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CONTROLLING CLASSICAL CARDINAL CHARACTERISTICS WHILE COLLAPSING CARDINALS

BY

MARTIN GOLDSTERN (Wien), JAKOB KELLNER (Wien),
DIEGO A. MEJÍA (Shizuoka) and
SAHARON SHELAH (Jerusalem and New Brunswick, NJ)

Abstract. We show how to force distinct values to $\mathfrak{m}$, $\mathfrak{p}$ and $\mathfrak{h}$ and the values in Cichoń's diagram, using the Boolean Ultrapower method. In our recent paper [J. Math. Logic 21 (2021)] the same was done for a newer Cichoń's Maximum construction which does not require large cardinals. The present version does need large cardinals, but allows one more value, in addition to the continuum, to be singular (either $\mathrm{cov}(\mathcal{M})$ or $\mathfrak{d}$).

We also show the following: Given a forcing notion P that forces certain values to several classical cardinal characteristics of the reals, we can compose P with a collapse (of a cardinal $\lambda > \kappa$ to κ) such that the composition still forces the previous values to these characteristics.

Introduction. Cichoń's diagram (see Figure 1) lists ten cardinal characteristics of the continuum, which we will call *Cichoń characteristics* (where we ignore the two "dependent" characteristics $\mathrm{add}(\mathcal{M}) = \min\{\mathrm{cov}(\mathcal{M}), \mathfrak{b}\}$ and $\mathrm{cof}(\mathcal{M}) = \max\{\mathrm{non}(\mathcal{M}), \mathfrak{d}\}$).

In many constructions that force given values to such characteristics we actually get something stronger, which we call "strong witnesses" (the objects $\bar{f}$ and $\bar{g}$ in Definition 1.8).

In this paper, we show how to collapse cardinals while preserving the strongly witnessed values for Cichoń characteristics (and certain other types of characteristics).

We also continue the investigation of forcing constructions that result in *Cichoń's Maximum*, i.e., in "all Cichoń characteristics (including $\aleph_1$ and the continuum) are pairwise different".

This investigation was started in [GKS19] with a construction using large cardinals, and continued in [GKMS21b] (without using large cardinals). Based on the latter construction (and accordingly also avoiding large cardinals),

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in [GKMS21a] we investigated how to preserve and how to change classical cardinal characteristics of the continuum in NNR extensions, i.e., extensions that do not add reals; and we showed how this gives 13 pairwise different ones: ten from Cichoń's Maximum, plus $\mathfrak{m}$, $\mathfrak{p}$ and $\mathfrak{h}$ (see Definition 1.1).

It turns out that it is possible to add $\mathfrak{m}$, $\mathfrak{p}$ and $\mathfrak{h}$ to the original (large cardinal) construction of [GKS19] as well (see Figure 2) and this is what we do in Section 3 of this paper. This result is weaker than the one in [GKMS21a] in the sense that we need large cardinals here; the advantage of the current result is that we can obtain singular values for $\operatorname{cov}(\mathcal{M})$ or $\mathfrak{d}$ (in addition to the singularity of $\mathfrak{c}$, which is easy to get in all constructions), something which does not seem to be possible with the elementary submodel method of [GKMS21b]. (As remarked in Fact 1.2, most of the Cichoń characteristics can "individually" be singular; but it seems hard to get them in Cichoń's Maximum setting, see Subsection 1.3.)

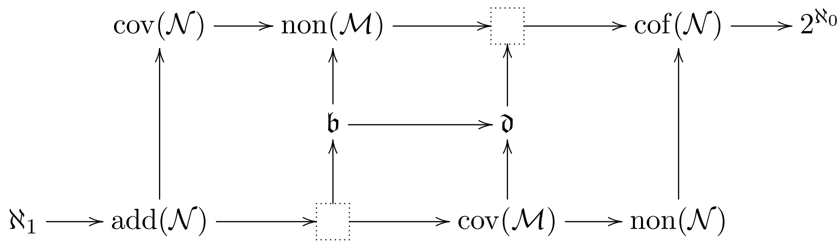

Fig. 1. Cichoń's diagram with the two "dependent" values removed, which are $\operatorname{add}(\mathcal{M}) = \min\{\mathfrak{b}, \operatorname{cov}(\mathcal{M})\}$ and $\operatorname{cof}(\mathcal{M}) = \max\{\operatorname{non}(\mathcal{M}), \mathfrak{d}\}$. An arrow $\mathfrak{x} \to \mathfrak{y}$ means that ZFC proves $\mathfrak{x} \leq \mathfrak{y}$.

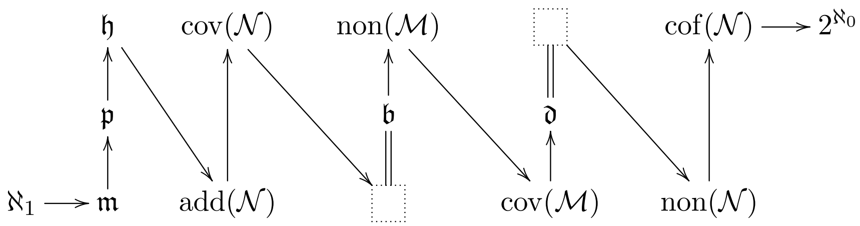

Fig. 2. The model we construct in this paper; here $\mathfrak{x} \to \mathfrak{y}$ means that $\mathfrak{x} < \mathfrak{y}$. When $\mathfrak{h}$ is omitted, any number of the $<$ signs can be replaced by $=$ as desired (and in each such constellation we can get $\mathfrak{p} = \mathfrak{h}$; see Remark 4.3 for details). This model corresponds to "Constellation A" (`cA`*, Fig. 3). We also realise another ordering of the Cichoń values, called "Constellation B" (`cB`*, Fig. 4).

Annotated contents. We will briefly review the Boolean ultrapower constructions in Section 1. We describe how we can start with several "initial forcings" (for the left hand side of Cichoń's diagram) and then extend each of them to a Cichoń's Maximum construction using three (or four) strongly compact cardinals.

Parts of the following sections are parallel to [GKMS21a], and we will regularly refer to that paper; this applies in particular to Section 2 (and parts of Subsection 1.4), where we describe some classes of cardinal characteristics, and their behaviour under no-new-reals extensions.

In Section 3 we show how to add $\mathfrak{m}$, $\mathfrak{p}$ and $\mathfrak{h}$ to the Boolean ultrapower construction.

Section 4: The Boolean ultrapower method produces large gaps between the Cichoń values of the left hand side: The κ_i in Figure 3 are strongly compact (in the ground model; so as cofinalities are preserved they are still weakly inaccessible in the extension). We can get rid of these gaps using the results of this section: We show how we can collapse cardinals while keeping values for characteristics that are either strongly witnessed or small.

In Section 5 we show how the known method of (simultaneously) adding many randoms at the end gives us models with several singular values.

We usually assume GCH in the ground model in our constructions. In Section 6 we observe that full GCH is usually not required, and give weaker sufficient conditions.

1. Preliminaries

1.1. The characteristics. In addition to the Cichoń characteristics we will consider the following ones, whose definitions are well known.

DEFINITION 1.1. Let $\mathcal{P}$ be a class of forcing notions.

(1) $\mathfrak{m}(\mathcal{P})$ denotes the minimal cardinal where Martin's axiom for the posets in $\mathcal{P}$ fails. More explicitly, it is the minimal κ such that, for some poset $Q \in \mathcal{P}$, there is a collection $\mathcal{D}$ of size κ of dense subsets of Q such that there is no filter in Q intersecting all the members of $\mathcal{D}$.
(2) $\mathfrak{m} := \mathfrak{m}(\text{ccc})$.
(3) Write $a \subseteq^* b$ iff $a \smallsetminus b$ is finite. Say that $a \in [\omega]^{\aleph_0}$ is a *pseudo-intersection* of $F \subseteq [\omega]^{\aleph_0}$ if $a \subseteq^* b$ for all $b \in F$.
(4) The *pseudo-intersection number* $\mathfrak{p}$ is the smallest size of a filter base of a free filter on ω that has no pseudo-intersection in $[\omega]^{\aleph_0}$.
(5) The *tower number* $\mathfrak{t}$ is the smallest order type of a $\subseteq^*$-decreasing sequence in $[\omega]^{\aleph_0}$ without pseudo-intersection.
(6) The *distributivity number* $\mathfrak{h}$ is the smallest size of a collection of open dense subsets of $([\omega]^{\aleph_0}, \subseteq^*)$ whose intersection is empty.
(7) A family $D \subseteq [\omega]^{\aleph_0}$ is *groupwise dense* if
    (i) $a \subseteq^* b$ and $b \in D$ implies $a \in D$, and
    (ii) whenever $(I_n : n < \omega)$ is an interval partition of ω, there is some $a \in [\omega]^{\aleph_0}$ such that $\bigcup_{n\in a} I_n \in D$.

The *groupwise density number* $\mathfrak{g}$ is the smallest size of a collection of groupwise dense sets whose intersection is empty.

The following is well known (for references see [GKMS21a]):

FACT 1.2. *ZFC proves*

$$\mathfrak{m} \le \mathfrak{p} = \mathfrak{t} \le \mathfrak{h} \le \mathfrak{g}, \quad \mathfrak{m} \le \mathrm{add}(\mathcal{N}), \quad \mathfrak{t} \le \mathrm{add}(\mathcal{M}), \quad \mathfrak{h} \le \mathfrak{b},$$
$$\max\{\mathfrak{b}, \mathfrak{g}\} \le \mathrm{cof}(\mathfrak{d}), \quad 2^{<\mathfrak{t}} = \mathfrak{c}, \quad \mathrm{cof}(\mathfrak{c}) \ge \mathfrak{g},$$

and all these cardinals are regular, with the possible exception of $\mathfrak{m}$, $\mathfrak{d}$ *and* $\mathfrak{c}$ (*which are consistently singular*). *In addition, all cardinals in Cichoń's diagram are consistently singular except* $\aleph_1$, $\mathfrak{b}$ *and the additivities.*

1.2. The old constructions. In this paper, we will build on two existing constructions of posets forcing different values to several (or all) entries of Cichoń's diagram. We will call them the "old constructions" and refer to them as "Constellation A" (in the variants $\mathtt{cA}$ and $\mathtt{cA}^*$) and "Constellation B" ($\mathtt{cB}$ and $\mathtt{cB}^*$). Here, $\mathtt{cA}$ and $\mathtt{cB}$ refer to the constructions for the left hand side (which do not require large cardinals), and $\mathtt{cA}^*$ and $\mathtt{cB}^*$ refer to Cichoń's Maximum.

Constellation A was introduced in [GKS19], and [BCM21] gives an improvement (requiring only three compacts and allowing $\mathfrak{d}$ to be singular instead of $\mathrm{cov}(\mathcal{M})$). Constellation B is from [KST19], and [Mej19b] notes that weaker assumptions on cardinal arithmetic in the ground model are sufficient.

We will not describe the old constructions in detail, but only state the results.

Note that in the following, we initially state the theorems about the existence of certain forcing notions assuming GCH (in the ground model). The only reason is that the theorems are better readable in this form. But only some very weak consequences of GCH are actually required, and we summarize the sufficient cardinal arithmetic assumptions in Section 6.

THEOREM 1.3. *Assume GCH and that* $\aleph_1 \le \lambda_1 \le \lambda_2 \le \lambda_3 \le \lambda_4 \le \lambda_5$ *are cardinals, with* λ_i *regular for* $i \ne 5$.

Constellation A: *Assume additionally* ([1]) *that either*

(i) λ_5 *is regular, and* $\mu \ge \lambda_5$ *is a cardinal with* $\mathrm{cof}(\mu) \ge \lambda_3$, *or*
(ii) $\mathrm{cof}(\lambda_5) \ge \lambda_4$, *and we set* $\mu := \lambda_5$.

Then there is a f.s. iteration $P^{\mathtt{cA}}$ *of length of size* μ *with cofinality* λ_4, *using iterands that are* (σ, k)*-linked for every* $k \in \omega$, *which forces the values of*

([1]) Assumptions (i) and (ii) are optimal in the sense that (i) ZFC proves $\mathrm{cof}(\mathfrak{d}) \ge \mathfrak{b}$ and (ii) for any ideal $\mathcal{I}$, if $\mathrm{cov}(\mathcal{I}) = \mathrm{cof}(\mathcal{I})$ then $\mathrm{non}(\mathcal{I}) \le \mathrm{cof}(\mathrm{cov}(\mathcal{I}))$ (see e.g. [BJ95, Lemma 5.1.16]).

λ_1–λ_5 *in Constellation A* (*Figure* 3)*:*

$$(\mathtt{cA})\qquad \begin{array}{lll} \mathrm{add}(\mathcal{N}) = \lambda_1, & \mathrm{cov}(\mathcal{N}) = \lambda_2, & \mathfrak{b} = \lambda_3, \\ \mathrm{non}(\mathcal{M}) = \lambda_4, & \mathrm{cov}(\mathcal{M}) = \lambda_5, & \mathfrak{d} = \mathfrak{c} = \mu. \end{array}$$

Constellation B: *Alternatively, assume additionally that* $\mathrm{cof}(\lambda_5) \geq \lambda_4$ *and either* $\lambda_2 = \lambda_3$*, or* λ_3 *is* $\aleph_1$*-inaccessible* (2)*. Then there is a f.s. iteration* $P^{\mathtt{cB}}$ *of length of size* λ_5 *with cofinality* λ_4*, using iterands that are* (σ, k)*-linked for every* $k \in \omega$*, which forces the values of* λ_1–λ_5 *in Constellation B* (*Figure* 4)*:*

$$(\mathtt{cB})\qquad \begin{array}{lll} \mathrm{add}(\mathcal{N}) = \lambda_1, & \mathfrak{b} = \lambda_2, & \mathrm{cov}(\mathcal{N}) = \lambda_3, \\ \mathrm{non}(\mathcal{M}) = \lambda_4, & \mathrm{cov}(\mathcal{M}) = \mathfrak{c} = \lambda_5. & \end{array}$$

All these constructions can then be extended with Boolean ultrapowers (more precisely: compositions of finitely many successive Boolean ultrapowers), to make all values simultaneously different:

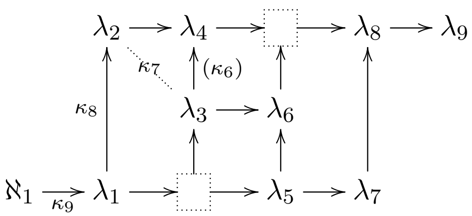

Fig. 3. Constellation A

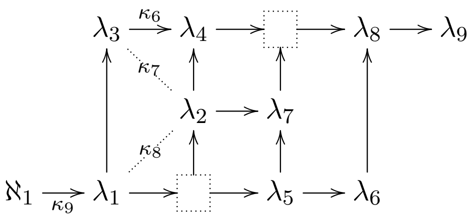

Fig. 4. Constellation B

THEOREM 1.4. *Assume GCH and* $\aleph_1 < \kappa_9 < \lambda_1 < \kappa_8 < \lambda_2 < \kappa_7 < \lambda_3 \leq \lambda_4 \leq \lambda_5 \leq \lambda_6 \leq \lambda_7 \leq \lambda_8 \leq \lambda_9$*,* λ_i *is regular for* $i \neq 5, 6$*, and* κ_j *is strongly compact for* $j = 7, 8, 9$*.*

Constellation A: *Assume additionally that either*

(i) λ_5 *is regular and* $\mathrm{cof}(\lambda_6) \geq \lambda_3$*, or*
(ii) λ_6 *is regular,* $\mathrm{cof}(\lambda_5) \geq \lambda_4$*, and there is a strongly compact* κ_6 *such that* $\lambda_3 < \kappa_6 < \lambda_4$*.*

Then there is a f.s. ccc iteration $P^{\mathtt{cA}*}$ (*a Boolean ultrapower of* $P^{\mathtt{cA}}$) *that forces Constellation A* (*Figure* 3)*:*

$$(\mathtt{cA}^*)\qquad \begin{array}{lllll} \mathrm{add}(\mathcal{N}) = \lambda_1, & \mathrm{cov}(\mathcal{N}) = \lambda_2, & \mathfrak{b} = \lambda_3, & \mathrm{non}(\mathcal{M}) = \lambda_4, & \\ \mathrm{cov}(\mathcal{M}) = \lambda_5, & \mathfrak{d} = \lambda_6, & \mathrm{non}(\mathcal{N}) = \lambda_7, & \mathrm{cof}(\mathcal{N}) = \lambda_8, & \mathfrak{c} = \lambda_9. \end{array}$$

(2) A cardinal λ is κ*-inaccessible* if $\mu^\nu < \lambda$ for any $\mu < \lambda$ and $\nu < \kappa$. Under GCH, this is equivalent to "$\lambda \geq \kappa$ and λ is not the (cardinal) successor of a cardinal with cofinality $< \kappa$".

Constellation B: *Alternatively, assume additionally* λ_3 *is* $\aleph_1$*-inaccessible and* (ii) (*of Constellation A*) *holds. Then there is a f.s. ccc iteration* $P^{\mathtt{cB}*}$ (*a Boolean ultrapower of* $P^{\mathtt{cB}}$) *that forces Constellation B* (*Figure* 4)*:*

$$(\mathtt{cB}^*)\quad \begin{array}{l} \mathrm{add}(\mathcal{N}) = \lambda_1, \quad \mathfrak{b} = \lambda_2, \quad \mathrm{cov}(\mathcal{N}) = \lambda_3, \quad \mathrm{non}(\mathcal{M}) = \lambda_4, \\ \mathrm{cov}(\mathcal{M}) = \lambda_5, \quad \mathrm{non}(\mathcal{N}) = \lambda_6, \quad \mathfrak{d} = \lambda_7, \quad \mathrm{cof}(\mathcal{N}) = \lambda_8, \quad \mathfrak{c} = \lambda_9. \end{array}$$

Sketch of proof of Theorems 1.3 and 1.4. In all versions, that is, A(i), A(ii) and B, we first construct a forcing for separating the characteristics on the left hand side, i.e., for Theorems 1.3. This initial forcing will be different for the different versions. To get the "full" result of Theorems 1.4, we always do the same: We apply Boolean ultrapowers to the initial forcing notion, as introduced in [GKS19].

For Constellation A(i) this result can be found explicitly in [BCM21, Thm. 5.3] (for the left hand side) and in [BCM21, Thm. 5.7] (for the full version); and for Constellation B in [Mej19b, Thms. A, B].

Constellation A(ii) is not explicitly described in the literature so far; [GKS19, Thm. 1.35] gives the result with a slightly stronger Assumptions 1.12 there. But these assumptions can be relaxed without too much work. This can be found in [Mej19b], which basically shows that you can relax the stronger assumptions of [KST19] to weaker ones for Constellation B; and exactly the same modification works for Constellation A(i) as well.

In particular, it shows:

- We can replace "λ_5 regular" with $\mathrm{cof}(\lambda_5) \geq \lambda_4$. (This is trivial as it requires no change in the proof whatsoever.)
  Alternatively we could first use a regular λ_5' and then use our Lemma 3.6 to get a (singular) λ_5 as value for the continuum.
  (Note that singular λ_5 allows us to get both $\mathrm{cov}(\mathcal{M})$ and $\mathfrak{c}$ singular in Theorem 3.10.)
- (For the left hand side only:) How each $\lambda_i < \lambda_j$ can be replaced by $\leq$. (This is rather obvious.)
- How to get rid of the assumption that λ_3 is successor of a regular, and that all λ_i are $\aleph_1$-inaccessible. (This requires some change in the construction and proof.)

We now give a *very* superficial overview of the Boolean ultrapower construction. We start with a suitable left hand side forcing P, forcing ($\mathtt{cA}$) or ($\mathtt{cB}$). For $i = 7, 8, 9$ (and also $i = 6$ in all versions apart from A(i)), we let j_i be a complete embedding associated with some suitable Boolean ultrapower of the complete Boolean algebra generated by $\mathrm{Fn}_{<\kappa_i}(\lambda_i, \kappa_i)$ (partial functions of cardinality $<\kappa_i$ from λ_i to κ_i), which yields $\mathrm{cr}(j_i) = \kappa_i$ and $\mathrm{cof}(j_i(\kappa_i)) = |j_i(\kappa_i)| = \lambda_i$. Then $P^* = j_9(j_8(j_7(P)))$ (in Constellation A(i)) is as required, and in the other versions we use $P^* = j_9(j_8(j_7(j_6(P))))$. At the end of Subsection 1.4 we give the argument why the Cichoń characteristics are forced to have the desired values. ■

REMARK 1.5. Whenever we change in Theorem 1.4 some strict inequalities on the right side to equalities, we may weaken the assumption by requiring fewer strongly compact cardinals.

For example, in Constellation A(i), if we want to end up with $\mathrm{non}(\mathcal{N}) = \lambda_7 = \lambda_8 = \mathrm{cof}(\mathcal{N})$, we can omit the compact κ_8, and it is enough to assume $\aleph_1 < \kappa_9 < \lambda_1 \leq \lambda_2 < \kappa_7 < \lambda_3 \leq \lambda_4 \leq \lambda_5 \leq \lambda_6 \leq \lambda_7 = \lambda_8 \leq \lambda_9$ (with the other requirements unchanged). The same construction and proof will work (where we omit the ultrapower corresponding to κ_8): After the first ultrapower (corresponding to κ_7), we know that the resulting forcing will force $\lambda_7 = \mathrm{non}(\mathcal{N}) = \mathrm{cof}(\mathcal{N}) = 2^{\aleph_0}$. And all the following ultrapowers (in this case there is only one, with critical point κ_9) will have critical point below λ_1 (the value for $\mathrm{add}(\mathcal{N})$), as we omit κ_8. Therefore these ultrapowers will keep the value forced to both $\mathrm{non}(\mathcal{N})$ and $\mathrm{cof}(\mathcal{N})$ (while increasing the value forced to the continuum).

See [GKMS21a, Subsec. 2.3] for details on the history of the results of this section (and more).

1.3. Singular values and Cichoń's Maximum. In the theorems in the previous section, we can force *either* $\mathfrak{d}$ (in Constellation A(i) only) *or* $\mathrm{cov}(\mathcal{M})$ singular (in the others).

Note that typically only the value of $\mathfrak{c}$ can be forced to be singular in forcing extensions produced by "usual" f.s. iterations of ccc posets (see e.g. [Bre91, Mej13]). Here, we start with such a "usual" left hand forcing P that (potentially) makes the continuum singular (which is equal to $\mathrm{cov}(\mathcal{M}) = \mathfrak{d}$, or just to $\mathfrak{d}$). After applying the Boolean ultrapowers, the resulting forcing P^* will still force the same (potentially singular) value to $\mathrm{cov}(\mathcal{M})$ or $\mathfrak{d}$, while increasing the values for the larger entries in Cichoń's diagram, including $\mathfrak{c}$, to regular values. We will see in Lemma 3.6 that we can further modify this P^* to force a singular value to the continuum as well; thus we can get two different singular entries in the diagram together with Cichoń Maximum (see e.g. Theorem 3.10).

We are aware of the following previously known examples of more than one singular value in Cichoń's diagram:

(1) Forcing with κ many Cohen reals, followed by simultaneously adding λ many randoms for $\kappa < \lambda$ both possibly singular, yields the consistency of $\aleph_1 = \mathrm{non}(\mathcal{N}) = \mathfrak{b} < \mathfrak{d} = \kappa < \mathrm{cov}(\mathcal{N}) = \mathfrak{c} = \lambda$.
(2) First adding κ many Cohens followed by $\lambda < \kappa$ many (simultaneous) randoms, where $\lambda = \lambda^{\aleph_0}$, we get $\aleph_1 = \mathrm{non}(\mathcal{N}) = \mathfrak{b} < \mathrm{cov}(\mathcal{N}) = \mathrm{non}(\mathcal{M}) = \lambda < \mathfrak{d} = \kappa = \mathfrak{c}$. See Section 5 for details.

(3) With non-ccc techniques, in [FGKS17] it can be forced that $\aleph_1 = \mathrm{cov}(\mathcal{N}) = \mathfrak{d} < \mathrm{non}(\mathcal{M}) < \mathrm{non}(\mathcal{N}) < \mathrm{cof}(\mathcal{N}) < \mathfrak{c}$ where all the values larger than $\aleph_1$ can be singular.
(4) Several examples of Cichoń's diagram constellations (not maximum though) with two different singular values are forced in [Mej19a] using matrix iterations with vertical support restrictions.

The first two examples are expanded in Section 5.

1.4. Blass-uniform cardinal characteristics, LCU and COB.

A more detailed discussion on the concepts reviewed in the rest of this section can be found in [GKMS21a, Sec. 2.1].

Definition 1.6 ([GKMS21a, Def. 2.1]). A *Blass-uniform cardinal characteristic* is a characteristic of the form

$$\mathfrak{d}_R := \min\{|D| : D \subseteq \omega^\omega \text{ and } (\forall x \in \omega^\omega)\,(\exists y \in D)\; xRy\}$$

for some Borel (3) R.

Its dual cardinal

$$\mathfrak{b}_R := \min\{|F| : F \subseteq \omega^\omega \text{ and } (\forall y \in \omega^\omega)\,(\exists x \in F)\; \neg xRy\}$$

is also Blass-uniform because $\mathfrak{b}_R = \mathfrak{d}_{R^\perp}$ where $xR^\perp y$ iff $\neg(yRx)$.

In practice, Blass-uniform cardinal characteristics are defined from a relation $R \subseteq X \times Y$ where X and Y are Polish spaces, but since we can translate such a relation to ω^ω using Borel isomorphisms, it is enough to discuss relations on ω^ω.

Systematic research on such cardinal characteristics started in the 1980s or possibly even earlier: see e.g. Fremlin [Fre84], Blass [Bla93, Bla10] and Vojtáš [Voj93].

Example 1.7. The following are pairs of dual Blass-uniform cardinals $(\mathfrak{b}_R, \mathfrak{d}_R)$ for natural Borel relations R:

(1) A cardinal on the left hand side of Cichoń's diagram and its dual on the right hand side: $(\mathfrak{b}, \mathfrak{d})$ and $(\mathrm{add}(\mathcal{N}), \mathrm{cof}(\mathcal{N}))$, $(\mathrm{cov}(\mathcal{N}), \mathrm{non}(\mathcal{N}))$, $(\mathrm{add}(\mathcal{M}), \mathrm{cof}(\mathcal{M}))$, $(\mathrm{non}(\mathcal{M}), \mathrm{cov}(\mathcal{M}))$.
(2) $(\mathfrak{s}, \mathfrak{r}) = (\mathfrak{b}_R, \mathfrak{d}_R)$ where $\mathfrak{s}$ is the splitting number, $\mathfrak{r}$ is the reaping number, and R is the relation on $[\omega]^{\aleph_0}$ defined by xRy iff "x does not split y".

Definition 1.8 ([GKMS21a, Def. 2.3]). Fix a Borel relation R, λ a regular cardinal and μ an arbitrary cardinal. We define two properties:

(3) More generally, it is just enough to assume that R is absolute between the extensions we consider.

Linearly cofinally unbounded: $\mathrm{LCU}_R(\lambda)$ means: There is a family $\bar{f} = (f_\alpha : \alpha < \lambda)$ of reals such that

$$(\forall g \in \omega^\omega)\,(\exists \alpha \in \lambda)\,(\forall \beta \in \lambda \setminus \alpha)\ \neg f_\beta R g. \tag{1.1}$$

Cone of bounds: $\mathrm{COB}_R(\lambda, \mu)$ means: There is a $<\!\lambda$-directed partial order $\trianglelefteq$ on μ (4) and a family $\bar{g} = (g_s : s \in \mu)$ of reals such that

$$(\forall f \in \omega^\omega)\,(\exists s \in \mu)\,(\forall t \trianglerighteq s)\ f R g_t. \tag{1.2}$$

FACT 1.9. $\mathrm{LCU}_R(\lambda)$ *implies* $\mathfrak{b}_R \le \lambda \le \mathfrak{d}_R$.
$\mathrm{COB}_R(\lambda, \mu)$ *implies* $\mathfrak{b}_R \ge \lambda$ *and* $\mathfrak{d}_R \le \mu$.

We often call the objects $\bar{f}$ in the definition of LCU and $(\trianglelefteq, \bar{g})$ for COB "strong witnesses", and we say that the corresponding cardinal inequalities (or equalities) are "strongly witnessed". For example, "$(\mathfrak{b}, \mathfrak{d}) = (\lambda_\mathfrak{b}, \lambda_\mathfrak{d})$ is strongly witnessed" means: for the natural relation R (namely, the relation $\le^*$ of eventual dominance), we have $\mathrm{COB}_R(\lambda_\mathfrak{b}, \lambda_\mathfrak{d})$, $\mathrm{LCU}_R(\lambda_\mathfrak{b})$ and there is some regular $\lambda_0 \le \lambda_\mathfrak{d}$ such that $\mathrm{LCU}_R(\lambda)$ for all regular $\lambda \in [\lambda_0, \lambda_\mathfrak{d}]$ (this is to allow $\lambda_\mathfrak{d}$ to be singular as in case (i) of `cA` and `cA`* of Theorems 1.3 and 1.4).

REMARK 1.10. The old constructions ((`cA`), (`cB`) in Theorem 1.3) use that we can first force strong witnesses to the left hand side, and then preserve strong witnesses in Boolean ultrapowers, so that in the final model all Cichoń characteristics are strongly witnessed. In more detail, for each dual pair $(\mathfrak{x}, \mathfrak{y})$ in Cichoń's diagram, there is a natural relation $R_\mathfrak{x}$ such that $(\mathfrak{x}, \mathfrak{y}) = (\mathfrak{b}_{R_\mathfrak{x}}, \mathfrak{d}_{R_\mathfrak{x}})$. We use these natural relations (with one exception (5)) as follows: The initial forcing (without Boolean ultrapowers) is a f.s. iteration P of length δ and forces $\mathrm{LCU}_{R_\mathfrak{x}}(\mu)$ for all regular $\lambda_\mathfrak{x} \le \mu \le |\delta|$, and $\mathrm{COB}_{R_\mathfrak{x}}(\lambda_\mathfrak{x}, |\delta|)$, where we either have $\lambda_{\mathrm{add}(\mathcal{N})} < \lambda_{\mathrm{cov}(\mathcal{N})} < \lambda_\mathfrak{b} < \lambda_{\mathrm{non}(\mathcal{M})}$ (`cA`, excluding $\lambda_{\mathrm{non}(\mathcal{M})}$ for case (i)), or $\lambda_{\mathrm{add}(\mathcal{N})} < \lambda_\mathfrak{b} < \lambda_{\mathrm{cov}(\mathcal{N})} < \lambda_{\mathrm{non}(\mathcal{M})}$ (`cB`), as in Theorem 1.3.

Once we know that the initial forcing P gives strong witnesses for the desired values $\lambda_\mathfrak{x}$ for all "left-hand" values $\mathfrak{x}$ in Cichoń's diagram and con-

(4) That is, every subset of μ of cardinality $<\!\lambda$ has a $\trianglelefteq$-upper bound.

(5) The exception is the following: In case (i) of `cA`, for the pair $(\mathfrak{x}, \mathfrak{y}) = (\mathrm{non}(\mathcal{M}), \mathrm{cov}(\mathcal{M}))$ it is forced $\mathrm{LCU}_{\neq^*}(\lambda_4)$, $\mathrm{LCU}_{\neq^*}(\lambda_5)$ and $\mathrm{COB}_{\neq^*}(\lambda_4, \lambda_5)$ (here $x \neq^* y$ iff $x(i) \neq y(i)$ for all but finitely many i); in `cB`, for $\mathfrak{x} = \mathrm{cov}(\mathcal{N})$, we use the natural relation $R_{\mathrm{cov}(\mathcal{N})}$ (defined as the set of all pairs (x, y) where the real y is in the F_σ set of full measure coded by x) only for COB. In this constellation, we do not know whether P forces $\mathrm{LCU}_{R_{\mathrm{cov}(\mathcal{N})}}(\lambda_3)$ (as we do not have sufficient preservation results for $R_{\mathrm{cov}(\mathcal{N})}$). Instead, we use another relation R' (which defines different, anti-localization characteristics $(\mathfrak{b}_{R'}, \mathfrak{d}_{R'})$), for which ZFC proves $\mathrm{cov}(\mathcal{N}) \le \mathfrak{b}_{R'}$ and $\mathrm{non}(\mathcal{N}) \ge \mathfrak{d}_{R'}$. We can then show that P forces $\mathrm{LCU}_{R'}(\mu)$ for all regular $\lambda_3 \le \mu \le |\delta|$.

tinuum for the right hand side values (6), we use the following theorem to separate all the entries.

Theorem 1.11 ([KTT18, GKS19]). *Let $\nu < \kappa$ and $\lambda \neq \kappa$ be uncountable regular cardinals, R a Borel relation, and let P be a ν-cc poset forcing that λ is regular. Assume that $j : V \to M$ is an elementary embedding into a transitive class M satisfying:*

(i) *The critical point of j is κ.*
(ii) *M is $<\kappa$-closed* (7).
(iii) *For any cardinal $\theta > \kappa$ and any $<\theta$-directed partial order I, $j''I$ is cofinal in $j(I)$.*

Then:

(a) *$j(P)$ is a ν-cc forcing.*
(b) *If $P \Vdash \mathrm{LCU}_R(\lambda)$, then $j(P) \Vdash \mathrm{LCU}_R(\lambda)$.*
(c) *If $\lambda < \kappa$ and $P \Vdash \mathrm{COB}_R(\lambda, \mu)$, then $j(P) \Vdash \mathrm{COB}_R(\lambda, |j(\mu)|)$.*
(d) *If $\lambda > \kappa$ and $P \Vdash \mathrm{COB}_R(\lambda, \mu)$, then $j(P) \Vdash \mathrm{COB}_R(\lambda, \mu)$.*

Proof. We include the proof for completeness. Property (a) is immediate by (ii). First note that j satisfies the following additional properties:

(iv) Whenever a is a set of size $<\kappa$, $j(a) = j''a$.
(v) If $\mathrm{cof}(\alpha) \neq \kappa$ then $\mathrm{cof}(j(\alpha)) = \mathrm{cof}(\alpha)$.
(vi) If $\theta > \kappa$, L is a set and $P \Vdash$ "$(L, \dot{\trianglelefteq})$ is $<\theta$-directed" then $j(P) \Vdash$ "$j''L$ is cofinal in $(j(L), j(\dot{\trianglelefteq}))$, and it is $<\theta$-directed".
(vii) $j(P) \Vdash$ "$\mathrm{cof}(j(\lambda)) = \lambda$".

Item (iv) follows from (i), and (v) follows from (iii). We show (vi). Let L^* be the set of nice P-names of members of L, and order it by $\dot{x} \leq \dot{y}$ iff $P \Vdash \dot{x} \dot{\trianglelefteq} \dot{y}$. It is clear that $\leq$ is $<\theta$-directed on L^*. On the other hand, since any nice $j(P)$-name of a member of $j(L)$ is already in M by (ii) and (a), $j(L^*)$ is equal to the set of nice $j(P)$-names of members of $j(L)$. Therefore, by (iii), $j''L^*$ is cofinal in $j(L^*)$. Note that $j''L^*$ is equal to the set of nice $j(P)$-names of members of $j''L$. Thus, (vi) follows.

For (vii), the case $\lambda < \kappa$ is immediate by (i) and (ii); when $\lambda > \kappa$, apply (vi) to $(L, \dot{\trianglelefteq}) = (\lambda, \leq)$ (the usual order) and $\theta = \lambda$.

To see (b), note that $M \vDash$ "$j(P) \Vdash \mathrm{LCU}_R(j(\lambda))$" and, by (a) and (ii), the same holds inside V (because any nice name of an ordinal, represented by a maximal antichain on P, belongs to M, hence any nice name of a real), which in fact means that $j(P) \Vdash \mathrm{LCU}_R(\mathrm{cof}(j(\lambda)))$. By (vii) we are done.

(6) More specifically: for the cardinals $\geq \mathfrak{d}, \mathrm{non}(\mathcal{N})$ in case (i) of cA or $\geq \mathrm{cov}(\mathcal{M})$ in case (ii) of cA and in cB.

(7) I.e., $M^{<\kappa} \subseteq M$.

Now assume $P \Vdash \mathrm{COB}_R(\lambda,\mu)$ witnessed by $(\dot{\trianglelefteq},\dot{\bar{g}})$. This implies that M thinks "$j(P) \Vdash (j(\dot{\trianglelefteq}), j(\dot{\bar{g}}))$ witnesses $\mathrm{COB}_R(j(\lambda), j(\mu))$". If $\lambda < \kappa$ then $j(\lambda) = \lambda$ and it follows that $V \models$ "$j(P) \Vdash \mathrm{COB}_R(\lambda, |j(\mu)|)$". In the case $\lambda > \kappa$ apply (vi) to conclude that $j(P)$ forces that $(j(\dot{\bar{g}}(\beta)) : \beta < \mu)$, with $j(\dot{\trianglelefteq})$ restricted to $j''\mu$, witnesses $\mathrm{COB}_R(\lambda,\mu)$. ∎

If κ is a strongly compact cardinal and $\theta^\kappa = \theta$, then there is an elementary embedding j associated with a Boolean ultrapower of the completion of $\mathrm{Fn}_{<\kappa}(\theta,\kappa)$ such that j satisfies (i)–(iii) of the preceding theorem and, in addition, for any cardinal $\lambda \geq \kappa$ such that either $\lambda \leq \theta$ or $\lambda^\kappa = \lambda$ holds, we have $\max\{\lambda,\theta\} \leq j(\lambda) < \max\{\lambda,\theta\}^+$ (see details in [KTT18, GKS19]). Therefore, using only Theorem 1.11, it is easy to see how to get from the old constructions (Theorem 1.3) to the Boolean ultrapowers (Theorem 1.4), as described in Remark 1.10 (see details in [BCM21, Thm. 5.7] and [GKS19, Thm. 3.1] for `cA`* and in [KST19, Thm. 3.1] for `cB`*). Note that also a potential singular left-hand value for $\mathrm{cov}(\mathcal{M})$ or $\mathfrak{d}$ is preserved by the ultrapowers: Theorem 1.11(d) does not require μ to be regular.

2. Cardinal characteristics in extensions without new $<\kappa$-sequences. This section summarizes the technical results introduced in [GKMS21a, Sect. 3].

Lemma 2.1 ([GKMS21a, Lemma 3.1]). *Assume that Q is θ-cc and $<\kappa$-distributive for κ regular uncountable, and let λ be a regular cardinal and R a Borel relation.*

(1) *If $\mathrm{LCU}_R(\lambda)$, then $Q \Vdash \mathrm{LCU}_R(\mathrm{cof}(\lambda))$. So if additionally $\lambda \leq \kappa$ or $\theta \leq \lambda$, then $Q \Vdash \mathrm{LCU}_R(\lambda)$.*
(2) *If $\mathrm{COB}_R(\lambda,\mu)$ and either $\lambda \leq \kappa$ or $\theta \leq \lambda$, then $Q \Vdash \mathrm{COB}_R(\lambda,|\mu|)$. So for any λ, $\mathrm{COB}_R(\lambda,\mu)$ implies $Q \Vdash \mathrm{COB}_R(\min\{|\lambda|,\kappa\},|\mu|)$.*

Lemma 2.2 ([GKMS21a, Lemma 3.2]). *Assume that R is a Borel relation, P' is a complete subforcing of P, λ regular and μ is a cardinal, both preserved in the P-extension.*

(a) *If $P \Vdash \mathrm{LCU}_R(\lambda)$ witnessed by some $\dot{\bar{f}}$, and $\dot{\bar{f}}$ is actually a P'-name, then $P' \Vdash \mathrm{LCU}_R(\lambda)$.*
(b) *If $P \Vdash \mathrm{COB}_R(\lambda,\mu)$ witnessed by some $(\dot{\trianglelefteq},\dot{\bar{g}})$, and $(\dot{\trianglelefteq},\dot{\bar{g}})$ is actually a P'-name, then $P' \Vdash \mathrm{COB}_R(\lambda,\mu)$.*

We now review three properties of cardinal characteristics.

Definition 2.3 ([GKMS21a, Def. 3.3]). Let $\mathfrak{x}$ be a cardinal characteristic.

(1) $\mathfrak{x}$ is $\mathfrak{t}$-*like* if it has the following form: There is a formula $\psi(x)$ (possibly with, e.g., real parameters) absolute between universe extensions that

do not add reals [8], such that $\mathfrak{x}$ is the smallest cardinality λ of a set A of reals such that $\psi(A)$.

All Blass-uniform characteristics are $\mathfrak{t}$-like; other examples are $\mathfrak{p}$, $\mathfrak{t}$, $\mathfrak{u}$, $\mathfrak{a}$ and $\mathfrak{i}$.

(2) $\mathfrak{x}$ is called $\mathfrak{h}$-*like* if it satisfies the same, but with A being a family of sets of reals (instead of just a set of reals).
Note that $\mathfrak{t}$-like implies $\mathfrak{h}$-like, as we can include "the family of sets of reals is a family of singletons" in ψ. Other examples are $\mathfrak{h}$ and $\mathfrak{g}$.

(3) $\mathfrak{x}$ is called $\mathfrak{m}$-*like* if it has the following form: There is a sentence φ (possibly with, e.g., real parameters) such that $\mathfrak{x}$ is the smallest cardinality λ such that $H(\leq\lambda) \vDash \varphi$.

Any infinite $\mathfrak{t}$-like characteristic is $\mathfrak{m}$-like: If ψ witnesses $\mathfrak{t}$-like, then we can use $\varphi = (\exists A)\,[\psi(A) \& (\forall a \in A)\ a \text{ is a real}]$ to get $\mathfrak{m}$-like (since $H(\leq\lambda)$ contains all reals). Other examples are [9] $\mathfrak{m}$, $\mathfrak{m}$(Knaster), etc.

Lemma 2.4 ([GKMS21a, Lemma 3.4]). *Let $V_1 \subseteq V_2$ be models (possibly classes) of set theory (or a sufficient fragment), with V_2 transitive and V_1 either transitive or an elementary submodel of $H^{V_2}(\chi)$ for some large enough regular χ, such that $V_1 \cap \omega^\omega = V_2 \cap \omega^\omega$.*

(a) *If $\mathfrak{x}$ is $\mathfrak{h}$-like, then $V_1 \vDash \mathfrak{x} = \lambda$ implies $V_2 \vDash \mathfrak{x} \leq |\lambda|$.*

In addition, whenever κ is uncountable regular in V_1 and $V_1^{<\kappa} \cap V_2 \subseteq V_1$:

(b) *If $\mathfrak{x}$ is $\mathfrak{m}$-like, then $V_1 \vDash \mathfrak{x} \geq \kappa$ iff $V_2 \vDash \mathfrak{x} \geq \kappa$.*
(c) *If $\mathfrak{x}$ is $\mathfrak{m}$-like and $\lambda < \kappa$, then $V_1 \vDash \mathfrak{x} = \lambda$ iff $V_2 \vDash \mathfrak{x} = \lambda$.*
(d) *If $\mathfrak{x}$ is $\mathfrak{t}$-like and $\lambda = \kappa$, then $V_1 \vDash \mathfrak{x} = \lambda$ implies $V_2 \vDash \mathfrak{x} = \lambda$.*

We apply this to three situations: Boolean ultrapowers, extensions by distributive forcings, and complete subforcings:

Corollary 2.5 ([GKMS21a, Cor. 3.5]). *Assume that κ is uncountable regular, $P \Vdash \mathfrak{x} = \lambda$, and either*

(i) *Q is a P-name for a $<\kappa$-distributive forcing, and we set $P^+ := P * Q$ and $j(\lambda) := \lambda$; or*
(ii) *P is ν-cc for some $\nu < \kappa$, $j : V \to M$ is a complete embedding into a transitive $<\kappa$-closed model M, $\mathrm{cr}(j) \geq \kappa$, and we set $P^+ := j(P)$; or*
(iii) *P is κ-cc, $M \preceq H(\chi)$ is $<\kappa$-closed, and we set $P^+ := P \cap M$ and $j(\lambda) := |\lambda \cap M|$. (So P^+ is a complete subposet of P; and if $\lambda \leq \kappa$ then $j(\lambda) = \lambda$.)*

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(8) Concretely, if $M_1 \subseteq M_2$ are transitive (possibly class) models of a fixed, large fragment of ZFC, with the same reals, then ψ is absolute between M_1 and M_2.

(9) $\mathfrak{m}$ can be characterized as the smallest λ such that there is in $H(\leq\lambda)$ a ccc forcing Q and a family $\bar{D}$ of dense subsets of Q such that "there is no filter $F \subseteq Q$ meeting all D_i" holds.

Then

(a) *If* $\mathfrak{x}$ *is* $\mathfrak{m}$*-like and* $\lambda \geq \kappa$*, then* $P^+ \Vdash \mathfrak{x} \geq \kappa$.
(b) *If* $\mathfrak{x}$ *is* $\mathfrak{m}$*-like and* $\lambda < \kappa$*, then* $P^+ \Vdash \mathfrak{x} = \lambda$.
(c) *If* $\mathfrak{x}$ *is* $\mathfrak{h}$*-like then* $P^+ \Vdash \mathfrak{x} \leq |j(\lambda)|$*. Concretely,*

for (i)*:* $P^+ \Vdash \mathfrak{x} \leq |\lambda|$*;*
for (ii)*:* $P^+ \Vdash \mathfrak{x} \leq |j(\lambda)|$*;*
for (iii)*:* $P^+ \Vdash \mathfrak{x} \leq |\lambda \cap M|$.

(d) *So if* $\mathfrak{x}$ *is* $\mathfrak{t}$*-like and* $\lambda = \kappa$*, then for* (i) *and* (iii) *we get* $P^+ \Vdash \mathfrak{x} = \kappa$.

2.1. On the role of large cardinals in our constructions. It is known that NNR (no-new-reals) extensions of proper classes (such as forcing extensions) will preserve Blass-uniform characteristics in the absence of at least some large cardinals. More specifically:

LEMMA 2.6. *Assume that* $0^\#$ *does not exist. Let* $V_1 \subseteq V_2$ *be transitive class models with the same reals, and assume* $V_1 \models \mathfrak{x} = \lambda$ *for some Blass-uniform* $\mathfrak{x}$*. Then* $V_2 \models \mathfrak{x} = |\lambda|$.

(This is inspired by the deeper observation [Mil98, Prop. 2.1] of Mildenberger, who uses the Covering Lemma [DJ82] for the Dodd–Jensen core model to show that in *cardinality preserving* NNR extensions, a measurable in an inner model is required to change the value of a Blass-uniform characteristic.)

Proof of Lemma 2.6. Fix a bijection in V_1 between the reals and some ordinal α. Assume that in V_2, $X \subseteq \omega^\omega$ witnesses that $\aleph_1 \leq \mathfrak{x} \leq \mu < |\lambda|$. Using the bijection, we can interpret X as a subset of α. According to Jensen's covering lemma in V_2, there is in L (and thus in V_1) some $X' \supseteq X$ such that $|X'| = |X|$ in V_2, in particular $|X'|^{V_2} < \lambda$. Therefore, $|X'|^{V_1} < \lambda$ as well; and, by absoluteness, V_1 thinks that X' witnesses $\mathfrak{x} < \lambda$, a contradiction. ■

Recall the "old" Boolean ultrapower construction `cA*` case (i): Assume that we start with a forcing notion P forcing $\mathfrak{d} = 2^{\aleph_0} = \lambda_6$. We now use the elementary embedding $j = j_7 : V \to M$ with critical point κ_7, and set $P' := j(P)$. As we have seen, P' still forces $\mathfrak{d} = \lambda_6$, but $2^{\aleph_0} = \lambda_7 = |j(\kappa_7)|$.

So let G be a P'-generic filter over V (which is also M-generic). Set $V_1 := M[G]$ and $V_2 := V[G]$. Then V_1 is a $<\kappa_7$-complete submodel of V_2. By elementaricity, $M \models j(P) \Vdash \mathfrak{d} = j(\lambda_6)$. So $V_1 \models \mathfrak{d} = j(\lambda_6)$, whereas $V_2 \models \mathfrak{d} = \lambda_6 < |j(\lambda_6)|$.

Hence, for this specific constellation of models, some large cardinals (at least $0^\#$) are required (for our construction we actually use strongly compact cardinals).

3. Applications

Notation 3.1. (1) Whenever we are investigating a characteristic $\mathfrak{x}$, we write $\lambda_{\mathfrak{x}}$ for the specific value we plan to force to it. For example, in Constellation A of any "old construction", $\lambda_{\mathrm{cov}(\mathcal{N})}$ would be λ_2, whereas in Constellation B it would be λ_3. We remark that we *do not* implicitly *assume* that $P \Vdash \mathfrak{x} = \lambda_{\mathfrak{x}}$ for the P under investigation; it is just an (implicit) declaration of intent.

(2) Whenever we base an argument on one of the old constructions, and say "we can modify the construction to additionally force ...", we implicitly assume that the desired values $\lambda_{\mathfrak{x}}$ for the "old" characteristics satisfy the assumptions we made in the "old" constructions (such as "$\lambda_{\mathfrak{x}}$ is regular").

Recall the following properties of posets.

Definition 3.2. Let $2 \le k < \omega$ and let Q be a poset.

(1) The poset Q is *k-Knaster* if for any uncountable $B \subseteq Q$ there is some uncountable *k-linked* $A \subseteq B$, i.e. any subset of A of size $\le k$ has a lower bound in Q.
(2) The poset Q has *precaliber* $\aleph_1$ if for any uncountable $B \subseteq Q$ there is some uncountable *centered* $A \subseteq B$, i.e. any finite subset of A has a lower bound in Q.

For notational convenience, we declare that "1-Knaster" means "ccc", and "ω-Knaster" means "precaliber $\aleph_1$".

Corollary 2.5 gives us 11 characteristics:

Lemma 3.3. *Given $\aleph_1 \le \lambda_{\mathfrak{m}} < \kappa_9$ regular and $1 \le k_0 \le \omega$, we can modify $P^{\mathsf{cA}*}$ (and also $P^{\mathsf{cB}*}$) so that we additionally force:*

(1) *$\mathfrak{m}(k\text{-Knaster}) = \aleph_1$ for $1 \le k < k_0$,*
(2) *$\mathfrak{m}(k\text{-Knaster}) = \lambda_{\mathfrak{m}}$ for $k \ge k_0$,*
(3) *$\mathfrak{p} \ge \kappa_9$.*

Proof. Start with an appropriate left hand side forcing P. We can modify it to construct a ccc poset P' as in [GKMS21a, Lemma 4.7] when $k_0 < \omega$, or as in [GKMS21a, Lemma 5.5] when $k_0 = \omega$, forcing the same as P and, in addition, $\mathfrak{p} = \mathfrak{b}$, and both (1) and (2) ([10]). Apply Boolean ultrapowers to P' just as in the "old" construction, resulting in P^*. We can apply Corollary 2.5(ii), more specifically the consequences (a) and (b): (b) implies that P^* forces (1) and (2), while (a) implies that P^* forces $\mathfrak{p} \ge \kappa_9$. And just as in the "old" construction, we can use Theorem 1.11 to show that P^* forces the desired values to the Cichoń characteristics. ∎

([10]) For Constellation A(i), $P = P^{\mathsf{cA}}$ is constructed by a matrix iteration, so the small posets in the modification P' should be inserted in a different way, specifically, as in the proof of [BCM21, Thm. 5.4].

If we use $\lambda_{\mathfrak{m}} = \kappa_9$, we already lose control of the Knaster number and only get the following (with the same construction):

LEMMA 3.4. *For $1 \le k_0 \le \omega$, we can modify $P^{\mathtt{cA}*}$ (and also $P^{\mathtt{cB}*}$) so that we additionally force*

(1) $\mathfrak{m}(k\text{-Knaster}) = \aleph_1$ *for* $1 \le k < k_0$,
(2) $\mathfrak{m}(k_0\text{-Knaster}) \ge \kappa_9$.

The following will be used to control $\mathfrak{g}$ in our construction:

LEMMA 3.5 (Blass [Bla89, Thm. 2]). *Let ν be an uncountable regular cardinal and let $(V_\alpha)_{\alpha\le\nu}$ be an increasing sequence of transitive models of ZFC such that*

(i) $\omega^\omega \cap (V_{\alpha+1} \setminus V_\alpha) \ne \emptyset$,
(ii) $(\omega^\omega \cap V_\alpha)_{\alpha<\nu} \in V_\nu$, *and*
(iii) $\omega^\omega \cap V_\nu = \bigcup_{\alpha<\nu} \omega^\omega \cap V_\alpha$.

Then, in V_ν, $\mathfrak{g} \le \nu$.

We now slightly expand ([11]) [GKMS21a, Lemma 6.3]. This lemma will be used to change the values forced to $\mathfrak{g}$ and $\mathfrak{c}$, while preserving the values for Blass-uniform characteristics.

LEMMA 3.6. *Assume the following:*

(1) $\aleph_1 \le \kappa \le \nu \le \mu$, *where κ and ν are regular and* $\mu = \mu^{<\kappa}$.
(2) *P is a κ-cc poset forcing* $\mathfrak{c} > \mu$.
(3) *For some Borel relations R^1_i ($i \in I_1$) on ω^ω and some regular cardinals λ^1_i, P forces* $\mathrm{LCU}_{R^1_i}(\lambda^1_i)$.
(4) *For some Borel relations R^2_i ($i \in I_2$) on ω^ω, λ^2_i regular and cardinals ϑ^2_i: P forces* $\mathrm{COB}_{R^2_i}(\lambda^2_i, \vartheta^2_i)$.
(5) *For some $\mathfrak{m}$-like characteristics $\mathfrak{y}_j$ ($j \in J$) and $\lambda_j < \kappa$, $P \Vdash \mathfrak{y}_j = \lambda_j$.*
(6) *For some $\mathfrak{m}$-like characteristics $\mathfrak{y}'_k$ ($k \in K$), $P \Vdash \mathfrak{y}'_k \ge \kappa$.*
(7) $|I_1 \cup I_2 \cup J \cup K| \le \mu$.

Then there is a complete subforcing P' of P of size μ that forces

(a) $\mathrm{LCU}_{R^1_i}(\lambda^1_i)$ *and* $\mathrm{COB}_{R^2_{i'}}(\lambda^2_{i'}, \vartheta^2_{i'})$ *for all $i \in I_1$ and $i' \in I_2$ such that* $\lambda^1_i, \lambda^2_{i'}, \vartheta^2_{i'} \le \mu$;
(b) $\mathrm{LCU}_{R^1_i}(\nu)$ *and* $\mathrm{COB}_{R^2_{i'}}(\nu,\nu)$ *for all $i \in I_1$ and $i' \in I_2$ such that* $\lambda^1_i, \lambda^2_{i'} > \mu$;
(c) $\mathrm{COB}_{R^2_{i'}}(\lambda^2_{i'}, \mu)$ *for all $i' \in I_2$ such that* $\lambda^2_{i'} \le \nu$, $\mu^{<\lambda^2_{i'}} = \mu$ *and* $\vartheta^2_{i'} > \mu$;

([11]) Compared to [GKMS21a, Lemma 6.3], we just added consequences (b) and (c). These are actually not used explicitly in the rest of the paper; but (c) is used implicitly in the sketch of proof of Theorem 1.3, where we claim that in Constellation A(ii) we can get $\mathfrak{c} = \lambda_5$ singular by applying Lemma 3.6.

(d) $\mathfrak{y}_j = \lambda_j$ *and* $\mathfrak{y}'_k \geq \kappa$ *for all* $j \in J$ *and* $k \in K$*;*
(e) $\mathfrak{c} = \mu$ *and* $\mathfrak{g} \leq \nu$.

Proof. As in the proof of [GKMS21a, Lemma 6.3], construct an increasing sequence $(M_\alpha : \alpha < \nu)$ of $<\kappa$-closed elementary submodels of $H(\chi)$ (for χ regular large enough) of size μ such that

(i) $\mu \cup \{\mu\} \subseteq M_0$ and M_0 contains (as elements) all the objects mentioned in the hypothesis of the lemma (i.e., in the case of an $\mathfrak{m}$-like characteristic, M_0 contains the parameters of the definition);
(ii) $(M_\xi : \xi \leq \alpha) \in M_{\alpha+1}$;
(iii) when $\lambda^1_i > \mu$ ($i \in I_1$), $\lambda^1_i \cap M_{\alpha+1}$ contains an upper bound of $\lambda^1_i \cap M_\alpha$;
(iv) when $\lambda^2_i > \mu$ ($i \in I_2$), $M_{\alpha+1}$ contains a P-name $\dot{\zeta}^i_\alpha$ of a member of ϑ^2_i that is forced to be a $\dot{\trianglelefteq}^i$-upper bound of $\vartheta^2_i \cap M_\alpha$, where $(\dot{\trianglelefteq}^i, \bar{\dot{g}}^i) \in M_0$ is a witness of $\mathrm{COB}_{R^2_i}(\lambda^2_i, \vartheta^2_i)$;
(v) when $\lambda^2_i \leq \nu$, $\mu^{<\lambda^2_i} = \mu$ and $\vartheta^2_i > \mu$ ($i \in I_2$): for any $C \subseteq \vartheta^2_i \cap M_\alpha$ of size $<\lambda^2_i$ there is some P-name in $M_{\alpha+1}$ of a member of ϑ^2_i that is forced to be a $\dot{\trianglelefteq}^i$-upper bound of C;
(vi) $M_{\alpha+1}$ contains a P-name of a real that is forced not to be in the $P \cap M_\alpha$-extension. (This is possible as P forces $\mathfrak{c} > \mu$.)

Set $M := M_\nu = \bigcup_{\alpha<\nu} M_\alpha$, which is also a $<\kappa$-closed elementary submodel of $H(\chi)$ of size μ. As P is κ-cc, $P_\alpha := P \cap M_\alpha$ is a complete subposet of P for any $\alpha \leq \nu$, and it is clear that $P' := P_\nu$ is the direct limit of $(P_\alpha : \alpha < \nu)$.

We show that P' is as required.

Item (a) follows from Lemma 2.2, and (d) follows from Corollary 2.5 (case (iii)).

For (b), note that by (iv), P' forces that $(\dot{h}^i_\alpha : \alpha < \nu)$ is a witness of $\mathrm{COB}_{R^2_i}(\nu, \nu)$ where ν has its usual order and $\dot{h}^i_\alpha := \dot{g}^i_{\dot{\zeta}^i_\alpha}$; and by (iii) $\mathrm{LCU}_{R^1_i}(\nu)$ is obtained "dually" (12).

For (c), by (v), P' forces that $\vartheta^2_i \cap M$ (of size μ) with the partial order $\dot{\trianglelefteq}^i \cap M$ (which is a P'-name) is $<\lambda^2_i$-directed and $(\dot{g}^i_\xi : \xi \in \vartheta^2_i \cap M)$ is a witness of $\mathrm{COB}_{R^2_i}(\lambda^2_i, \mu)$.

For (e), let V_ν be a P'-generic extension and, for each $\alpha < \nu$, let V_α be its intermediate P_α-extension. By (vi) the sequence $(V_\alpha : \alpha \leq \nu)$ satisfies the hypothesis of Lemma 3.5, so $V_\nu \models \mathfrak{g} \leq \nu$. On the other hand, it is clear that $V_\nu \models \mathfrak{c} = \mu$. ∎

Remark 3.7. We cannot preserve $\mathrm{COB}_R(\lambda, \theta)$ when $\lambda > \mu$. For example, if $\lambda > \mu$, then $\mathrm{COB}_R(\lambda, \theta)$ will fail in the P'-extension as it would imply $\mathfrak{b}_R \geq \lambda > \mu = \mathfrak{c}$.

(12) This argument comes from [GKMS21b, Lemma 1.6].

The following two results deal with $\mathfrak{p}$.

LEMMA 3.8 ([GKMS21a, Lemma 7.2]). *Assume $\xi^{<\xi} = \xi$, P is ξ-cc, and set $Q := \xi^{<\xi}$ (ordered by extension). Then P forces that Q^V preserves all cardinals and cofinalities. Assume $P \Vdash \mathfrak{x} = \lambda$ (in particular λ is a cardinal), and let R be a Borel relation.*

(a) *If $\mathfrak{x}$ is $\mathfrak{m}$-like, then $\lambda < \xi$ implies $P \times Q \Vdash \mathfrak{x} = \lambda$, while $\lambda \geq \xi$ implies $P \times Q \Vdash \mathfrak{x} \geq \xi$.*
(b) *If $\mathfrak{x}$ is $\mathfrak{h}$-like, then $P \times Q \Vdash \mathfrak{x} \leq \lambda$.*
(c) *$P \Vdash \mathrm{LCU}_R(\lambda)$ implies $P \times Q \Vdash \mathrm{LCU}_R(\lambda)$.*
(d) *$P \Vdash \mathrm{COB}_R(\lambda, \mu)$ implies $P \times Q \Vdash \mathrm{COB}_R(\lambda, \mu)$.*

LEMMA 3.9 ([DS], [GKMS21a, Lemma 7.3]). *Assume that $\xi = \xi^{<\xi}$ and P is a ξ-cc poset that forces $\xi \leq \mathfrak{p}$. In the P-extension V', let $Q = (\xi^{<\xi})^V$. Then:*

(a) *$P \times Q = P * Q$ forces $\mathfrak{p} = \xi$;*
(b) *if in addition P forces $\xi \leq \mathfrak{p} = \mathfrak{h} = \kappa$, then $P \times Q$ forces $\mathfrak{h} = \kappa$.*

We are now ready to prove the consistency of 13 pairwise different classical characteristics. Note that the following result allows both $\mathfrak{c}$ and either $\mathrm{cov}(\mathcal{M})$ or $\mathfrak{d}$ to be singular.

THEOREM 3.10. *Assume GCH and $\aleph_1 \leq \lambda_{\mathfrak{m}} \leq \lambda_{\mathfrak{p}} \leq \lambda_{\mathfrak{h}} \leq \kappa_9 < \lambda_1 < \kappa_8 < \lambda_2 < \kappa_7 < \lambda_3 \leq \lambda_4 \leq \lambda_5 \leq \lambda_6 \leq \lambda_7 \leq \lambda_8 \leq \lambda_9$ are such that the assumptions of Theorem 1.4 (Constellation A or B) are met, except for the regularity requirement on λ_9, and additionally $\lambda_{\mathfrak{m}}$, $\lambda_{\mathfrak{p}}$ and $\lambda_{\mathfrak{h}}$ are regular and* [13] *$\mathrm{cof}(\lambda_9) \geq \lambda_{\mathfrak{h}}$. Then there is a $\lambda_{\mathfrak{p}}^+$-cc poset P (if $\lambda_{\mathfrak{p}} = \lambda_{\mathfrak{h}}$ we even get ccc) which preserves cofinalities and forces (1) and (2) of Lemma 3.3, and*

$$\mathfrak{p} = \lambda_{\mathfrak{p}}, \quad \mathfrak{h} = \mathfrak{g} = \lambda_{\mathfrak{h}},$$

as well as the old values for the Cichoń characteristics, that is, (cA*) *or* (cB*).

For the convenience of the reader, we repeat here "the assumptions of Theorem 1.4 without λ_9 regular": λ_i regular for $i \neq 5, 6, 9$, and κ_j is strongly compact for $j = 7, 8, 9$, and additionally

Constellation A: either

(a) λ_5 is regular and $\mathrm{cof}(\lambda_6) \geq \lambda_3$, or
(b) λ_6 is regular, $\mathrm{cof}(\lambda_5) \geq \lambda_4$, and there is a strongly compact κ_6 such that $\lambda_3 < \kappa_6 < \lambda_4$.

Constellation B: λ_3 is $\aleph_1$-inaccessible and (ii) holds.

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(13) The $\mathrm{cof}(\lambda_9) \geq \lambda_{\mathfrak{h}}$ is optimal in our situation $\mathfrak{g} = \lambda_{\mathfrak{h}}$, as $\mathrm{cof}(\mathfrak{c}) \geq \mathfrak{g}$.

Proof of Theorem 3.10. Let P^* be the suitable ccc poset obtained in the proof of Lemma 3.3 (or Lemma 3.4 if $\lambda_{\mathfrak{m}} = \kappa_9$), but not for the given λ_9 as value for the continuum, but $\xi := (\lambda_9^{\kappa_9})^+$ instead.

This is a ccc poset of size ξ that forces strong witnesses for the desired values of the Cichoń characteristics (but $\mathfrak{c} = \xi$), and gives the results of Lemma 3.3 (or Lemma 3.4) on the Knaster numbers (and $\mathfrak{p}$).

We now apply Lemma 3.6 with $\kappa = \nu = \lambda_{\mathfrak{h}}$ and $\mu = \lambda_9$. This gives us a complete subposet P' of P^*.

If $\lambda_{\mathfrak{m}} < \lambda_{\mathfrak{h}}$ (and so in particular $\lambda_{\mathfrak{m}} < \kappa_9$), we still get (1) and (2) of Lemma 3.3, and $\mathfrak{p}$ (which was forced by P^* to be $\geq \kappa_9$) is forced to be $\geq \kappa = \lambda_{\mathfrak{h}}$; also $\mathfrak{g}$ is forced to be $\leq \nu = \lambda_{\mathfrak{h}}$, and so it is forced that $\mathfrak{p} = \mathfrak{h} = \mathfrak{g} = \lambda_{\mathfrak{h}}$.

If $\lambda_{\mathfrak{m}} = \lambda_{\mathfrak{h}}$, then we only get $\mathfrak{m}(k_0\text{-Knaster}) \geq \kappa = \lambda_{\mathfrak{h}} = \lambda_{\mathfrak{m}}$, so we get $\lambda_{\mathfrak{m}} \leq \mathfrak{m}(k_0\text{-Knaster}) \leq \mathfrak{p} \leq \mathfrak{g} = \nu = \lambda_{\mathfrak{h}} = \lambda_{\mathfrak{m}}$.

In any case $\mathfrak{c} = \lambda_9$ and the values forced by P' of the other cardinals in Cichoń's diagram are the same values forced by P^*, again with strong witnesses.

If $\lambda_{\mathfrak{p}} = \lambda_{\mathfrak{h}}$, then we are done. So assume that $\lambda_{\mathfrak{p}} < \lambda_{\mathfrak{h}}$. Hence, by Lemmas 3.8 and 3.9, $P := P' \times (\lambda_{\mathfrak{p}}^{<\lambda_{\mathfrak{p}}})$ is as required. It is clear that P forces $\mathfrak{m}(k_0\text{-Knaster}) = \mathfrak{m}(\text{precaliber}) = \lambda_{\mathfrak{m}}$ when $\lambda_{\mathfrak{m}} < \lambda_{\mathfrak{p}}$, but the same happens when $\lambda_{\mathfrak{m}} = \lambda_{\mathfrak{p}}$ because P would force $\lambda_{\mathfrak{m}} \leq \mathfrak{m}(k_0\text{-Knaster}) \leq \mathfrak{m}(\text{precaliber}) \leq \mathfrak{p} \leq \lambda_{\mathfrak{m}}$. ■

4. Reducing gaps (or getting rid of them). As mentioned in Remark 1.5, we can choose right side Cichoń characteristics rather arbitrarily or even choose them to be equal (equality allows a construction from fewer compact cardinals). However, large gaps were required between some left side cardinals. We deal with this problem now, and show that we can assign reasonably arbitrary regular values to all characteristics (such as $\lambda_i = \aleph_{i+1}$), and in particular set any "reasonable selection" of them equal.

Let us introduce notation to describe this effect:

DEFINITION 4.1. Let $\bar{\mathfrak{x}} = (\mathfrak{x}_i : i < n)$ be a finite sequence of cardinal characteristics (i.e., of definitions). Say that $\bar{\mathfrak{x}}$ is a *$<$-consistent sequence* if the statement $\mathfrak{x}_0 < \cdots < \mathfrak{x}_{n-1}$ is consistent with ZFC (perhaps modulo large cardinals).

A consistent sequence $\bar{\mathfrak{x}}$ is *$\leq$-consistent* if, in the previous chain of inequalities, it is consistent to replace any desired instance or instances of $<$ with $=$. More formally, for any interval partition $(I_k : k < m)$ of $\{0, \ldots, n-1\}$, it is consistent that $\mathfrak{x}_i = \mathfrak{x}_j$ for any $i, j \in I_k$, and $\mathfrak{x}_i < \mathfrak{x}_j$ whenever $i \in I_k$, $j \in I_{k'}$ and $k < k' < m$.

For example, the sequence

$$(\aleph_1, \mathrm{add}(\mathcal{N}), \mathrm{cov}(\mathcal{N}), \mathfrak{b}, \mathrm{non}(\mathcal{M}), \mathrm{cov}(\mathcal{M}), \mathfrak{d})$$

is $\leq$-consistent, as also is

$$(\aleph_1, \mathrm{add}(\mathcal{N}), \mathfrak{b}, \mathrm{cov}(\mathcal{N}), \mathrm{non}(\mathcal{M}), \mathrm{cov}(\mathcal{M}))$$

(see Theorem 1.3). Previously, it had not been known whether the sequences of ten Cichoń characteristics from [GKS19, BCM21, KST19] are $\leq$-consistent: It is not immediate that cardinals on the left side can be equal while separating everything on the right side. As described in Remark 1.5, the reason is that to separate cardinals on the right side, it is necessary to have a strongly compact cardinal between the dual pair of cardinals on the left, thus the left side gets separated as well. But thanks to the collapsing method of this section, we can equalize cardinals on the left as well. As a result, we obtain the following ([14]):

LEMMA 4.2. *The sequences*

$$(\aleph_1, \mathfrak{m}, \mathfrak{p}, \mathrm{add}(\mathcal{N}), \mathrm{cov}(\mathcal{N}), \mathfrak{b}, \mathrm{non}(\mathcal{M}), \mathrm{cov}(\mathcal{M}), \mathfrak{d}, \mathrm{non}(\mathcal{N}), \mathrm{cof}(\mathcal{N}), \mathfrak{c}) \text{ and}$$
$$(\aleph_1, \mathfrak{m}, \mathfrak{p}, \mathrm{add}(\mathcal{N}), \mathfrak{b}, \mathrm{cov}(\mathcal{N}), \mathrm{non}(\mathcal{M}), \mathrm{cov}(\mathcal{M}), \mathrm{non}(\mathcal{N}), \mathfrak{d}, \mathrm{cof}(\mathcal{N}), \mathfrak{c})$$

are $\leq$-consistent (*modulo large cardinals*).

REMARK 4.3. Note that we do not claim (nor conjecture) that the collapse forcings we use for this result will preserve the value of $\mathfrak{h}$ (which is neither $\mathfrak{m}$-like nor $\mathfrak{t}$-like).

We only know, by Lemma 2.4(a), that the collapse will not increase $\mathfrak{h}$. Accordingly, if we start out with $\mathfrak{p} = \mathfrak{h}$, then the resulting model will satisfy this as well (as $\mathfrak{p} \leq \mathfrak{h}$ in ZFC). So for any constellation of the characteristics in Lemma 4.2 we can additionally get $\mathfrak{p} = \mathfrak{h}$.

In contrast, using the methods of this paper we do not know how to get $\mathfrak{p} < \mathfrak{h}$ (and in particular $\mathfrak{p} < \mathfrak{h} = \mathrm{add}(\mathcal{N})$) in these constellations (apart of course from the constellations we already dealt with in the preceding section). Note that we cannot just apply Lemma 3.9 *after* collapsing to get the desired Cichoń values, as the collapses are not ξ-cc (where ξ is the desired value for $\mathfrak{p}$).

We start with the following well-known result:

LEMMA 4.4 (Easton's lemma). *Let κ be an uncountable cardinal, P a κ-cc poset and Q a $<\kappa$-closed poset. Then P forces that Q is $<\kappa$-distributive.*

Proof. For successor cardinals, this is proved in [Jec03, Lemma 15.19], but the same argument is valid for any regular cardinal. Singular cardinals are also fine because, for κ singular, $<\kappa$-closed implies $<\kappa^+$-closed. ■

([14]) Each sequence yields 2^{11} many consistency results (not all of them new, obviously; CH is one of them).

To prove Lemma 4.2, we use the following:

ASSUMPTION 4.5.

(1) κ is regular uncountable.
(2) $\theta \geq \kappa$, $\theta = \theta^{<\kappa}$.
(3) P is κ-cc and forces that $\mathfrak{x} = \lambda$ for some characteristic $\mathfrak{x}$ (so in particular λ is a cardinal in the P-extension).
(4) Q is $<\kappa$-closed.
(5) $P \Vdash Q$ is θ^+-cc ([15]).
(6) We set $P^+ := P \times Q = P * Q$. We call the P^+-extension V'' and the intermediate P-extension V'. (We will actually have $|Q| = \theta$, which implies (5)).

Let us list a few simple facts:

(P1) In V', all V-cardinals $\geq\kappa$ are still cardinals, and Q is a $<\kappa$-distributive forcing (due to Easton's lemma). So we can apply Lemma 2.1 and Corollary 2.5.
(P2) Let μ be the successor (in V or equivalently in V') of θ. So in V', Q is μ-cc and preserves all cardinals $\leq \kappa$ as well as all cardinals $\geq \mu$.
(P3) So if $V \models$ "$\kappa \leq \nu \leq \theta$", then in V'', $\kappa \leq |\nu| < \mu$. The V'' successor of κ is $\leq \mu$.

We now apply it to a collapse:

LEMMA 4.6. *Let R be a Borel relation, κ be regular, $\theta > \kappa$, $\theta^{<\kappa} = \theta$, P κ-cc, and set $Q :=$ Coll(κ, θ), i.e., the set of partial functions $f : \kappa \to \theta$ of size $< \kappa$. Then:*

(a) *$P \times Q$ forces $|\theta| = \kappa$.*
(b) *If P forces that λ is a cardinal then*

$$P \times Q \Vdash |\lambda| = \begin{cases} \kappa & \textit{if (in } V\textit{) } \kappa \leq \lambda \leq \theta, \\ \lambda & \textit{otherwise.} \end{cases}$$

(c) *If $\mathfrak{x}$ is $\mathfrak{m}$-like, $\lambda < \kappa$ and $P \Vdash \mathfrak{x} = \lambda$, then $P \times Q \Vdash \mathfrak{x} = \lambda$. In the case when $\mathfrak{x}$ is $\mathfrak{t}$-like, it is enough to assume $\lambda \leq \kappa$.*
(d) *If $\mathfrak{x}$ is $\mathfrak{m}$-like and $P \Vdash \mathfrak{x} \geq \kappa$, then $P \times Q \Vdash \mathfrak{x} \geq \kappa$.*
(e) *If R is a Borel relation then*
   (i) *$P \Vdash$ "λ regular and $\mathrm{LCU}_R(\lambda)$" implies $P \times Q \Vdash \mathrm{LCU}_R(|\lambda|)$.*
   (ii) *$P \Vdash$ "λ is regular and $\mathrm{COB}_R(\lambda, \mu)$" implies $P \times Q \Vdash \mathrm{COB}_R(|\lambda|, |\mu|)$.*

Proof. As mentioned, Assumption 4.5 is met; in particular, P forces that $\check{Q}$ is $< \kappa$-distributive (by 4.5(P1)), so we can use Lemma 2.1 and Corol-

([15]) That is, P forces that all antichains of Q have size $\leq\theta$.

lary 2.5. Also note that whenever $\kappa < \lambda \leq \theta$ and $P \Vdash$ "λ is regular", $P \times Q$ forces $\mathrm{cof}(\lambda) = \kappa = |\lambda|$. ■

So we can start, e.g., with a forcing P_0 as in Theorem 3.10: As we can just set $\mathfrak{h} := \mathfrak{p}$, we can assume P_0 is ccc, and P_0 forces strictly increasing values to the characteristics in the first, say, sequence of Lemma 4.2.

We now pick some $\kappa_0 < \theta_0$, satisfying $\lambda_{\mathfrak{p}} < \kappa_0$ and the assumptions of the previous lemma, i.e., κ_0 is regular and $\theta_0^{<\kappa_0} = \theta_0$. Let Q_0 be the collapse of θ_0 to κ_0, a forcing of size θ_0. So $P_1 := P_0 \times Q_0$ is θ_0^+-cc and, according to the previous lemma, still forces the "same" values (and in fact strong witnesses) to the Cichoń characteristics (including the case that any value λ_i with $\kappa_0 < \lambda_i \leq \theta_0$ is collapsed to $|\lambda_i| = \kappa_0$). The $\mathfrak{m}$-like characteristics below κ_0, that is, $\mathfrak{p}$ and, e.g., $\mathfrak{m}$, are also unchanged.

We now pick another pair $\theta_0 < \kappa_1 < \theta_1$ (with the same requirements) and take the product of P_1 with the collapse Q_1 of θ_1 to κ_1, etc.

In the end, we get $P_0 \times Q_0 \times \cdots \times P_n \times Q_n$. Each characteristic which by P was forced to have value λ is now forced to have value $|\lambda|$, which is κ_m if $\kappa_m \leq \lambda \leq \theta_m$ for some m, and λ otherwise. This immediately gives

Proof of Lemma 4.2. We start with GCH, and construct an initial ccc forcing P_0, according to Theorem 3.10, to already result in the desired (in)equalities between $\aleph_1, \mathfrak{m}, \mathfrak{p} = \mathfrak{h}$, such that we get pairwise different regular Cichoń values λ_i, and $\mathfrak{p} < \mathrm{add}(\mathcal{N})$.

Let $(I_m)_{m \in M}$ be the interval partition of the sequence $(\mathfrak{p}, \mathrm{add}(\mathcal{N}), \ldots, \mathfrak{c})$ indicating which characteristics we want to identify, and let $S := \{m \in M : |I_m| > 1\}$. For each $m \in M$, let κ_m be the value of the smallest characteristic in I_m, and θ_m the largest. Note that $\kappa_m \leq \theta_m < \kappa_{m+1}$, and $\kappa_m < \theta_m$ iff $m \in S$. Then $P_0 \times \prod_{m \in S} Q_m$ forces that all characteristics in I_m (for all $m < M$) have value κ_m, as desired.

The only case that might require some elaboration is that one of the intervals contains $\mathfrak{p}$; i.e., we desire $\mathfrak{p} = \mathrm{add}(\mathcal{N}) = \cdots = \mathfrak{x}$, where $\mathfrak{x}$ is the largest Cichoń characteristic in the interval, which gets assigned some value λ_i by the initial forcing; whereas $\mathfrak{p}$ is assigned some value $\lambda_{\mathfrak{p}}$. So we use the collapse from $\theta = \lambda_i$ to $\kappa = \lambda_{\mathfrak{p}}$. This collapse results in $\mathfrak{p} = \lambda_{\mathfrak{p}}$ by Lemma 4.6(c) (recall that $\mathfrak{p}$ is $\mathfrak{t}$-like), and in $\mathrm{add}(\mathcal{N}) = \mathfrak{x} = \lambda_{\mathfrak{p}}$ by Lemma 4.6(e). ■

We can use the same method to assign specific values to the characteristics. We start with a simple example, and then give a more general theorem.

Example 4.7. We can assign the values $\aleph_1, \ldots, \aleph_{12}$ to the first sequence of Lemma 4.2 (as in Figure 5). We can do the same for the second sequence.

Proof. Again, start with GCH and P_0 forcing the desired values for $\mathfrak{m}$ and $\mathfrak{p}$ (now $\aleph_2$ and $\aleph_3$) and pairwise distinct regular Cichoń values λ_i. Then pick $\kappa_0 = \lambda_{\mathfrak{p}}^+ = \aleph_4$ and $\theta_0 = \lambda_1$ (which then becomes $\aleph_4$ after the collapse).

Then set $\kappa_1 = \lambda_1^+$ (which would be $\aleph_5$ after the first collapse), and $\theta_1 = \lambda_2$, etc. ∎

We can of course just as well assign the values $(\aleph_{\omega\cdot m+1})_{1\le m\le 12}$ instead of $(\aleph_m)_{1\le m\le 12}$, and also get certain singular values for $\mathfrak{d}$ and $\mathfrak{c}$. It is a bit awkward to make precise the (not entirely correct) claim "we can assign whatever reasonable value we want"; nevertheless we will try to do just that in the following (at first for the case $\alpha_1 < \alpha_2 < \alpha_3$; as explained in Remark 4.10, there are variants of the theorem which allow $\alpha_1 = \alpha_2$ and/or $\alpha_2 = \alpha_3$).

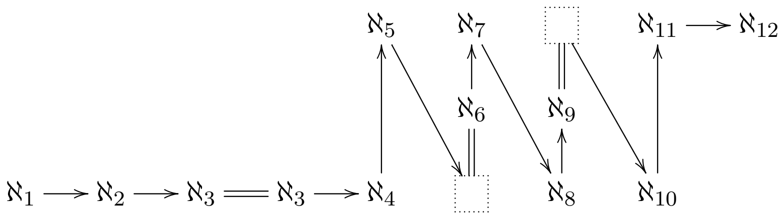


Fig. 5. A possible assignment for Figure 2: $\mathfrak{m} = \aleph_2$, $\mathfrak{p} = \mathfrak{h} = \aleph_3$, $\lambda_i = \aleph_{3+i}$ for $i = 1, \dots, 9$. (Note that with the method of this section we cannot get $\mathfrak{p} < \mathfrak{h}$.)

We first give the result for Constellation A(i). See below (Theorems 4.14, 4.15) for the other constellations.

THEOREM 4.8. *Assume GCH and* $1 \le k_0 \le \omega$. *Let* $1 \le \alpha_{\mathfrak{m}} \le \alpha_{\mathfrak{p}} \le \alpha_1 < \alpha_2 < \alpha_3 \le \alpha_4 \le \dots \le \alpha_9$ *be ordinals and assume that there are strongly compact cardinals* $\kappa_9 < \kappa_8 < \kappa_7$ *such that*

(i) $\aleph_{\alpha_{\mathfrak{p}}} \le \kappa_9$, $\aleph_{\alpha_1} < \kappa_8$ *and* $\aleph_{\alpha_2} < \kappa_7$;
(ii) *for* $i = 1, 2, 3$, $\aleph_{\beta_{i-1}+(\alpha_i-\alpha_{i-1})}$ *is regular* (16), *where*

$$\beta_i := \max\{\alpha_i, \kappa_{10-i} + 1\} \quad \textit{and} \quad \alpha_0 = \beta_0 = 0;$$

(iii) *for* $i \ge 4$, $i \ne 6, 9$, $\aleph_{\beta_3+(\alpha_i-\alpha_3)}$ *is regular;*
(iv) $\operatorname{cof}(\aleph_{\beta_3+(\alpha_6-\alpha_3)}) \ge \aleph_{\beta_3}$;
(v) $\aleph_{\alpha_{\mathfrak{m}}}$ *and* $\aleph_{\alpha_{\mathfrak{p}}}$ *are regular; and*
(vi) $\operatorname{cof}(\aleph_{\beta_3+(\alpha_9-\alpha_3)}) \ge \aleph_{\alpha_{\mathfrak{p}}}$.

Then we get a poset P *which forces* (1) *and* (2) *of Lemma* 3.3 *for* $\lambda_{\mathfrak{m}} = \aleph_{\alpha_{\mathfrak{m}}}$, *and*

$$\mathfrak{p} = \mathfrak{g} = \aleph_{\alpha_{\mathfrak{p}}}, \quad \operatorname{add}(\mathcal{N}) = \aleph_{\alpha_1}, \quad \operatorname{cov}(\mathcal{N}) = \aleph_{\alpha_2}, \quad \mathfrak{b} = \aleph_{\alpha_3}, \quad \operatorname{non}(\mathcal{M}) = \aleph_{\alpha_4},$$
$$\operatorname{cov}(\mathcal{M}) = \aleph_{\alpha_5}, \quad \mathfrak{d} = \aleph_{\alpha_6}, \quad \operatorname{non}(\mathcal{N}) = \aleph_{\alpha_7}, \quad \operatorname{cof}(\mathcal{N}) = \aleph_{\alpha_8}, \quad \mathfrak{c} = \aleph_{\alpha_9},$$

as well as

(16) See Discussion 4.9 for an analysis of this assumption.

$$\aleph_\xi = \begin{cases} (\aleph_\xi)^V & \textit{if } \xi \leq \alpha_1, \\ (\aleph_{\beta_1+(\xi-\alpha_1)})^V & \textit{if } \alpha_1 < \xi \leq \alpha_2, \\ (\aleph_{\beta_2+(\xi-\alpha_2)})^V & \textit{if } \alpha_2 < \xi \leq \alpha_3, \\ (\aleph_{\beta_3+(\xi-\alpha_3)})^V & \textit{if } \alpha_3 < \xi. \end{cases}$$

Before giving the proof, we more verbosely describe some aspects of the hypotheses:

Discussion 4.9. (1) In (ii), for $i = 1, 2, 3$, "$\aleph_{\beta_{i-1}+(\alpha_i-\alpha_{i-1})}$ is regular" is equivalent to saying that α_i is either a successor ordinal or a weakly inaccessible larger than β_{i-1}. In this case β_i is either a successor ordinal or weakly inaccessible, so $\aleph_{\beta_i}$ is regular.

(2) In (iii), "$\aleph_{\beta_3+(\alpha_i-\alpha_3)}$ is regular" is equivalent to saying that one of the following cases holds:

- $\alpha_i > \alpha_3$ and α_i is either a successor ordinal or a weakly inaccessible larger than β_3; or
- $\alpha_i = \alpha_3$ (since then $\aleph_{\beta_3}$ is regular due to (1)).

(3) In relation to (iv) and (vi), whenever $\aleph_{\beta_3} \geq \kappa$, $\operatorname{cof}(\aleph_{\beta_3+(\alpha_i-\alpha_3)}) \geq \kappa$ is equivalent to saying that one of the following cases holds:

- $\alpha_i > \alpha_3$ and α_i is either a successor ordinal or a limit ordinal with cofinality $\geq \kappa$; or
- $\alpha_i = \alpha_3$.

Proof of Theorem 4.8. For $4 \leq i \leq 9$ put $\beta_i := \beta_3 + (\alpha_i - \alpha_3)$. Also set $\lambda_\mathfrak{m} := \aleph_{\alpha_\mathfrak{m}}$, $\lambda_\mathfrak{p} := \aleph_{\alpha_\mathfrak{p}}$ and $\lambda_i := \aleph_{\beta_i}$ for $1 \leq i \leq 9$. Note that λ_i is regular for $i \neq 6, 9$ (see (1) and (2) above), $\operatorname{cof}(\lambda_6) \geq \lambda_3$, $\operatorname{cof}(\lambda_9) \geq \lambda_\mathfrak{p}$ and $\lambda_\mathfrak{m} \leq \lambda_\mathfrak{p} \leq \kappa_9 < \lambda_1 < \kappa_8 < \lambda_2 < \kappa_7 < \lambda_3 \leq \lambda_4 \leq \cdots \leq \lambda_9$. Let P be the ccc poset corresponding to Theorem 3.10 (the modification of $P^{\mathtt{cA}*}$ with $\lambda_\mathfrak{h} = \lambda_\mathfrak{p}$).

Step 1. In the case $\kappa_9 < \alpha_1$ we have $\beta_1 = \alpha_1$, so let $P_1 := P$; in the case $\alpha_1 \leq \kappa_9$ we have $\beta_1 = \kappa_9 + 1$ and $\lambda_1 = \kappa_9^+$. Put $\kappa_1 := \aleph_{\alpha_1}$ and $P_1 := P \times \operatorname{Coll}(\kappa_1, \lambda_1)$. It is clear that κ_1 is regular and $\kappa_1 < \lambda_1$, so, by Lemma 4.6, P_1 forces $\operatorname{add}(\mathcal{N}) = \aleph_{\alpha_1}$ and that the values of the other cardinal characteristics are the same as in the P-extension (also note that P_1 forces $\mathfrak{g} \leq |\lambda_\mathfrak{p}| = \lambda_\mathfrak{p}$ by Corollary 2.5(i)(c), so equality holds since $\mathfrak{p} \leq \mathfrak{g}$ in ZFC). Moreover, P_1 forces

$$\aleph_\xi = \begin{cases} (\aleph_\xi)^V & \text{if } \xi \leq \alpha_1, \\ (\aleph_{\beta_1+(\xi-\alpha_1)})^V & \text{if } \alpha_1 < \xi. \end{cases}$$

Note that this is also valid in the case $\kappa_9 < \alpha_1$ (where $\beta_1 = \alpha_1$). In particular, for any $\xi \geq \kappa_8$, P_1 forces $\aleph_\xi = \aleph_\xi^V$ because, in the ground model, κ_8 is an $\aleph$-fixed point larger than β_1.

Step 2. In the case $\kappa_8 < \alpha_2$ put $P_2 := P_1$; otherwise, we have $\beta_2 = \kappa_8 + 1$ and $\lambda_2 = \kappa_8^+$. Set $\kappa_2 := (\aleph_{\beta_1+(\alpha_2-\alpha_1)})^V$ and $P_2 := P_1 \times \mathrm{Coll}(\kappa_2, \lambda_2)$. It is clear that $\kappa_2 < \lambda_2$, so Lemma 4.6 applies, i.e., P_2 forces $\mathrm{cov}(\mathcal{N}) = \kappa_2$ and that the values of the other characteristics are the same as in the P_1-extension. Also note that P_1 forces $\kappa_2 = \aleph_{\alpha_2}$, and this value remains unaltered in the P_2-extension. Furthermore P_2 forces

$$\aleph_\xi = \begin{cases} (\aleph_\xi)^{V^{P_1}} & \text{if } \xi \leq \alpha_2, \\ (\aleph_{\beta_2+(\xi-\alpha_2)})^{V^{P_1}} & \text{if } \alpha_2 < \xi, \end{cases}$$

hence it forces

$$\aleph_\xi = \begin{cases} (\aleph_\xi)^V & \text{if } \xi \leq \alpha_1, \\ (\aleph_{\beta_1+(\xi-\alpha_1)})^V & \text{if } \alpha_1 < \xi \leq \alpha_2, \\ (\aleph_{\beta_1+((\beta_2+(\xi-\alpha_2))-\alpha_1)})^V & \text{if } \alpha_2 < \xi. \end{cases}$$

This is also valid in the case $\kappa_8 < \alpha_2$. In fact, since $\alpha_1 < \kappa_8$ we find in V that $\beta_2 - \alpha_1 = \beta_2$ and $\beta_1 + \beta_2 = \beta_2$, so $\beta_1 + (\beta_2 + (\xi - \alpha_2)) - \alpha_1 = \beta_2 + (\xi - \alpha_2)$. Hence, in the case $\xi > \alpha_2$, P_2 forces $\aleph_\xi = (\aleph_{\beta_2+(\xi-\alpha_2)})^V$. In particular, P_2 forces $\aleph_\xi = \aleph_\xi^V$ for any $\xi \geq \kappa_7$.

Step 3. In the case $\kappa_7 < \alpha_3$, set $P_3 := P_2$. Otherwise, set $\kappa_3 := (\aleph_{\beta_2+(\alpha_3-\alpha_2)})^V$ and $P_3 := P_2 \times \mathrm{Coll}(\kappa_3, \lambda_3)$.

Note that P_3 forces $\mathfrak{b} = \kappa_3 = \aleph_{\alpha_3}$ and that the other values are the same as the ones forced by P_2. Hence, P_3 is as desired, e.g., $\mathrm{non}(\mathcal{M}) = \lambda_4 = \aleph_{\beta_4}^V = \aleph_{\alpha_4}$. Moreover, P_3 forces

$$\aleph_\xi = \begin{cases} (\aleph_\xi)^{V^{P_2}} & \text{if } \xi \leq \alpha_3, \\ (\aleph_{\beta_3+(\xi-\alpha_3)})^{V^{P_2}} & \text{if } \alpha_3 < \xi, \end{cases}$$

and therefore

$$\aleph_\xi = \begin{cases} (\aleph_\xi)^V & \text{if } \xi \leq \alpha_1, \\ (\aleph_{\beta_1+(\xi-\alpha_1)})^V & \text{if } \alpha_1 < \xi \leq \alpha_2, \\ (\aleph_{\beta_2+(\xi-\alpha_2)})^V & \text{if } \alpha_2 < \xi \leq \alpha_3, \\ (\aleph_{(\beta_2+((\beta_3+(\xi-\alpha_3))-\alpha_2)})^V & \text{if } \alpha_3 < \xi. \end{cases}$$

Note that this is also valid in the case $\kappa_7 < \alpha_3$. Since $\alpha_2 < \kappa_7$ we have in V that $\beta_3 - \alpha_2 = \beta_3$ and $\beta_2 + \beta_3 = \beta_3$, so whenever $\xi > \alpha_3$, P_3 forces $\aleph_\xi = (\aleph_{\beta_3+(\xi-\alpha_3)})^V$. ■

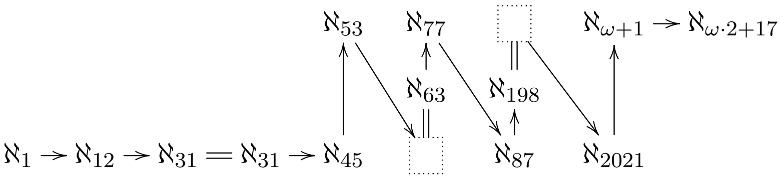
REMARK 4.10. Theorem 4.8 also holds when $\alpha_1 \leq \alpha_2 \leq \alpha_3$, but depending on the equalities the proof changes a bit. For example, in the case $\alpha_1 = \alpha_2 < \alpha_3$, the idea is first to collapse $\lambda_2 := \aleph_{\beta_2}$ to $\kappa_1 := \aleph_{\alpha_1}$ (as in Step 1) and then (possibly) collapse $\lambda_3 := \aleph_{\beta_3}$ to κ_3 (as in Step 3).

For successor cardinals, the assumptions of this theorem are trivially met, so we get the following simpler form:

COROLLARY 4.11. *Assume GCH. Let* $1 \leq k_0 \leq \omega$, *let* $1 \leq \alpha_{\mathfrak{m}} \leq \alpha_{\mathfrak{p}} \leq \alpha_1 \leq \cdots \leq \alpha_9$ *be a sequence of successor ordinals, and* $\kappa_9 < \kappa_8 < \kappa_7$ *compact cardinals with* $\kappa_9 \geq \alpha_3$. *Then there is a poset* P *forcing values to the various characteristics as in the previous theorem.*

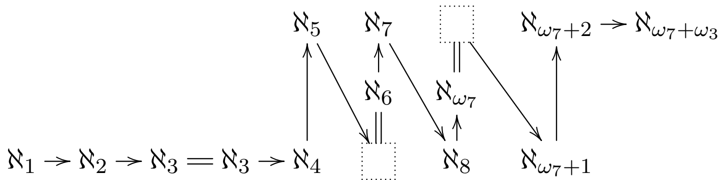
(Note that in this case, $\beta_i = \kappa_{10-i} + 1$ for $i = 1, 2, 3$.)

Let us give some concrete examples, where we give concrete values for the diagram of Figure 2. The simple corollary shows that, e.g., the following is consistent:



Using the more general theorem, we also get examples with singular $\mathfrak{d}$ and $\mathfrak{c}$:

EXAMPLE 4.12. The following is consistent:



Instead of ω_7, we could also use, e.g., any ω_n for $n \geq 7$, $n \in \omega$, and $\omega_n + \omega_k$ instead of $\omega_7 + \omega_3$ for $k \geq 3$.

Proof. We show the general case assuming $7 \leq n < \omega$ and $3 \leq k < \omega$. We use the following parameters for the theorem:

- $\alpha_{\mathfrak{m}} := 2$, $\alpha_{\mathfrak{p}} := 3$, $\alpha_j := 3 + j$ for $1 \leq j \leq 5$.
- From these values we already know that in the extension we will get (according to the last part of the theorem, and as $\alpha_3 = 6$ and $\beta_3 = \kappa_7 + 1$)
$$\aleph_n = \aleph^V_{\kappa_7+1+(n-6)} = \left(\aleph_{\kappa_7+(n-5)}\right)^V =: \alpha_6,$$
therefore
$$\aleph_{\omega_n} = \aleph_{\alpha_6} = \left(\aleph_{\kappa_7+1+(\alpha_6-6)}\right)^V = \left(\aleph_{\alpha_6-6}\right)^V = \left(\aleph_{\alpha_6}\right)^V.$$
Note that this satisfies the condition $\mathrm{cof}(\alpha_6) = \alpha_6 \geq \aleph_{\beta_3} = \kappa_7^+$ (in the ground model); and in the extension we get $\mathfrak{d} = \aleph_{\alpha_6} = \aleph_{\omega_n}$.
- $\alpha_7 := \alpha_6 + 1$ and $\alpha_8 := \alpha_7 + 1$.

- Calculate the ordinal $\aleph_k$ of the extension and call it β (which has cofinality $\geq \aleph_3$ in V, as the cofinality in the extension is $\geq \aleph_3$), and set $\alpha_9 := \alpha_6+\beta$, which is equal to $\omega_n+\omega_k$ in the final extension. ∎

REMARK 4.13. In this example, $\aleph_{\omega_n}$ for $n<6$ is impossible as value for $\mathfrak{d}$, as $\mathrm{cof}(\mathfrak{d}) \geq \mathfrak{b}$ in ZFC.

This leaves the case $\mathfrak{d}=\aleph_{\omega_6}$, which is probably consistent but which we cannot get with the theorem: Using calculations as above we find that the $\aleph_{\omega_6}$ in the extension is $(\aleph_\gamma)^V$ for $\gamma=(\kappa_8^{++})^V$, which does not satisfy $\mathrm{cof}(\gamma)>\kappa_7$ (in the ground model).

We could set $\alpha_6 := \aleph_\gamma$ for $\gamma=\kappa_7^+$ (this has sufficient cofinality), but note that this γ is collapsed in the extension, so in the extension $\mathfrak{d}$ will have the form $\aleph_\gamma$ with γ of cofinality and cardinality ω_6, but $\gamma \neq \omega_6=(\kappa_8^{++})^V$.

We now add the variants of the theorem for Constellations $\texttt{cA}^*$ (ii) and $\texttt{cB}^*$ (the same remarks about $\leq$-consistency apply).

THEOREM 4.14. *Assume GCH and* $1 \leq k_0 \leq \omega$. *Let* $1 \leq \alpha_{\mathfrak{m}} \leq \alpha_{\mathfrak{p}} \leq \alpha_1 < \alpha_2 < \alpha_3 \leq \alpha_4 \leq \cdots \leq \alpha_9$ *be ordinals and assume that there are strongly compact cardinals* $\kappa_9<\kappa_8<\kappa_7<\kappa_6$ *such that*

(i) $\alpha_{\mathfrak{p}} \leq \kappa_9$, $\alpha_1<\kappa_8$, $\alpha_2<\kappa_7$ *and* $\alpha_3<\kappa_6$;
(ii) *for* $i=1,2,3,4$, $\aleph_{\beta_{i-1}+(\alpha_i-\alpha_{i-1})}$ *is regular, where* $\beta_i := \max\{\alpha_i, \kappa_{10-i}+1\}$ *and* $\alpha_0=\beta_0=0$;
(iii) *for* $i=6,7,8$, $\aleph_{\beta_4+(\alpha_i-\alpha_4)}$ *is regular;*
(iv) $\mathrm{cof}(\aleph_{\beta_4+(\alpha_5-\alpha_4)}) \geq \aleph_{\beta_4}$;
(v) $\aleph_{\alpha_{\mathfrak{m}}}$ *and* $\aleph_{\alpha_{\mathfrak{p}}}$ *are regular; and*
(vi) $\mathrm{cof}(\aleph_{\beta_4+(\alpha_9-\alpha_4)}) \geq \aleph_{\alpha_{\mathfrak{p}}}$.

Then we get a poset P *as in the previous theorem, which also forces*

$$\aleph_\xi = \begin{cases} (\aleph_\xi)^V & \text{if } \xi \leq \alpha_1, \\ (\aleph_{\beta_1+(\xi-\alpha_1)})^V & \text{if } \alpha_1<\xi\leq\alpha_2, \\ (\aleph_{\beta_2+(\xi-\alpha_2)})^V & \text{if } \alpha_2<\xi\leq\alpha_3, \\ (\aleph_{\beta_3+(\xi-\alpha_3)})^V & \text{if } \alpha_3<\xi\leq\alpha_4, \\ (\aleph_{\beta_4+(\xi-\alpha_4)})^V & \text{if } \alpha_4<\xi. \end{cases}$$

THEOREM 4.15. *With the same assumptions as in Theorem* 4.14, *if in addition* β_3 *is* <u>*not*</u> *the successor of an ordinal with countable cofinality then there is a poset that forces* (1) *and* (2) *of Lemma* 3.3 *for* $\lambda_{\mathfrak{m}}=\aleph_{\alpha_{\mathfrak{m}}}$ *and*

$$\mathfrak{p}=\mathfrak{g}=\aleph_{\alpha_{\mathfrak{p}}},\quad \mathrm{add}(\mathcal{N})=\aleph_{\alpha_1},\quad \mathfrak{b}=\aleph_{\alpha_2},\quad \mathrm{cov}(\mathcal{N})=\aleph_{\alpha_3},\quad \mathrm{non}(\mathcal{M})=\aleph_{\alpha_4},$$
$$\mathrm{cov}(\mathcal{M})=\aleph_{\alpha_5},\quad \mathrm{non}(\mathcal{N})=\aleph_{\alpha_6},\quad \mathfrak{d}=\aleph_{\alpha_7},\quad \mathrm{cof}(\mathcal{N})=\aleph_{\alpha_8},\quad \mathfrak{c}=\aleph_{\alpha_9}.$$

Moreover, this poset forces that $\aleph_\xi$ *(for any ordinal* ξ*) is evaluated as in the conclusion of Theorem* 4.14.

5. Getting singular values by adding randoms. We can now come back to examples (1) and (2) of Subsection 1.3.

Brendle (private communication) proved: If λ is uncountable then adding λ-many random reals forces $\mathrm{cov}(\mathcal{N}) \geq \mathrm{cof}([\lambda]^{\leq\aleph_0}) := \mathrm{cof}([\lambda]^{\leq\aleph_0}, \subseteq)$ ([17]), and thus (using results from [Paw86, BRS96])

(i) $\mathrm{non}(\mathcal{N}) = \aleph_1$;
(ii) $\mathfrak{b} = \mathfrak{b}^V$ and $\mathfrak{d} = \mathfrak{d}^V$;
(iii) $\mathrm{cov}(\mathcal{N}) \geq \max\{\mathrm{cof}([\lambda]^{\leq\aleph_0}), \mathrm{cov}(\mathcal{N})^V\}$; and
(iv) $\mathrm{non}(\mathcal{M}) = \max\{\mathrm{cof}([\lambda]^{\leq\aleph_0}), \mathrm{non}(\mathcal{M})^V\}$, and similarly for $\mathrm{cof}(\mathcal{M})$, $\mathrm{cof}(\mathcal{N})$ and $\mathfrak{c}$.

Assume that V is a model of Cichoń's Maximum as in the final extension of Theorem 4.8, where $\alpha_p := 1$, $\mu_i := \aleph_{\alpha_i}$ for $1 \leq i \leq 9$, with μ_6 and μ_9 possibly singular and satisfying $\mathrm{cof}(\mu_6) \geq \mu_3$ and $\mathrm{cof}(\mu_9) \geq \aleph_1$. Now, if $\lambda^{\aleph_0} = \lambda$ then, after adding λ many random reals, depending on the position of λ with respect to the μ_i's, we obtain the following constellations of Cichoń's diagram:

(1) $\mathrm{non}(\mathcal{N}) = \aleph_1 \leq \mathfrak{b} = \mu_3 \leq \mathfrak{d} = \mu_6 \leq \mathrm{cov}(\mathcal{N}) = \mathfrak{c} = \lambda$ when $\lambda \geq \mu_9$;
(2) $\mathrm{non}(\mathcal{N}) = \aleph_1 \leq \mathfrak{b} = \mu_3 \leq \mathfrak{d} = \mu_6 \leq \mathrm{cov}(\mathcal{N}) = \mathrm{cof}(\mathcal{N}) = \lambda < \mathfrak{c} = \mu_9$ when $\mu_8 \leq \lambda < \mu_9$;
(3) $\mathrm{non}(\mathcal{N}) = \aleph_1 \leq \mathfrak{b} = \mu_3 \leq \mathfrak{d} = \mu_6 \leq \mathrm{cov}(\mathcal{N}) = \mathrm{cof}(\mathcal{M}) = \lambda < \mathrm{cof}(\mathcal{N}) = \mu_8 \leq \mathfrak{c} = \mu_9$ when $\mu_6 \leq \lambda < \mu_8$;
(4) $\mathrm{non}(\mathcal{N}) = \aleph_1 \leq \mathfrak{b} = \mu_3 \leq \mathrm{cov}(\mathcal{N}) = \mathrm{non}(\mathcal{M}) = \lambda < \mathfrak{d} = \mathrm{cof}(\mathcal{M}) = \mu_6 \leq \mathrm{cof}(\mathcal{N}) = \mu_8 \leq \mathfrak{c} = \mu_9$ when $\mu_4 \leq \lambda < \mu_6$; and
(5) when $\lambda < \mu_4$, we have $\mathrm{non}(\mathcal{N}) = \aleph_1 \leq \mathfrak{b} = \mu_3 \leq \mathrm{non}(\mathcal{M}) = \mu_4 \leq \mathfrak{d} = \mathrm{cof}(\mathcal{M}) = \mu_6 \leq \mathrm{cof}(\mathcal{N}) = \mu_8 \leq \mathfrak{c} = \mu_9$, but the best we can say about $\mathrm{cov}(\mathcal{N})$ is $\mathrm{cov}(\mathcal{N}) \geq \max\{\lambda, \mu_2\}$.

(2)–(4) are examples of constellations of Cichoń's diagram with three possible singular values, namely $\mathrm{cov}(\mathcal{N})$, $\mathfrak{d}$ and $\mathfrak{c}$.

6. Relaxing the GCH requirement. So far, we often assumed GCH in the ground model to make the theorems easier to read. But the full power of this assumption is not required. In fact, finitely many assumptions about the cardinals at hand are enough, without requiring any changes in the proof.

In the following, we list the relevant theorems with the weaker assumptions. While this does not immediately give any new independence results, we still think that it can be useful, as it allows us to, e.g., construct and use forcings such as in Theorem 1.4 after a preparatory forcing that does something useful, e.g., on cardinals much smaller than λ_1, and by doing so

([17]) When $\lambda^{\aleph_0} = \lambda$ it is clear that $\mathrm{cof}([\lambda]^{\leq\aleph_0}) = \lambda$ and $\mathrm{cov}(\mathcal{N}) \geq \lambda$.

destroys GCH below λ_1. (It is easy to see that cardinal arithmetic below λ_1 is irrelevant for Theorem 1.4.)

THEOREM 1.3^+. *The conclusion of Theorem* 1.3 *holds under the following assumptions:* $\aleph_1 \leq \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \lambda_4 \leq \lambda_5$ *are cardinals, with* λ_i *regular for* $i \neq \lambda_5$, *and*

Constellation A: either

(i) λ_5 *is regular and there is some* $\mu \geq \lambda_5$ *with* $\mu^{<\lambda_3} = \mu$; *or*
(ii) $\lambda_5 = \lambda_5^{<\lambda_4}$, $\lambda_3 = \lambda_3^{<\lambda_3}$, $\lambda_4^{\aleph_0} = \lambda_4$, *and we set* $\mu := \lambda_5$.

Constellation B: $\lambda_5 = \lambda_5^{<\lambda_4}$ *and either*

(iii) $\lambda_2 = \lambda_3$; *or*
(iv) λ_3 *is* $\aleph_1$*-inaccessible,* $\lambda_2 = \lambda_2^{<\lambda_2}$ *and* $\lambda_4^{\aleph_0} = \lambda_4$.

Proof. For Constellation B, note that [Mej19b, Thm. A] does not assume GCH, and the same holds for Constellation A(i) by [BCM21, Thm. 5.3]. For Constellation A(ii), the assumptions can be weakened in the same way as in [Mej19b] for Constellation B. ∎

THEOREM 1.4^+. *The conclusion of Theorem* 1.4 *holds under the following assumptions:* $\aleph_1 < \kappa_9 < \lambda_1 < \kappa_8 < \lambda_2 < \kappa_7 < \lambda_3 \leq \lambda_4 \leq \lambda_5 \leq \lambda_6 \leq \lambda_7 \leq \lambda_8 \leq \lambda_9$, λ_i *regular for* $i \neq 5, 6$, κ_j *strongly compact for* $j = 7, 8, 9$, $\lambda_j^{\kappa_j} = \lambda_j$ *for all* $7 \leq j \leq 9$, *and*

Constellation A: either

(i) λ_5 *is regular and* $\lambda_6^{<\lambda_3} = \lambda_6$; *or*
(ii) $\lambda_3 = \lambda_3^{<\lambda_3}$, $\lambda_4^{\aleph_0} = \lambda_4$ *and*

(∗) λ_6 is regular, there is a strongly compact κ_6 with $\lambda_3 < \kappa_6 < \lambda_4$, $\lambda_6^{\kappa_6} = \lambda_6$ and $\lambda_5 = \lambda_5^{<\lambda_4}$.

Constellation B: (∗) *holds, and* λ_3 *is* $\aleph_1$*-inaccessible,* $\lambda_2 = \lambda_2^{<\lambda_2}$ *and* $\lambda_4^{\aleph_0} = \lambda_4$.

Proof. Again, for Constellation A(i) this can be found in [BCM21, Thm. 5.7], for Constellation B in [Mej19b, Thm. B]; and again apply the modifications of [Mej19b] to [GKS19]. ∎

The constructions in this paper then also give Theorem 3.10 under these weaker conditions (with the same proofs):

THEOREM 3.10^+. *The conclusion of Theorem* 3.10 *holds under the same assumptions as in Theorem* 1.4^+ *with the exception that* λ_9 *may be singular, and additionally* $\aleph_1 \leq \lambda_{\mathfrak{m}} \leq \lambda_{\mathfrak{p}} \leq \lambda_{\mathfrak{h}} \leq \kappa_9$, $\lambda_{\mathfrak{p}}^{<\lambda_{\mathfrak{p}}} = \lambda_{\mathfrak{p}}$, $\lambda_{\mathfrak{m}}$ *and* $\lambda_{\mathfrak{h}}$ *are regular, and* $\lambda_9^{<\lambda_{\mathfrak{h}}} = \lambda_9$.

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Martin Goldstern, Jakob Kellner
Institut für Diskrete Mathematik
und Geometrie
Technische Universität Wien
Wiedner Hauptstrasse 8-10/104
1040 Wien, Austria
E-mail: martin.goldstern@tuwien.ac.at
kellner@fsmat.at
http://www.tuwien.ac.at/goldstern/
http://dmg.tuwien.ac.at/kellner/

Saharon Shelah
Einstein Institute of Mathematics
Edmond J. Safra Campus, Givat Ram
The Hebrew University of Jerusalem
Jerusalem, 91904, Israel
and
Department of Mathematics
Rutgers University
New Brunswick, NJ 08854, USA
E-mail: shelah@math.huji.ac.il
http://shelah.logic.at

Diego A. Mejía
Faculty of Science
Creative Science Course (Mathematics)
Shizuoka University
Ohya 836, Suruga-ku
Shizuoka-shi, Japan 422-8529
E-mail: diego.mejia@shizuoka.ac.jp
http://www.researchgate.com/profile/Diego_Mejia2