc for some $\nu < \kappa$, $j : V \to M$ is a complete embedding into a transitive $<\kappa$-closed model M, $\mathrm{cr}(j) \geq \kappa$, and we set $P^+ := j(P)$,*
(iii) *<u>or</u> P is κ-cc, $M \preceq H(\chi)$ is $<\kappa$-closed, and we set $P^+ := P \cap M$ and $j(\lambda) := |\lambda \cap M|$ (so P^+ is a complete subposet of P; and if $\lambda \leq \kappa$ then $j(\lambda) = \lambda$).*

Then we get

(a) *If $\mathfrak{x}$ is $\mathfrak{m}$-like and $\lambda \geq \kappa$, then $P^+ \Vdash \mathfrak{x} \geq \kappa$.*
(b) *If $\mathfrak{x}$ is $\mathfrak{m}$-like and $\lambda < \kappa$, then $P^+ \Vdash \mathfrak{x} = \lambda$.*
(c) *If $\mathfrak{x}$ is $\mathfrak{h}$-like then $P^+ \Vdash \mathfrak{x} \leq |j(\lambda)|$. Concretely,*
   for (i): $P^+ \Vdash \mathfrak{x} \leq |\lambda|$;
   for (ii): $P^+ \Vdash \mathfrak{x} \leq |j(\lambda)|$;
   for (iii): $P^+ \Vdash \mathfrak{x} \leq |\lambda \cap M|$.
(d) *So if $\mathfrak{x}$ is $\mathfrak{t}$-like and $\lambda = \kappa$, then for* (i) *and* (iii) *we get $P^+ \vDash \mathfrak{x} = \kappa$.*

Proof. Case (i). Follows directly from Lemma 3.4.

Case (ii). Since M is $<\kappa$-closed and P is ν-cc, P (or rather: the isomorphic image $j''P$) is a complete subforcing of $j(P)$. Let G be a $j(P)$-generic filter over V. As $j(P)$ is in M (and M is transitive), G is generic over M as well. Then $V_1 := M[G]$ is $<\kappa$ closed in $V_2 := V[G]$.

First note that V_1 and V_2 have the same $<\kappa$-sequences of ordinals. Let $\dot{\bar{x}} = (\dot{x}_i)_{i\in\mu}$ be a sequence of $j(P)$-names for members of M with $\mu < \kappa$. Each $\dot{x}_i$ is determined by an antichain, which has size $<\nu$ and therefore is in M, so each $\dot{x}_i$ is in M. Hence, $\dot{\bar{x}}$ is in M.

By elementaricity, $P \Vdash \mathfrak{x} = \lambda$ implies $M \models$"$j(P) \Vdash \mathfrak{x} = j(\lambda)$". So $V_1 \models \mathfrak{x} = j(\lambda)$, and we can apply Lemma 3.4: In the case that $\mathfrak{x}$ is $\mathfrak{m}$-like, if $\lambda \geq \kappa$, then $j(\lambda) \geq j(\kappa) \geq \kappa$, so $V_2 \models \mathfrak{x} \geq \kappa$; If $\lambda < \kappa$, then $j(\lambda) = \lambda$, so $V_2 \models \mathfrak{x} = \lambda$; if $\mathfrak{x}$ is $\mathfrak{h}$-like, then $V_2 \models \mathfrak{x} \leq |j(\lambda)|$.

Case (iii). Let $\pi^0 : M \to \bar{M}$ be the transitive collapse. Set $\bar{P} := \pi^0(P) \in \bar{M}$. Note that $\pi^0(\kappa) = \kappa$ and that $\bar{M}$ is $<\kappa$-closed. Also, any condition in P is M-generic since, for any antichain A in P, $A \in M$ iff $A \subseteq M$ (by $<\kappa$-closedness).

Let G^+ be P^+-generic over V. We can extend G^+ to a P-generic G over V (as P^+ is a complete subforcing of P), and we get $G^+ = G \cap P^+ = G \cap M$. Now, work in $V[G]$. Note that $M[G]$ is an elementary submodel of $H^{V[G]}(\chi)$ (and obviously not transitive), and that the transitive collapse $\pi : M[G] \to V_1$ extends π^0 (as there are no new elements of V in $M[G]$). We claim that $V_1 = \bar{M}[\bar{G}^+]$, where $\bar{G}^+ := \pi^{0\prime\prime}G^+$ (which is $\bar{P}$-generic over $\bar{M}$, also $\bar{G}^+ = \pi(G)$), and that $\bar{\tau}[\bar{G}^+] = \pi(\tau[G])$ for any P-name $\tau \in M$, where $\bar{\tau} := \pi^0(\tau)$.[1] So in particular, V_1 is a subset of $V_2 := V[G^+]$ (the P^+-generic extension of V) because π^0 and M (and therefore $\bar{M}$) are elements of V, so G^+ (and therefore $\bar{G}^+$) are elements of $V[G^+]$. In fact, $\bar{G}^+$ is $\bar{P}$-generic over V because $\bar{M}$ is $<\kappa$-closed and $\bar{P}$ is κ-cc, moreover, $V_2 = V[\bar{G}^+]$ (this is reflected by the fact that, in V, $\pi^0 {\restriction} P^+$ is an isomorphism between P^+ and $\bar{P}$).

We claim:

$$V_2 \text{ is an NNR extension of } V_1, \text{ moreover } V_1 \text{ is } {<}\kappa\text{-closed in } V_2. \qquad (*)$$

To show this, work in V. We argue with $\bar{P}$. Let τ be a $\bar{P}$-name of an element of $V_1 = \bar{M}[\bar{G}^+]$. So we can find a maximal antichain A in $\bar{P}$ and, for each $a \in A$, a $\bar{P}$-name σ_a in $\bar{M}$ such that $a \Vdash_{\bar{P}} \tau = \sigma_a$. Since $|A| < \kappa$ and $\bar{P} \subseteq \bar{M}$ and $\bar{M}$ is $<\kappa$-closed, A, as well as the function $a \mapsto \sigma_a$, are in $\bar{M}$. Mixing the names σ_a along A to a name $\sigma \in \bar{M}$, we get $\bar{M} \vDash a \Vdash_{\bar{P}} \sigma_a = \sigma$ for all $a \in A$, which implies $V \vDash a \Vdash_{\bar{P}} \sigma_a = \sigma$ because the forcing relation of atomic formulas is absolute. So $\bar{P} \Vdash \tau = \sigma$.

Now, fix a $\bar{P}$ name $\vec{\tau} = (\tau_\alpha)_{\alpha<\mu}$ of a sequence of elements of V_1, with $\mu < \kappa$. Again we use closure of $\bar{M}$ and get a sequence $(\sigma_\alpha)_{\alpha<\mu}$ in $\bar{M}$ such that $\bar{P}$ forces that $\tau_\alpha = \sigma_\alpha[\bar{G}^+]$, and so the evaluation of the sequence $\vec{\tau}$ is in $\bar{M}[\bar{G}^+] = V_1$. This proves $(*)$.

[1]This can be proved by induction on the rank of τ, and uses that $M[G] \preceq H^{V[G]}(\chi)$.

Now assume that $\mathfrak{x}$ is either $\mathfrak{h}$-like or $\mathfrak{m}$-like, and $P \Vdash \mathfrak{x} = \lambda$. By elementaricity, this holds in M, so $\bar{M} \models \bar{P} \Vdash \mathfrak{x} = \pi^0(\lambda)$. Now, let $\bar{G}^+$ be $\bar{P}$-generic over V, $V_1 := \bar{M}[\bar{G}^+]$ and $V_2 := V[\bar{G}^+]$, so $V_1 \models \mathfrak{x} = \pi^0(\lambda)$. If $\mathfrak{x}$ is $\mathfrak{h}$-like then, by Lemma 3.4(a), $V_2 \models \mathfrak{x} \leq |\pi^0(\lambda)| = |\lambda \cap M|$; if $\mathfrak{x}$ is $\mathfrak{m}$-like and $\lambda < \kappa$, then $V_1 \models \mathfrak{x} = \lambda$ and so the same is satisfied in V_2 by Lemma 3.4(c); otherwise, if $\lambda \geq \kappa$ then $V_1 \models \mathfrak{x} = \pi^0(\lambda) \geq \pi^0(\kappa) = \kappa$, so $V_2 \models \mathfrak{x} \geq \kappa$ by Lemma 3.4(b).

In any of the cases above, (d) is a direct consequence of (a) and (c). □

4. Dealing with $\mathfrak{m}$

We show how to deal with $\mathfrak{m}$. It is easy to check that the Cichoń's Maximum construction from [20] forces $\mathfrak{m} = \aleph_1$, and can easily be modified to force $\mathfrak{m} = \mathrm{add}(\mathcal{N})$ (by forcing with all small ccc forcings during the iteration). With a bit more work it is also possible to get $\aleph_1 < \mathfrak{m} < \mathrm{add}(\mathcal{N})$.

Let us start by recalling the definitions of some well-known classes of ccc forcings.

Definition 4.1. Let λ be an infinite cardinal, $k \geq 2$ and let Q be a poset.

(1) Q is (λ, k)-*Knaster* if, for every $A \in [Q]^\lambda$, there is a $B \in [A]^\lambda$ which is k-linked (i.e. every $c \in [B]^k$ has a lower bound in Q). We write k-*Knaster* for $(\aleph_1, k)$-Knaster; *Knaster* means 2-Knaster; $(\lambda, 1)$-*Knaster* denotes λ-cc and 1-*Knaster* denotes ccc.[m]
(2) Q *has precaliber* λ if, for every $A \in [Q]^\lambda$, there is a $B \in [A]^\lambda$ which is centered, i.e. every finite subset of B has a lower bound in Q. We sometimes shorten "precaliber $\aleph_1$" to "precaliber".
(3) Q is (σ, k)-*linked* if there is a function $\pi : Q \to \omega$ such that $\pi^{-1}(\{n\})$ is k-linked for each n.
(4) Q is σ-*centered* if there is a function $\pi : Q \to \omega$ such that each $\pi^{-1}(\{n\})$ is centered.

The implications between these notions (for $\lambda = \aleph_1$) are listed in Fig. 3. To each class C of forcing notions, we can define the Martin's Axiom number $\mathfrak{m}(C)$ in the usual way (recall Definition 1.1). An implication $C_1 \leftarrow C_2$ in the diagram corresponds to a ZFC inequality $\mathfrak{m}(C_1) \leq \mathfrak{m}(C_2)$. Recall that $\mathfrak{m}(\sigma\text{-centered}) = \mathfrak{p} = \mathfrak{t}$. Also recall that, in the old constructions, all iterands were (σ, k)-linked for all k.

Lemma 4.2. (1) *If there is a Suslin tree, then* $\mathfrak{m} = \aleph_1$.
(2) *After adding a Cohen real* c *over* V, *in* $V[c]$ *there is a Suslin tree.*

$$\begin{tikzcd}
& \sigma\text{-2-linked} \arrow[d] & \sigma\text{-3-linked} \arrow[l] \arrow[d] & \cdots \arrow[l] \arrow[d] & \sigma\text{-centered} \arrow[l] \arrow[d] \\
\text{ccc} & \text{2-Knaster} \arrow[l] & \text{3-Knaster} \arrow[l] & \cdots \arrow[l] & \text{precaliber} \arrow[l]
\end{tikzcd}$$

Fig. 3. Some classes of ccc forcings.

[m]This is just an abuse of notation that turns out to be convenient for stating our results.

(3) *Any Knaster poset preserves Suslin trees.*
(4) *The result of any finite support iteration of* (λ, k)*-Knaster posets* (λ *uncountable regular and* $k \geq 1$) *is again* (λ, k)*-Knaster.*
(5) *In particular, when* $k \geq 1$, *if* P *is a f.s. iteration of forcings such that all iterands are either* (σ, k)*-linked or smaller than* λ, *then* P *is* (λ, k)*-Knaster.*
(6) *Let* C *be any of the forcing classes of Fig.* 3, *and assume* $\mathfrak{m}(C) = \lambda > \aleph_1$ (*or just assume that* C *is a class of ccc forcings closed under* $Q \mapsto Q^{<\omega}$, *the finite support product of countably many copies of* Q, *and under* $(Q, p) \mapsto \{q : q \leq p\}$ *for* $p \in Q$).

If $Q \in C$, *then every subset* A *of* Q *of size* $<\lambda$ *is "*σ*-centered in* Q*"* (*i.e. there is a function* $\pi : A \to \omega$ *such that every finite* π*-homogeneous subset of* A *has a common lower bound in* Q).

So in particular, for all $\mu < \lambda$ *of uncountable cofinality,* Q *has precaliber* μ *and is* (μ, ℓ)*-Knaster for all* $\ell \geq 2$.
(7) $\mathfrak{m} > \aleph_1$ *implies* $\mathfrak{m} = \mathfrak{m}(\text{precaliber})$.
$\mathfrak{m}(k\text{-Knaster}) > \aleph_1$ *implies* $\mathfrak{m}(k\text{-Knaster}) = \mathfrak{m}(\text{precaliber})$.

Proof. (1) Clear.

(2) See [37, 40] or Velleman [41].

(3) Recall that the product of a Knaster poset with a ccc poset is still ccc. Hence, if P is Knaster and T is a Suslin tree, then $P \times T = P * \check{T}$ is ccc, i.e. T remains Suslin in the P-extension.

(4) Well-known, see, e.g. Kunen [30, Lemma V.4.10] for $(\aleph_1, 2)$-Knaster. The proof for the general case is the same, see, e.g. [31, Sec. 5].

(5) Clear, as (σ, k)-linked implies (μ, k)-Knaster (for all uncountable regular μ), and since every forcing of size $<\mu$ is (μ, k)-Knaster (for any k).

(6) First note that it is well known[n] that $\mathrm{MA}_{\aleph_1}(\text{ccc})$ implies that every ccc forcing is Knaster, and hence that the class C of ccc forcings is closed under $Q \mapsto Q^{<\omega}$ (for the other classes C in Fig. 3, the closure is immediate).

So let C be a closed class, $\mathfrak{m}(C) = \lambda > \aleph_1$, $Q \in C$ and $A \in [Q]^{<\lambda}$. Given a filter G in $Q^{<\omega}$ and $q \in Q$, set $c(q) = n$ iff n is minimal such that there is a $\bar{p} \in G$ with $p(n) = q$. Note that for all q, the set

$$D_q = \{p \in Q^{<\omega} : (\exists\, n \in \omega)\, q = p(n)\}$$

is dense, and that $c(q)$ is defined whenever G intersects D_q. Pick a filter G meeting all D_q for $q \in A$. This defines $c : A \to \omega$ such that $c(a_0) = c(a_1) = \cdots = c(a_{\ell-1}) = n$ implies that all a_i appear in $G(n)$ and thus they are compatible in Q. Hence, A is the union of countably many centered (in Q) subsets of Q.

(7) Follows as a corollary. □

[n]See, e.g. Jech [24, Theorem 16.21] (and the historical remarks, where the result is attributed to (independently) Kunen *et al.*) or [3, Lemma 1.4.14] or Galvin [17, p. 34].

This shows that it is not possible to simultaneously separate more than two Knaster numbers. More specifically: ZFC proves that there is a (unique) $1 \leq k^* \leq \omega$ and, if $k^* < \omega$, a (unique) $\lambda > \aleph_1$, such that for all $1 \leq \ell < \omega$

$$\mathfrak{m}(\ell\text{-Knaster}) = \begin{cases} \aleph_1 & \text{if } \ell < k^*, \\ \lambda & \text{otherwise} \end{cases} \tag{4.1}$$

(recall that $\mathfrak{m}(1\text{-Knaster}) = \mathfrak{m}(\text{ccc})$ by our definition). So the case $k^* = \omega$ means that all Knaster numbers are $\aleph_1$.

In this section, we will show how these constellations can be realized together with the previous values for the Cichoń-characteristics.

In the case $k^* < \omega$, we know that $\mathfrak{m}(\text{precaliber}) = \lambda$ as well. We briefly comment that $\mathfrak{m}(\text{precaliber}) = \aleph_1$ (in connection with the Cichoń-values) is possible too. In the next section, we will deal with the remaining case: $k^* = \omega$, i.e., all Knaster numbers are $\aleph_1$, while $\mathfrak{m}(\text{precaliber}) > \aleph_1$.

The central observation is the following, see [39, 40; 2, Sec. 3].

Lemma 4.3. *Let $k \in \omega$, $k \geq 2$ and λ be uncountable regular. Let C be the finite support iteration of λ many copies of Cohen forcing. Assume that C forces that P is $(\lambda, k+1)$-Knaster. Then $C * P$ forces $\mathfrak{m}(k\text{-Knaster}) \leq \lambda$.*

The same holds for $k = 1$ and $\lambda = \aleph_1$.

For $k = 1$ this trivially follows from Lemma 4.2: The first Cohen forcing adds a Suslin tree, which is preserved by the rest of the Cohen posets composed with P. So we get $\mathfrak{m} = \aleph_1$. The proof for $k > 1$ is done in the following two lemmas.

Remark 4.4. Adding the Cohen reals first is just for notational convenience. The same holds, e.g. in a f.s. iteration where we add Cohen reals on a subset of the index set of order type λ; and we assume that the (limit of the) whole iteration is $(\lambda, k+1)$-Knaster.

Lemma 4.5. *Under the assumption of Lemma* 4.3, *for $k \geq 1$: We interpret each Cohen real η_α $(\alpha \in \lambda)$ as an element of $(k+1)^\omega$. $C * P$ forces: For all $X \in [\lambda]^\lambda$,*

$$(\exists\, \nu \in (k+1)^{<\omega})\ (\exists\, o\alpha_0, \ldots, \alpha_k \in X)\, (\forall 0 \leq i \leq k) \nu^\frown i \lhd \eta_{\alpha_i}. \tag{**}$$

Proof. Let $p^* \in C * P$ force that $X \in [\lambda]^\lambda$. By our assumption, first note that $p^* {\restriction} \lambda$ forces that there is some $X' \in [\lambda]^\lambda$ and a $k+1$-linked set $\{r_\alpha : \alpha \in X'\}$ of conditions in P below $p^*(\lambda)$ such that $r_\alpha \Vdash_P \alpha \in X$ for any $\alpha \in X'$.

Since X' is a C-name, there is some $Y \in [\lambda]^\lambda$ and, for each $\alpha \in Y$, some $p_\alpha \leq p^* {\restriction} \lambda$ in C forcing $\alpha \in X'$. We can assume that $\alpha \in \text{dom}(p_\alpha)$ and, by thinning out Y, that $\text{dom}(p_\alpha)$ forms a Δ-system with heart a below each $\alpha \in Y$, $\langle p_\alpha {\restriction} a : \alpha \in Y \rangle$ is constant, and that $p_\alpha(\alpha)$ is always the same Cohen condition $\nu \in (k+1)^{<\omega}$.

For each $\alpha \in Y$ let $q_\alpha \in C * P$ such that $q_\alpha {\restriction} \lambda = p_\alpha$ and $q_\alpha(\lambda) = r_\alpha$. It is clear that $\langle q_\alpha : \alpha \in Y \rangle$ is $k+1$-linked and that $q_\alpha \Vdash \alpha \in X$. Pick $\alpha_0, \ldots, \alpha_k \in Y$ and $q \leq q_{\alpha_0}, \ldots, q_{\alpha_k}$. We can assume that $q {\restriction} \lambda$ is just the union of the $q_{\alpha_i} {\restriction} \lambda$. In particular, we can extend $q(\alpha_i) = \nu$ to $\nu^\frown i$, satisfying $(**)$ after all. This proves the claim. $\square$

Lemma 4.6. *Under the assumption of Lemma* 4.3, *for* $k \geq 2$: *In* V^C *define* $R_{K,k}$ *to be the set of finite partial functions* $p : u \to \omega$, $u \subseteq \lambda$ *finite, such that* $(**)$ *fails for all* p*-homogeneous* $X \subseteq u$.[o] *Then* P *forces the following*:

(a) *There is no filter on* $R_{K,k}$ *meeting all dense* D_α $(\alpha \in \lambda)$, *where we set* $D_\alpha = \{p : \alpha \in \mathrm{dom}(u)\}$.
(b) $R_{K,k}$ *is* k*-Knaster.*

Note that this proves Lemma 4.3, as $R_{K,k}$ is a witness.

Proof. Clearly each D_α is dense (as we can just use a hitherto unused color). If G is a filter meeting all D_α, then G defines a total function $p^* : \lambda \to \omega$, and there is some $n \in \omega$ such that $X := {p^*}^{-1}(\{n\})$ has size λ. So $(**)$ holds for X, witnessed by some $\alpha_0, \ldots, \alpha_k$. Now, pick some $q \in G$ such that all α_i are in the domain of q. Then q contradicts the definition of $R_{K,k}$.

$R_{K,k}$ is k-Knaster: Given $(r_\alpha : u_\alpha \to \omega)_{\alpha \in \omega_1}$, we thin out so that u_α forms a Δ-system of sets of the same size and such that each r_α has the same "type", independent of α, where the type contains the following information: The color assigned to the n-the element of u_α; the (minimal, say) h such that all $\eta_\beta \restriction h$ are distinct for $\beta \in u_\alpha$, and $\eta_\beta \restriction h + 1$.

We claim that the union of k many such r_α is still in $R_{K,k}$: Assume towards a contradiction that there is a $\bigcup_{i<k} r_i$-homogeneous set $\alpha_0, \ldots, \alpha_k$ in $\bigcup_{i<k} u_i$ such that $(**)$ holds for $\nu \in (k+1)^H$ for some $H \in \omega$. Assume $H \geq h$. Note that $\eta_\beta \restriction h$ are already distinct for the different β in the same u_i, so all $k+1$ many α_j have to be the n^*th element of different u_i (n^* fixed), which is impossible as there are only k many u_i. So assume $H < h$. But then $\eta_\beta \restriction H + 1$ and the color of β both are determined by the position of β within u_i; so without loss of generality all the α_j are in the same u_i, which is impossible as $r_i : u_i \to \omega$ was a valid condition.

To summarize: P forces that there is a k-Knaster poset $R_{K,k}$ and λ many dense sets not met by any filter. Therefore, P forces that $\mathfrak{m}(k\text{-Knaster}) \leq \lambda$. $\square$

Let P^{pre} be the initial forcing of Lemma 2.6; recall that it forces $\mathrm{add}(\mathcal{N}) = \nu_1$ and $\mathfrak{b} = \nu_3$.

Lemma 4.7. *For each of the following items* (1)–(3), *and* $\aleph_1 \leq \lambda \leq \nu_1$ *regular,* P^{pre} *can be modified to some forcing* P' *which still strongly witnesses the Cichoń-characteristics, and additionally satisfies*:

(1) *Each iterand in* P' *is* (σ, ℓ)*-linked for all* $\ell \geq 2$; *and* P' *forces*

$$\aleph_1 = \mathfrak{m} = \mathfrak{m}(\text{precaliber}) \leq \mathfrak{p} = \mathfrak{b}.$$

[o]Say that $X \subseteq u$ is *p-homogeneous* if $p \restriction X$ is a constant function.

(2) *Fix $k \geq 1$. Each iterand in P' is $k+1$-Knaster, and additionally either (σ, ℓ)-linked for all ℓ or of size less than λ; and P' forces*

$$\aleph_1 = \mathfrak{m} = \mathfrak{m}(k\text{-Knaster}) < \mathfrak{m}(k+1\text{-Knaster})$$

$$= \mathfrak{m}(\text{precaliber}) = \lambda \leq \mathfrak{p} = \mathfrak{b}.$$

(3) *Each iterand in P' is either (σ, ℓ)-linked for all ℓ, or ccc of size less than λ; and P' forces*

$$\mathfrak{m} = \mathfrak{m}(\text{precaliber}) = \lambda \leq \mathfrak{p} = \mathfrak{b}.$$

Proof. An argument like in [9] works. We first modify P^{pre} as follows:

We construct an iteration P with the same index set δ as P^{pre}; we partition δ into two cofinal sets $\delta = S_{\text{old}} \cup S_{\text{new}}$ of the same size. For $\alpha \in S_{\text{old}}$ we define Q_α as we defined Q^*_α for P^{pre}. For $\alpha \in S_{\text{new}}$, pick (by suitable book-keeping) a small (less than ν_3, the value for $\mathfrak{b}$) σ-centered forcing Q_α.

As $\text{cof}(\delta) \geq \lambda_{\mathfrak{b}}$, we get that P forces $\mathfrak{p} \geq \nu_3$. Also, P still adds strong witnesses for the Cichoń-characteristics, according to Claim 2.7. All new iterands are smaller than ν_3 and σ-centered.

Note that all iterands are still (σ, k)-linked for all k (as the new ones are even σ-centered).

To deal with ℓ-Knaster, recall that the first λ_∞ iterands are Cohen forcings; and we call these Cohen reals η_α ($\alpha \in \lambda_\infty$). Given ℓ, we can (and will) interpret the Cohen real η_α as an element of $(\ell+1)^\omega$.

(1) Recall from [2, Sec. 2] that, after a Cohen real, there is a precaliber ω_1 poset Q^* such that no σ-linked poset adds a filter intersecting certain $\aleph_1$-many dense subsets of Q^*.[p] Therefore, the P we just constructed forces $\mathfrak{m}(\text{precaliber}) = \aleph_1$.

(2) Just as with the modification from P^{pre} to P, we now further modify P to force (by some bookkeeping) with all small (smaller than λ) $k+1$-Knaster forcings. So the resulting iteration obviously forces $\mathfrak{m}(k+1\text{-Knaster}) \geq \lambda$.

Note that now all iterands are either smaller than $\lambda \leq \nu_1$ or σ-linked (so we can again use Claim 2.7); and additionally all iterands are $k+1$-Knaster. So P is both $(\aleph_1, k+1)$-Knaster and (λ, ℓ)-Knaster for any ℓ. Again by Lemma 4.3, P forces both $\mathfrak{m}(k\text{-Knaster}) = \aleph_1$ and $\mathfrak{m}(\ell\text{-Knaster}) \leq \lambda$ for any ℓ (which implies $\mathfrak{m}(k+1\text{-Knaster}) = \lambda$).

(3) This is very similar, but this time we use all small ccc forcings (not just the $k+1$-Knaster ones). This obviously results in $\mathfrak{m} \geq \lambda$; and the same argument as above shows that still $\mathfrak{m}(\ell\text{-Knaster}) \leq \lambda$ for all ℓ. □

[p]To be more precise, after one Cohen real there is a sequence $\bar{r} = \langle r_\alpha : \omega \to 2 : \alpha \in \omega_1 \text{ limit}\rangle$ such that, for any ladder system $\bar{c}$ from the ground model, the pair $(\bar{c}, \bar{r})$, as a ladder system coloring, cannot be uniformized in any stationary subset of ω_1. Furthermore, this property is preserved after any σ-linked poset. Also recall from [14] (with Devlin) that $\mathfrak{m}(\text{precaliber}) > \aleph_1$ implies that any ladder system coloring can be uniformized.

This, together with Corollary 3.5 and Lemma 2.10 gives us 11 characteristics. However, we postpone this collorally until a time we can also add $\mathfrak{h} = \mathfrak{g} = \mathfrak{p} = \kappa$ in Lemma 6.4.

5. Dealing with the Precaliber Number

Recall the possible constellations for the Knaster numbers and the definition of k^* and λ given in (4.1) (and recall that if $k^* < \omega$, i.e. then $\mathfrak{m}(\text{precaliber}) = \lambda$ as well).

In this section, we construct models for $k^* = \omega$, i.e. all Knaster numbers being $\aleph_1$, while $\mathfrak{m}(\text{precaliber}) = \lambda$ for some given regular $\aleph_1 < \lambda \leq \text{add}(\mathcal{N})$ (and for the "old" values for the Cichoń-characteristics, as in the previous section).

Definition 5.1. Let $\lambda > \aleph_1$ be regular. A condition $p \in P_{\text{cal}} = P_{\text{cal},\lambda}$ consists of

(i) finite sets $u_p, F_p \subseteq \lambda$,
(ii) a function $c_p : [u_p]^2 \to 2$,
(iii) for each $\alpha \in F_p$, a function $d_{p,\alpha} : \mathcal{P}(u_p \cap \alpha) \to \omega$ satisfying
  ($\star$) if $\alpha \in F_p$ and s_1, s_2 are 1-homogeneous (with respect to c_p)[q] subsets of $u_p \cap \alpha$ with $d_{p,\alpha}(s_1) = d_{p,\alpha}(s_2)$, then $s_1 \cup s_2$ is 1-homogeneous.

The order is defined by $q \leq p$ iff $u_p \subseteq u_q$, $F_p \subseteq F_q$, $c_p \subseteq c_q$ and $d_{p,\alpha} \subseteq d_{q,\alpha}$ for any $\alpha \in F_p$.

Lemma 5.2. *P_{cal} has precaliber ω_1 (and in fact precaliber μ for any regular uncountable μ) and forces the following:*

(1) *The generic functions $c : [\lambda]^2 \to \{0, 1\}$ and $d_\alpha : [\alpha]^{<\aleph_0} \to \omega$ for $\alpha < \lambda$ are totally defined.*
(2) *Whenever $(s_i)_{i\in I}$ is a family of finite, 1-homogeneous (with respect to c) subsets of α, and $d_\alpha(s_i) = d_\alpha(s_j)$ for $i, j \in I$, then $\bigcup_{i\in I} s_i$ is 1-homogeneous.*
(3) *If $A \subseteq [\lambda]^{<\aleph_0}$ is a family of size λ of pairwise disjoint sets, then there are two sets $u \neq v$ in A such that $c(\xi, \eta) = 0$ for any $\xi \in u$ and $\eta \in v$.*
(4) *Whenever $u \in [\lambda]^{<\aleph_0}$, the set $\{\eta < \lambda : \forall\, \xi \in u(c(\xi, \eta) = 1)\}$ is unbounded in λ.*

Proof. *For any $\alpha < \lambda$, the set of conditions $p \in P_{\text{cal}}$ such that $\alpha \in F_p$ is dense.*

Starting with p such that $\alpha \notin F_p$, we set $u_q = u_p$, $F_q = F_p \cup \{\alpha\}$, and we pick new and unique values for all $d_{q,\alpha}(s)$ for $s \subseteq u_q \cap \alpha = u_p \cap \alpha$, as well as new and unique values for all $d_{q,\beta}(s)$ for $s \subseteq u_q \cap \beta$ with $\alpha \in s$. We have to show that $q \in P_{\text{cal}}$, i.e. that it satisfies ($\star$): Whenever s_1, s_2 satisfy the assumptions of ($\star$), then $\alpha \notin s_i$ (for $i = 1, 2$), as we would otherwise have chosen different values. So we can use that ($\star$) holds for p.

(1) and (4) *For any $\xi < \lambda$, the set of $q \in P_{\text{cal}}$ such that $\xi \in u_q$ is dense.*

Starting with p with $\xi \notin u_p$, we set $u_q = u_p \cup \{\xi\}$ and $F_q = F_p$. Again, pick new (and different) values for all $d_{q,\alpha}(s)$ with $\xi \in s$, and we can set $c(x, \xi)$ to whatever

[q]Say that $s \subseteq u_p \cap \alpha$ is 1-*homogeneous with respect to* c_p if $c_p(\xi, \zeta) = 1$ for any $\xi \neq \zeta$ in s.

we want. The same argument as above shows that $q \in P_{\mathrm{cal}}$. In particular we can set all $c(x,\xi) = 1$, which shows that P_{cal} forces (4).

(2) follows from ($\star$) for $I = \{1,2\}$, and this trivially implies the case for arbitrary I (for $x_1, x_2 \in \bigcup_{i\in I} s_i$, pick $i_1, i_2 \in I$ such that $x_1 \in s_{i_1}$ and $x_2 \in s_{i_2}$; then apply ($\star$) to $\{i_1, i_2\}$).

Precaliber. P_{cal} *has precaliber* μ *for any uncountable regular* μ.

Let $A \subseteq P_{\mathrm{cal}}$ have size μ. We can assume that the u_p's and F_p's for $p \in A$ form Δ-systems with roots u and F, respectively, and we can assume that all $p \in A$ have the same "type (over u, F)", which is defined as follows.

Let i be the order-preserving bijection (Mostowski's collapse) of $u_p \cup F_p$ to some $N \in \omega$. This induces sets $\bar{u} \subseteq \bar{u}_p \subseteq N$ and $\bar{F} \subseteq \bar{F}_p \subseteq N$ and partial functions $\bar{c}_p : [\bar{u}_p]^2 \to 2$, $\bar{d}_{p,\bar{\alpha}} : \mathcal{P}(\bar{u}_p \cap \bar{\alpha}) \to \omega$ (for $\bar{\alpha} \in \bar{F}_p$), such that i is an isomorphism between the structures $p = (u_p \cup F_p, u_p, F_p, u, F, c_p, (d_{p,\alpha})_{\alpha\in F_p})$ and $\bar{p} = (N, \bar{u}_p, \bar{F}_p, \bar{u}, \bar{F}, \bar{c}_p, (\bar{d}_{p,\bar{\alpha}})_{\bar{\alpha}\in\bar{F}_p})$; the latter structure is called the *type* of p (over u, F).

Let us note some trivial facts: There are only countably many different types; between any two conditions with same type there is a natural isomorphism; and if p and q have the same type (over $u = u_p \cap u_q$ and $F = F_p \cap F_q$), then c_p and c_q agree on the common domain, and the same holds for d_p and d_q.[r]

To summarize: Given $A \subseteq P_{\mathrm{cal}}$ of size μ, we can find a μ-sized subset B forming a Δ-system such that all elements have the same type (over the root). We claim that then any finite subset of B has a common lower bound q (which implies that P_{cal} has precaliber μ, as required). This is done by amalgamation, as follows.

Amalgamation. Fix $p_0, \ldots, p_{n-1}$ of the same type over (u, F), such that $u_i \cap u_j = u$ and $F_i \cap F_j = F$ for all i, j in n (where we set $u_i := u_{p_i}$, etc.). We define an "amalgam" q of these conditions as follows: $u_q := \bigcup_{i\in n} u_i$, $F_q := \bigcup_{i\in n} F_i$, d_q extends all d_i and has a unique new value for each new element in its domain, c_q extends all c_i; and yet undefined $c_q(x,y)$ are set to 0 if $x, y > \max(F)$ (and 1 otherwise).

To see that $q \in P_{\mathrm{cal}}$, assume that $\alpha \in F_q$ and s_1, s_2 are as in ($\star$) of Definition 5.1. This implies that $d_{q,\alpha}(s_k)$ for both $k = 1, 2$ were already defined[s] by one of the p_i (for $i \in n$), otherwise we would have picked a new value.

If they are both defined by the same p_i, we can use ($\star$) for p_i. So assume otherwise, and for notational simplicity assume that s_i is defined by p_i; and let $x_i \in s_i$. We have to show $c_q(x_1, x_2) = 1$. Note that $\alpha \in F_1 \cap F_2 = F$. If x_1 or x_2 are not in u, then we have set $c_q(x_1, x_2)$ to 1 (as $x_i < \alpha \in F$), so we are done. So assume $x_1, x_2 \in u$. The natural isomorphism between p_1 and p_2 maps s_1 onto some $s'_1 \subseteq u_2$, and we get that s'_1 is 1-homogeneous and that $d_{2,\alpha}(s_2) = d_{1,\alpha}(s_1) = d_{2,\alpha}(s'_1)$. So

[r]i.e. for $\alpha < \beta$ in u, $c_p(\alpha,\beta) = c_q(\alpha,\beta)$, and for $\alpha \in F$ and $s \subseteq u$, $d_{p,\alpha}(s) = d_{q,\alpha}(s)$.

[s]By which we mean $\alpha \in F_i$ and $s_k \subseteq u_i$ for both $k = 1, 2$.

we use that p_2 satisfies $(\star)$ to get that $c_2(a,b) = 1$ for all $a \in s_2$ and $b \in s'_1$. As the isomorphism does not move x_1, we can use $a = x_2$ and $b = x_1$.

(3) Let $p \in P_{\mathrm{cal}}$ and assume that p forces that $\dot{A} \subseteq [\lambda]^{<\aleph_0}$ is a family of size λ of pairwise disjoint sets. We can find, in the ground model, a family $A' \subseteq [\lambda]^{<\aleph_0}$ of size λ and conditions $p_v \leq p$ for $v \in A'$ such that $v \subseteq u_{p_v}$, and p_v forces $v \in \dot{A}$. We again thin out to a Δ-system as above; this time we can additionally assume that the heart of the F_v is below the non-heart parts of all u_v, i.e. that $\max(F)$ is below $u_v \backslash u$ for all v.

Pick any two p_v, p_w in this Δ-system, and let q be the amalgam defined above. Then q witnesses that p_v, p_w are compatible, which implies $v \cap w = 0$, i.e. v, w are outside the heart; which by construction of q implies that c_q is constantly zero on $v \times w$ (as their elements are above $\max(F)$). □

The poset $P_{\mathrm{cal},\lambda}$ adds generic functions c and d_α. We now use them to define a precaliber ω_1 poset Q_{cal} witnessing $\mathfrak{m}(\text{precaliber}) \leq \lambda$.

Lemma 5.3. *In $V^{P_{\mathrm{cal}}}$, define the poset $Q_{\mathrm{cal}} := \{u \in [\lambda]^{<\aleph_0} : u \text{ is } 1\text{-homogeneous}\}$, ordered by $\supseteq$ (by 1-homogeneous, we mean 1-homogeneous with respect to c). Then the following is satisfied (in $V^{P_{\mathrm{cal}}}$):*

(1) *Q_{cal} is an increasing union of length λ of centered sets (so in particular it has precaliber $\aleph_1$).*
(2) *For $\alpha < \lambda$, the set $D_\alpha := \{u \in Q_{\mathrm{cal}} : u \nsubseteq \alpha\}$ is open dense. So Q_{cal} adds a cofinal generic 1-homogeneous subset of λ.*
(3) *There is no 1-homogeneous set of size λ (in $V^{P_{\mathrm{cal}}}$). In other words, there is no filter meeting all D_α.*

Proof. For (1) set $Q^\alpha_{\mathrm{cal}} = Q_{\mathrm{cal}} \cap [\alpha]^{<\aleph_0}$. Then $d_\alpha : Q^\alpha_{\mathrm{cal}} \to \omega$ is a centering function, according to Lemma 5.2(2). Precaliber $\aleph_1$ is a consequence of $\lambda_{\mathrm{cal}} > \aleph_1$.

Property (2) is a direct consequence of Lemma 5.2(4), and (3) follows from Lemma 5.2(3). □

This shows that $P_{\mathrm{cal},\lambda} \Vdash \mathfrak{m}(\text{precaliber}) \leq \lambda$. We now show that this is preserved in further Knaster extensions.

Lemma 5.4. *In $V^{P_{\mathrm{cal}}}$, assume that P' is a ccc λ-Knaster poset. Then, in $V^{P_{\mathrm{cal}} * P'}$, $\mathfrak{m}(\text{precaliber}) \leq \lambda$.*

Proof. We claim that in $V^{P_{\mathrm{cal}} * P'}$, Q_{cal} still has precaliber $\aleph_1$, and there is no filter meeting each open dense subset $D_\alpha \subseteq Q_{\mathrm{cal}}$ for $\alpha < \lambda$.

Precaliber follows from Lemma 5.3(1). So we have to show that λ has no 1-homogeneous set (with respect to c) of size λ in $V^{P_{\mathrm{cal}} * P'}$.

Work in $V^{P_{\mathrm{cal}}}$ and assume that $\dot{A}$ is a P'-name and $p \in P'$ forces that $\dot{A}$ is in $[\lambda]^\lambda$. By recursion, find $A' \in [\lambda]^\lambda$ and $p_\zeta \leq p$ for each $\zeta \in A'$ such that $p_\zeta \Vdash \zeta \in \dot{A}$. Since P' is λ-Knaster, we may assume that $\{p_\zeta : \zeta \in A'\}$ is linked. By

Lemma 5.2(3), there are $\zeta \neq \zeta'$ in A' such that $c(\zeta,\zeta') = 0$. So there is a condition q stronger that both p_ζ and $p_{\zeta'}$ forcing that $\zeta,\zeta' \in \dot{A}$ and $c(\zeta,\zeta') = 0$, i.e. that $\dot{A}$ is not 1-homogeneous. $\square$

We can now add another case to Lemma 4.7.

Lemma 5.5. *For $\aleph_1 \leq \lambda \leq \nu_1$ regular, P^{pre} can be modified to some forcing P' which still strongly witnesses the Cichoń-characteristics, and additionally satisfies: For all $k \in \omega$, $\mathfrak{m}(k\text{-Knaster}) = \aleph_1$; $\mathfrak{m}(\text{precaliber}) = \lambda$; and $\mathfrak{p} = \mathfrak{b}$.*

Proof. The case $\lambda = \aleph_1$ was already dealt with in the previous section, so we assume $\lambda > \aleph_1$.

We modify P^{pre} as follows: We start with the forcing $P_{\text{cal},\lambda}$. From then on, use (by bookkeeping) all precaliber forcings of size $<\lambda$, all σ-centered ones of size $<\nu_3$, the value for $\mathfrak{b}$ (and in between we use all the iterands required for the original construction). So each new iterand either has precaliber $\aleph_1$ and is of size $<\lambda$, or is (σ,k)-linked for any $k \geq 2$. Therefore, the limits are $k+1$-Knaster (for any k). Accordingly, the limit forces that each k-Knaster number is $\aleph_1$.

Also, each iterand is either of size $<\lambda$ or σ-linked; so the limit is λ-Knaster, and by Lemma 5.4 it forces that the precaliber number is $\leq \lambda$; our bookkeeping gives $\geq \lambda$. As before, we get $\mathfrak{p} \geq \nu_3$ by bookkeeping. $\square$

6. Dealing with $\mathfrak{h}$

The following is a very useful tool to deal with $\mathfrak{g}$.

Lemma 6.1 (Blass [5, Theorem 2]). *Let ν be an uncountable regular cardinal and let $(V_\alpha)_{\alpha\leq\nu}$ be an increasing sequence of transitive models of ZFC such that*

(i) $\omega^\omega \cap (V_{\alpha+1} \setminus V_\alpha) \neq \emptyset$,
(ii) $(\omega^\omega \cap V_\alpha)_{\alpha<\nu} \in V_\nu$, *and*
(iii) $\omega^\omega \cap V_\nu = \bigcup_{\alpha<\nu} \omega^\omega \cap V_\alpha$.

Then, in V_ν, $\mathfrak{g} \leq \nu$.

This result gives an alternative proof of the well known.

Corollary 6.2. $\mathfrak{g} \leq \text{cof}(\mathfrak{c})$.[t]

Proof. Put $\nu := \text{cof}(\mathfrak{c})$ and let $(\mu_\alpha)_{\alpha<\nu}$ be a cofinal increasing sequence in $\mathfrak{c}$ formed by limit ordinals. By recursion, we can find an increasing sequence $(V_\alpha)_{\alpha<\nu}$ of transitive models of (a large enough fragment of) ZFC such that (i) of Lemma 6.1 is satisfied, $\mu_\alpha \in V_\alpha$, $|V_\alpha| = |\mu_\alpha|$ and $\bigcup_{\alpha<\nu} \omega^\omega \cap V_\alpha = \omega^\omega$. Set $V_\nu := V$, so Lemma 6.1 applies, i.e. $\mathfrak{g} \leq \nu = \text{cof}(\mathfrak{c})$. $\square$

[t]A more elementary proof can be found in [7, Theorem 8.6, Corollary 8.7].

The following lemma is our main tool to modify the values of $\mathfrak{g}$ and $\mathfrak{c}$ via a complete subposet of some forcing, while preserving $\mathfrak{m}$-like and Blass-uniform values from the original poset. This is a direct consequence of Lemmas 3.2 and 6.1 and Corollary 3.5. As we are only interested in finitely many characteristics, the index sets I_1, I_2, J and K will be finite when we apply the lemma.

Lemma 6.3. *Assume the following*:

(1) $\aleph_1 \le \kappa \le \nu \le \mu$, *where* κ *and* ν *are regular and* $\mu = \mu^{<\kappa} \ge \nu$.
(2) P *is a* κ-*cc poset forcing* $\mathfrak{c} > \mu$.
(3) *For some Borel relations* R^1_i $(i \in I_1)$ *on* ω^ω *and some regular* $\lambda^1_i \le \mu$: P *forces* $\mathrm{LCU}_{R^1_i}(\lambda^1_i)$.
(4) *For some Borel relations* R^2_i $(i \in I_2)$ *on* ω^ω, $\lambda^2_i \le \mu$ *regular and a cardinal* $\vartheta^2_i \le \mu$: P *forces* $\mathrm{COB}_{R^2_i}(\lambda^2_i, \vartheta^2_i)$.
(5) *For some* $\mathfrak{m}$-*like characteristics* $\mathfrak{y}_j$ $(j \in J)$ *and* $\lambda_j < \kappa$: $P \Vdash \mathfrak{y}_j = \lambda_j$.
(6) *For some* $\mathfrak{m}$-*like characteristics* $\mathfrak{y}'_k$ $(k \in K)$: $P \Vdash \mathfrak{y}'_k \ge \kappa$.
(7) $|I_1 \cup I_2 \cup J \cup K| \le \mu$.

Then there is a complete subforcing P' *of* P *of size* μ *forcing*

(a) $\mathfrak{y}_j = \lambda_j$, $\mathfrak{y}'_k \ge \kappa$, $\mathrm{LCU}_{R^1_i}(\lambda^1_i)$ *and* $\mathrm{COB}_{R^2_{i'}}(\lambda^2_{i'}, \vartheta^2_{i'})$ *for all* $i \in I_1$, $i' \in I_2$, $j \in J$ *and* $k \in K$;
(b) $\mathfrak{c} = \mu$ *and* $\mathfrak{g} \le \nu$.

Proof. Construct an increasing sequence of elementary submodels $(M_\alpha : \alpha < \nu)$ of some $(H(\chi), \in)$ for some sufficiently large χ, where each M_α is $<\kappa$-closed with cardinality μ, in a way that $M := M_\nu = \bigcup_{\alpha<\nu} M_\alpha$ satisfies:

(i) $\mu \cup \{\mu\} \subseteq M_0$,
(ii) $I_1 \cup I_2 \cup J \cup K \subseteq M_0$,
(iii) M_0 contains all the definitions of the characteristics we use,
(iv) M_0 contains all the P-names of witnesses of each $\mathrm{LCU}_{R^1_i}(\lambda^1_i)$ $(i \in I_1)$,
(v) for each $i \in I_2$ and some chosen name $(\dot{\trianglelefteq}^i, \dot{\bar{g}}^i)$ of a witness of $\mathrm{COB}_{R^2_i}(\lambda^2_i, \vartheta^2_i)$: for all $(s,t) \in \vartheta^2_i \times \vartheta^2_i$, $\dot{g}^i_s \in M_0$ and the maximal antichain deciding "$s \dot{\trianglelefteq}^i t$" belongs to M_0,
(vi) $M_{\alpha+1}$ contains P-names of reals that are forced not to be in the $P \cap M_\alpha$-extension (this is because P forces $\mathfrak{c} > \mu$).

Note that M is also a $<\kappa$-closed elementary submodel of $H(\chi)$ of size μ, and that $P_\alpha := P \cap M_\alpha$ (for $\alpha \le \nu$) is a complete subposet of P. Put $P' := P_\nu$.

According to Corollary 3.5, in the P'-extension, each $\mathfrak{m}$-like characteristic below κ is preserved (as in the P-extension) and for the others "$\mathfrak{y}'_k \ge \kappa$" is preserved; and according to Lemma 3.2 the LCU and COB statements are preserved as well. This shows (a).

It is clear that P_α is a complete subposet of P_β for every $\alpha < \beta \leq \nu$, and that P' is the direct limit of the P_α. Therefore, if V' denotes the P'-extension and V_α denotes the P_α-intermediate extensions, then $\omega^\omega \cap V_{\alpha+1} \setminus V_\alpha \neq \emptyset$ (by (vi)) and $\omega^\omega \cap V' \subseteq \bigcup_{\alpha<\nu} V_\alpha$. Hence, by Lemma 6.1, $V' \models \mathfrak{g} \leq \nu$. Clearly, $V' \models \mathfrak{c} = \mu$. $\square$

We are now ready to add $\mathfrak{h} = \mathfrak{g} = \mathfrak{p}$ to our characteristics.

Lemma 6.4. *For $\aleph_1 \leq \lambda_\mathfrak{m} \leq \kappa \leq \nu_1$ regular, P^{pre} can be modified to some forcing P' which still strongly witnesses the Cichoń-characteristics, and additionally satisfies*:

$$\mathfrak{m} = \lambda_\mathfrak{m} \leq \mathfrak{h} = \mathfrak{g} = \mathfrak{p} = \kappa.$$

In addition to $\mathfrak{m} = \lambda_\mathfrak{m}$ we can get $\mathfrak{m} = \mathfrak{m}$(precaliber), which is Case 3 of Lemma 4.7; and instead of $\mathfrak{m} = \lambda_\mathfrak{m}$ we can alternatively force Case 1 or 2 of Lemma 4.7, or the situation of Lemma 5.5.

Proof. We start with the (appropriate) P from Lemma 4.7 (or from Lemma 5.5); but for the "inflated" continuum θ_∞^+ instead of θ_∞.

We then apply Lemma 6.3 for $\mu := \theta_\infty$, and $\nu := \kappa$. This gives a subforcing P' which still forces:

- Strong witnesses for all the Cichoń-characteristics;
  as they fall under Lemma 6.3(3,4).
- $\mathfrak{p} \geq \kappa$; an instance of Lemma 6.3(6) as P forces $\mathfrak{p} = \nu_3 \geq \kappa$.
- $\mathfrak{g} \leq \nu$; according to Lemma 6.3(b).
  As ZFC proves $\mathfrak{p} \leq \mathfrak{h} \leq \mathfrak{g}$ and $\nu = \kappa$, this implies $\mathfrak{p} = \mathfrak{h} = \mathfrak{g} = \kappa$.
- If $\lambda_\mathfrak{m} < \kappa$, we get $\mathfrak{m} = \lambda_\mathfrak{m} < \kappa$ as instance of Lemma 6.3(5).
- If $\lambda_\mathfrak{m} = \kappa$, we get $\mathfrak{m} \geq \kappa$ by Lemma 6.3(6);
  but as $\mathfrak{m} \leq \mathfrak{p}$ this also implies $\mathfrak{m} = \lambda_\mathfrak{m}$.
- Alternatively: The same argument for $\mathfrak{m}$(precaliber) and/or $\mathfrak{m}$(k-Knaster) instead of / in addition to $\mathfrak{m}$; as required by the desired case of Lemma 4.7 or 5.5. $\square$

We can now get twelve different characteristics.

Corollary 6.5. *Under Assumption* 2.8, *and for $\aleph_1 \leq \lambda_\mathfrak{m} \leq \kappa$ regular, we can get a* ccc *poset P'' which forces, in addition to Theorem* 2.9,

$$\mathfrak{m} = \lambda_\mathfrak{m} \leq \mathfrak{h} = \mathfrak{g} = \mathfrak{p} = \kappa$$

(the comment after Lemma 6.4 regarding various Martins axiom numbers applies here as well).

Proof. The resulting P' we just constructed still satisfies the requirements for Lemma 2.10, so we apply this lemma and get $P'' := P' \cap N^*$ (for a $<\kappa$-closed N^*) which forces the desired values to all Cichoń-characteristics. Additionally P'' forces:

- $\mathfrak{p} \geq \kappa$, by Corollary 3.5(iii)(a), as P' forces $\mathfrak{p} = \kappa$.
- $\mathfrak{g} \leq \kappa$, by Corollary 3.5(iii)(c), as P' forces $\mathfrak{g} = \kappa$.

- $\mathfrak{p} = \mathfrak{h} = \mathfrak{g} = \kappa$, as ZFC proves $\mathfrak{p} \leq \mathfrak{g}$.
- In case $\lambda_m < \kappa$: $\mathfrak{m} = \lambda_m$ by Corollary 3.5(iii)(b).
- In case $\lambda_m = \kappa$: $\mathfrak{m} \geq \kappa$ by Corollary 3.5(iii)(a), which again implies $\mathfrak{m} = \lambda_m$, as ZFC proves $\mathfrak{m} \leq \mathfrak{p} = \kappa$. $\square$

7. Products, Dealing with $\mathfrak{p}$

We start reviewing a basic result in forcing theory.

Lemma 7.1 (Easton's lemma). *Let ξ be an uncountable cardinal, P a ξ-cc poset and let Q be a $<\xi$-closed poset. Then P forces that Q is $<\xi$-distributive.*

Proof. See, e.g. [24, Lemma 15.19]. Note that there the lemma is proved for successor cardinals only, but literally the same proof works for any regular cardinal; for singular cardinals ξ note that $<\xi$-closed implies $<\xi^+$-closed so we even get $<\xi^+$-distributive. $\square$

Lemma 7.2. *Assume $\xi^{<\xi} = \xi$, P is ξ-cc, and set $Q = \xi^{<\xi}$ (ordered by extension). Then P forces that Q^V preserves all cardinals and cofinalities. Assume $P \Vdash \mathfrak{x} = \lambda$ (in particular that λ is a cardinal), and let R be a Borel relation.*

(a) *If $\mathfrak{x}$ is $\mathfrak{m}$-like: $\lambda < \xi$ implies $P \times Q \Vdash \mathfrak{x} = \lambda$; $\lambda \geq \xi$ implies $P \times Q \Vdash \mathfrak{x} \geq \xi$.*
(b) *If $\mathfrak{x}$ is $\mathfrak{h}$-like: $P \times Q \Vdash \mathfrak{x} \leq \lambda$.*
(c) *$P \Vdash \mathrm{LCU}_R(\lambda)$ implies $P \times Q \Vdash \mathrm{LCU}_R(\lambda)$.*
(d) *$P \Vdash \mathrm{COB}_R(\lambda, \mu)$ implies $P \times Q \Vdash \mathrm{COB}_R(\lambda, \mu)$.*

Proof. We call the P^+-extension V'' and the intermediate P-extension V'.

In V', all V-cardinals $\geq\xi$ are still cardinals, and Q is a $<\xi$-distributive forcing (due to Easton's lemma). So we can apply Lemma 3.1 and Corollary 3.5. $\square$

The following is shown in [15].

Lemma 7.3. *Assume that $\xi = \xi^{<\xi}$ and P is a ξ-cc poset that forces $\xi \leq \mathfrak{p}$. In the P-extension V', let $Q = (\xi^{<\xi})^V$. Then,*

(a) *$P \times Q = P * Q$ forces $\mathfrak{p} = \xi$.*
(b) *If in addition P forces $\xi \leq \mathfrak{p} = \mathfrak{h} = \kappa$ then $P \times Q$ forces $\mathfrak{h} = \kappa$.*

Proof. Work in the P-extension V'. Q preserves cardinals and cofinalities, and it forces $\mathfrak{p} \geq \xi$ by Lemma 7.2.

There is an embedding F from $\langle Q, \subsetneq\rangle$ into $\langle [\omega]^{\aleph_0}, \supsetneq^*\rangle$ preserving the order and incompatibility (using the fact that $\xi \leq \mathfrak{p} = \mathfrak{t}$ and that every infinite set can be split into ξ many almost disjoint sets). Now, Q adds a new sequence $z \in \xi^\xi \setminus V'$ and forces that $\dot{T} = \{F(z \restriction \alpha) : \alpha < \xi\}$ is a tower (hence $\mathfrak{t} \leq \xi$). If this were not the case, some condition in Q would force that $\dot{T}$ has a pseudo-intersection a, but actually $a \in V'$ and it determines uniquely a branch in ξ^ξ, and this branch would be in fact z, i.e. $z \in V'$, a contradiction. So we have shown $P \times Q \Vdash \mathfrak{t} = \xi$.

For (b), we already know that $Q \Vdash \mathfrak{h} \leq \kappa$. To show that $\mathfrak{h}$ does not decrease, again work in V'. Note that $\langle [\omega]^{\aleph_0}, \subseteq^* \rangle$ is $<\kappa$-closed (as $\mathfrak{t} = \kappa$). We claim that Q forces that $\langle [\omega]^{\aleph_0}, \subseteq^* \rangle$ is $<\kappa$-distributive (which implies $Q \Vdash \mathfrak{h} \geq \kappa$).

If $\kappa = \xi$ then $\langle [\omega]^{\aleph_0}, \subseteq^* \rangle$ is still $<\xi$-closed because Q is $<\xi$-distributive; so assume $\xi < \kappa$. Then Q is κ-cc (because $|Q| = \xi$), so $\langle [\omega]^{\aleph_0}, \subseteq^* \rangle$ is forced to be $<\kappa$-distributive by Easton's Lemma (recall that Q does not add new reals). $\square$

We are now ready to formulate the main theorem, the consistency of 13 different values (see Fig. 2).

Theorem 7.4. *Assume GCH, and that*

$$\aleph_1 \leq \lambda_{\mathfrak{m}} \leq \xi \leq \kappa \leq \lambda_{\mathrm{add}(\mathcal{N})} \leq \lambda_{\mathrm{cov}(\mathcal{N})} \leq \lambda_{\mathfrak{b}} \leq \lambda_{\mathrm{non}(\mathcal{M})}$$
$$\leq \lambda_{\mathrm{cov}(\mathcal{M})} \leq \lambda_{\mathfrak{d}} \leq \lambda_{\mathrm{non}(\mathcal{N})} \leq \lambda_{\mathrm{cof}(\mathcal{N})} \leq \lambda_\infty$$

are regular cardinals, with the possible exception of λ_∞, for which we only require $\lambda_\infty^{<\kappa} = \lambda_\infty$. Then we can force that

$$\aleph_1 \leq \lambda_{\mathfrak{m}} \leq \mathfrak{p} = \xi \leq \mathfrak{h} = \mathfrak{g} = \kappa$$
$$\leq \mathrm{add}(\mathcal{N}) = \lambda_{\mathrm{add}(\mathcal{N})} \leq \mathrm{cov}(\mathcal{N}) = \lambda_{\mathrm{cov}(\mathcal{N})} \leq \mathfrak{b} = \lambda_{\mathfrak{b}} \leq \mathrm{non}(\mathcal{M}) = \lambda_{\mathrm{non}(\mathcal{M})}$$
$$\leq \mathrm{cov}(\mathcal{M}) = \lambda_{\mathrm{cov}(\mathcal{M})} \leq \mathfrak{d} = \lambda_{\mathfrak{d}} \leq \mathrm{non}(\mathcal{N}) = \lambda_{\mathrm{non}(\mathcal{N})} \leq \mathrm{cof}(\mathcal{N}) = \lambda_{\mathrm{cof}(\mathcal{N})}$$
$$\leq 2^{\aleph_0} = \lambda_\infty$$

and we can additionally chose any one of the following:

- $\mathfrak{m} = \mathfrak{m}(\text{precaliber}) = \lambda_{\mathfrak{m}}$.
- *For a fixed* $1 \leq k < \omega$, $\mathfrak{m}(k\text{-Knaster}) = \aleph_1$ *and* $\mathfrak{m}(k+1\text{-Knaster}) = \lambda_{\mathfrak{m}}$.
- $\mathfrak{m}(k\text{-Knaster}) = \aleph_1$ *for all* $k < \omega$, *and* $\mathfrak{m}(\text{precaliber}) = \lambda_{\mathfrak{m}}$.

Proof. Start with the appropriate forcing P'' of Corollary 6.5. Then $P'' \times \xi^{<\xi}$ forces:

- Strong witnesses to all Cichoń-characteristics; by Lemma 7.2(c,d).
- $\mathfrak{p} = \xi$ and $\mathfrak{h} = \kappa$; by Lemma 7.3.
- $\mathfrak{g} \leq \kappa$ by Lemma 7.2(b) as P'' forces $\mathfrak{g} = \kappa$ and $\mathfrak{g}$ is $\mathfrak{h}$-like. This implies $\mathfrak{g} = \kappa$, as ZFC proves $\mathfrak{h} \leq \mathfrak{g}$.
- The desired values to the Martin axiom numbers; by Lemma 7.2(a) (and by the fact that $\mathfrak{m} \leq \mathfrak{p}$, in case $\lambda_m = \xi$). $\square$

8. Alternatives

The methods of this paper can be used for other initial forcings on the left-hand side and for the Boolean ultrapower method instead of the method of intersections with elementary submodels. Also, they allow us to compose many forcing notions with collapses while preserving cardinal characteristics.

All these topics are described in more detail in [18]; in the following we just give an overview.

8.1. *Another order*

In [28], another ordering of Cichoń's maximum is shown to be consistent (using large cardinals), namely, the ordering shown in Fig. 4.

The initial (left-hand side) forcing is based on ideas from [38], and in particular the notion of finite additive measure (FAM) limit introduced there for random forcing. In addition, a creature forcing Q^2 similar to the one defined in [23] (with Horowitz) is introduced, which forces $\mathrm{non}(\mathcal{M}) \geq \lambda_{\mathrm{non}(\mathcal{M})}$ and which has FAM-limits similar to random forcing (which is required to keep $\mathfrak{b}$ small).

In [19], we show that we can remove the large cardinal assumptions for this ordering as well (using the same method).

It is straightforward to check that the method in this paper allows us to add $\mathfrak{m}$, $\mathfrak{p}$, $\mathfrak{h}$ to this ordering as well; so we get Theorem 7.4 with both ($\mathfrak{b}$ and $\mathrm{cov}(\mathcal{N})$) and ($\mathfrak{d}$ and $\mathrm{non}(\mathcal{N})$) exchanged. In particular, we get (see Fig. 4).

Theorem 8.1. *Consistently,*

$$\aleph_1 < \mathfrak{m} < \mathfrak{p} < \mathfrak{h} < \mathrm{add}(\mathcal{N}) < \mathfrak{b} < \mathrm{cov}(\mathcal{N}) < \mathrm{non}(\mathcal{M})$$
$$< \mathrm{cov}(\mathcal{M}) < \mathrm{non}(\mathcal{N}) < \mathfrak{d} < \mathrm{cof}(\mathcal{N}) < 2^{\aleph_0}.$$

8.2. *Boolean ultrapowers*

As mentioned in Sec. 2.3, the original Cichoń Maximum construction [20] uses four strongly compact cardinals: First, the left side of Cichoń's diagram is separated with P^{pre} of 2.6, where we assume that there are compacts between each of $\aleph_1 < \nu_1 < \nu_2 < \nu_3 < \nu_4$. Then four Boolean ultrapowers are applied to this poset (one for each compact cardinal) to construct a forcing P^* that separates, in addition, the right-hand side, while preserving the left side values already forced by P^{pre}.

In view of Corollary 3.5(ii), we can use the methods of Secs. 4–7 to force, in addition, $\mathfrak{m} < \mathfrak{p} < \mathfrak{h} < \mathrm{add}(\mathcal{N})$.

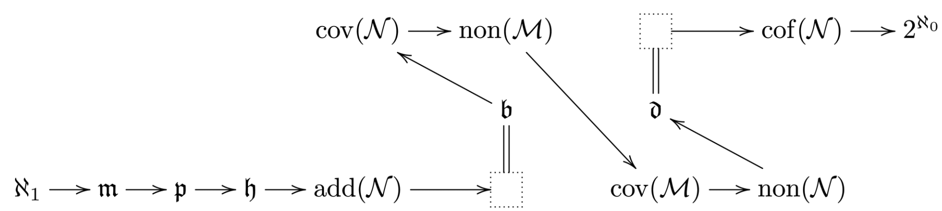

Fig. 4. An alternative order that we get when we start with the initial forcing from [28] (any $\rightarrow$ can be interpreted as either $<$ or $=$ as desired).

In contrast with Theorem 7.4, we can now force not only the continuum to be singular, but also $\mathrm{cov}(\mathcal{M})$. The reason is that the poset for the left side can force $\mathrm{cov}(\mathcal{M}) = \mathfrak{c}$ singular,[u] and the value of $\mathrm{cov}(\mathcal{M})$ is not changed after Boolean ultrapowers (and the other methods). The same applies to the alternate order from [28] as well.

8.3. *Alternative left-hand side forcings*

According to Sec. 2.3, [10] provides an alternative proof of Cichoń's maximum, using three strongly compact cardinals. As in [20], this results from applying Boolean ultrapowers to a ccc poset that separates the left side, but the new initial forcing additionally gives $\mathrm{cov}(\mathcal{M}) < \mathfrak{d} = \mathrm{non}(\mathcal{N}) = \mathfrak{c}$, where this value of $\mathfrak{d}$ can be singular. The methods of this work also apply, and we can obtain a consistency result as in Theorem 7.4, but there $\mathfrak{d}$ and $\mathfrak{c}$ are forced to be singular.

8.4. *Reducing gaps with collapsing forcing*

To be able to apply Boolean ultrapowers, it is necessary to have strongly compact cardinals between the left-hand side values. Accordingly these values have to have large gaps. The methods of this paper allow to collapse these gaps; and more generally to compose collapses with a large family of forcing notions.

For example, if P forces $\mathfrak{x} = \lambda < \mathfrak{y} = \kappa$, and λ and κ are far apart; but you would prefer to have $\mathfrak{x} = \lambda < \mathfrak{y} = \lambda^+$, then the methods of [18] allow us to compose P with a collapse of κ to λ^+, provided $\mathfrak{x}$ is reasonably well behaved.

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